Elliptic PDEs

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Handwritten lecture notes will appear on Moodle & www.dpmms.cam.ac.uk/ngt306/teaching.html

<u>Prerequisites</u>: Part III Analysis of PDEs.

<u>Reading</u>:
- Gilbarg & Trudinger, "Elliptic PDEs of Second Order"
- L. Simon, "Schauder estimates by scaling", Calc. Var. & PDE 5, 1997, pp. 391–407.
- minterscompactness.wordpress.com/lecture-notes/ (Paul Minter)
- Folland, "Introduction to PDEs"
- Evans & Gariepy, "Measure Theory & Fine Properties of Functions".

§ 0 Introduction

We will study 2^{nd} order elliptic PDEs on a domain $\Omega \subset \mathbb{R}^n$, as arising from e.g. variational problems, with the aim of building towards nonlinear PDEs. First need to understand the linear theory.

<u>Setup</u> Consider for $\Omega \subset \mathbb{R}^n$ open & bounded the "Lagrangian"

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x, z, p) \longmapsto F(x, z, p)$$

and the associated functional

will sometimes write ∂, ∇, or D

$$\mathcal{F}[u] := \int_\Omega F(x, u(x), Du(x))\, dx.$$

Assume F suff. regular. Let S be a suitable space of functions and $u \in S$,

$$u : \Omega \longrightarrow \mathbb{R}$$

(frequently $S = H^1(\Omega) = \{ f \in L^2(\Omega) : \nabla f \in L^2(\Omega) \}$, or e.g. $S = C^{1,\alpha}(\Omega)$ — to be introduced later). Suppose $u \in S$ minimizes $\mathcal{F}[u]$ subject to $u|_{\partial\Omega} = g$, $g : \partial\Omega \to \mathbb{R}$ given. That is, $\forall \varphi \in S$

$$\mathcal{F}[u + t\varphi] \geq \mathcal{F}[u] \quad \forall t \in \mathbb{R},$$

i.e.

$$\left(\frac{d}{dt} \mathcal{F}[u + t\varphi] \right) \Big|_{t=0} = 0,$$

or

$$\frac{d}{dt}\Big|_{t=0} \int_\Omega F(x, u + t\varphi, Du + tD\varphi)\, dx = 0.$$

Assuming enough regularity to exchange $\frac{d}{dt}$ & $\int$, we get

$$\int_\Omega (D_z F)(x, u, Du)\,\varphi + (D_i\varphi)(D_{p_i}F)(x, u, Du)\, dx$$

$$= 0 . \qquad (0.1)$$

To ensure the perturbed $u + t\varphi$ has the correct BC, we need $\varphi|_{\partial\Omega} = 0$, so integrating (0.1) by parts gives

$$\int_\Omega \varphi(x) \left(D_z F - D_i D_{p_i} F \right)(x, u, Dx) \, dx = 0$$

$\forall \varphi \in S$, and so (by the Fundamental Lemma of Calculus of Variations)

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial p_i} \right) = 0$$

in Ω. This is the Euler-Lagrange equation for $\mathcal{F}$. We can rewrite this as

$$\frac{\partial F}{\partial z} - \partial_i \partial_j u \frac{\partial^2 F}{\partial p_i \partial p_j} = 0, \qquad (0.2)$$

which is a quasilinear 2nd order PDE.

"coefficients of $\partial^2 u$ do not depend on $\partial^2 u$"

More generally, let us consider

$$a^{ij}(x, u, Du)\, D^2_{ij} u \quad - \quad b(x, u, Du) = 0 \qquad (0.3)$$

Definition 0.1

We say (0.3) is elliptic in Ω if $a^{ij}(x, u, Du)$ is a positive-definite matrix in Ω.

Remark In the case of (0.2), ellipticity is therefore equivalent to F convex in the p variable.

Example 0.2 (Dirichlet Energy)

When $F(x, z, p) = |p|^2$, one obtains

$$\Delta u = 0. \qquad (0.4)$$

Solutions to (0.4), i.e extremizers of $\mathcal{F}[u]$ for this F, are called "harmonic functions".

Example 0.3 (Minimal Surfaces)

When $F(x, z, p) = \sqrt{1+|p|^2}$ (exercise: write down $\mathcal{F}[u]$ and interpret it), one gets

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0. \quad (0.5)$$

Eq. (0.5) is the minimal surface equation.

Remark Locally, $\nabla u \sim$ const., so (0.5) looks similar to (0.4). However, the existence theory for (0.5) is very different & may fail — very different global properties.

Indeed, for entire solutions (ie defined on all of $\mathbb{R}^n$) we have the following.

Theorem 0.4 (Liouville)

If $u: \mathbb{R}^n \to \mathbb{R}$ is C^2 & $\Delta u = 0$ and bounded, then $u =$ const.

Theorem 0.5 (Bernstein)

The only entire solutions to (0.5) in $\mathbb{R}^n$ are planar (ie u linear) iff $n \le 7$.

Remark For further very recent developments see arXiv: 2401.01492.

§1 Harmonic Functions

§1.1 Basic Properties

Let $\Omega \subset \mathbb{R}^n$ be a domain, ie open and connected.

Definition 1.1 A function $u \in C^2(\Omega)$ is called

<u>harmonic</u> if $\Delta u = 0$

<u>subharmonic</u> if $\Delta u \geq 0$

<u>superharmonic</u> if $\Delta u \leq 0$

in Ω.

Write $B_r(y) = \{ x : |x-y| < r \}$.

Theorem 1.2 (Mean Value Property, MVP)

If $u \in C^2(\Omega)$ is subharmonic in Ω and $B_r(y) \subset \Omega$, then

$$u(y) \leq \frac{1}{\omega_n r^n} \int_{B_r(y)} u(x)\, dx \qquad (1.1)$$

and

$$u(y) < \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(y)} u(x)\, dx, \qquad (1.2)$$

where $\omega_n = |B_1(0)|$. If u is superharmonic, then the inequalities are reversed. If u is harmonic, then (1.1) & (1.2) are equalities.

Proof. We have for $\rho > 0$

$$0 \leq \int_{B_\rho(y)} \Delta u \, dx$$

$$\overset{\text{parts}}{=} \int_{\partial B_\rho(y)} \nabla u \cdot \underbrace{w}_{\substack{\text{outward unit} \\ \text{normal } w = \frac{x-y}{\rho}}} \, dx$$

$$\overset{\rho w = x-y}{=} \rho^{n-1} \int_{S^{n-1}} w \cdot \nabla u \, (y+\rho w) \, dw$$

$$\overset{\text{chain rule}}{=} \rho^{n-1} \int_{S^{n-1}} \frac{\partial}{\partial \rho} \big(u(y+\rho w) \big) \, dw.$$

This is true $\forall \rho > 0$, so

$$0 \leq \int_{S^{n-1}} \frac{\partial}{\partial \rho} \big(u(y+\rho w) \big) \, dw = \frac{\partial}{\partial \rho} \int_{S^{n-1}} u(y+\rho w) \, dw,$$

ie the map $\rho \longmapsto \int_{S^{n-1}} u(y+\rho w) \, dw$ is

<u>increasing</u>, ie for $\rho \leq r$

$$\int_{S^{n-1}} u(y+\rho w) \, dw \leq \int_{S^{n-1}} u(y+rw) \, dw.$$

By continuity, letting $\rho \to 0$ we get

$$n \omega_n u(y) \leq \frac{1}{r^{n-1}} \int_{\partial B_r(y)} u(x) \, dx$$

after changing back to x on the RHS. This gives (1.2). To get (1.1), integrate in r. The superharmonic case is similar, and the harmonic case follows by combining the sub- and super-harmonic cases. $\square$

<u>Remark</u> In fact, MVP characterizes harmonic functions — see Example Sheet 1.

Last time: $u \in C^2(\Omega)$ harmonic $\Leftrightarrow$ u satisfies MVP

Theorem 1.3 (Strong Maximum Principle (SMP) for Δ)

Suppose $u \in C^2(\Omega)$ is subharmonic in Ω, i.e. $\Delta u \geq 0$, and suppose that $\exists\, y_0 \in \Omega$ s.t. $u(y_0) = \sup_\Omega u$. Then $u = \text{const}$ in Ω.

Remark If u is superharmonic, then replace "sup" $\longrightarrow$ "inf". If u is harmonic, either "sup" or "inf" will work.

Proof Let $M = \sup_\Omega u < \infty$ and let $\Sigma = \{ y \in \Omega : u(y) = M \}$. By assumption, $\Sigma \neq \emptyset$ as $y_0 \in \Sigma$, and Σ is closed as u is cts.
As Ω connected, suffices to show Σ is open; then $\Sigma = \Omega$.

Pick $y \in \Sigma$. By MVP, for $e > 0$

s.t. $\overline{B_\rho(y)} \subset \Omega$ we have

$$u(y) \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u(x)\, dx$$

$$\begin{array}{c} \| \\ M \end{array}$$

$$\Rightarrow \int_{B_\rho(y)} (M - u(x))\, dx \leq 0$$

But by definition of M, $M - u \geq 0$, so we must have $u \equiv M$ on $B_\rho(y)$, and so $B_\rho(y) \subset \Sigma$.

Hence Σ is open. $\square$

Here SMP is easy since we have the MVP. For more general PDEs this is not the case. Must first obtain a weaker statement.

<u>Theorem 1.4</u> (Weak Maximum Principle (WMP) for Δ)

Suppose $\Omega \subset \mathbb{R}^n$ is bounded, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If u is subharmonic on Ω, then $\sup_\Omega u = \sup_{\partial\Omega} u$.

Remark If u is superharmonic, replace "sup" $\longrightarrow$ "inf". If u is harmonic, both hold.

Proof This is immediate from SMP: since Ω is bounded, $\sup_{\overline{\Omega}} u$ & $\inf_{\overline{\Omega}} u$ are attained in $\overline{\Omega}$. By the SMP, either $u \equiv \text{const}$ or the points at which $\sup_{\overline{\Omega}} u$ & $\inf_{\overline{\Omega}} u$ are attained are not in $\mathring{\Omega}$, ie on $\partial\Omega$.

$\square$

The MVP suggests that u cannot vary too much since it is always an average of itself. Turns out we can relate $\sup u$ & $\inf u$.

Theorem 1.5 (Harnack Inequality)

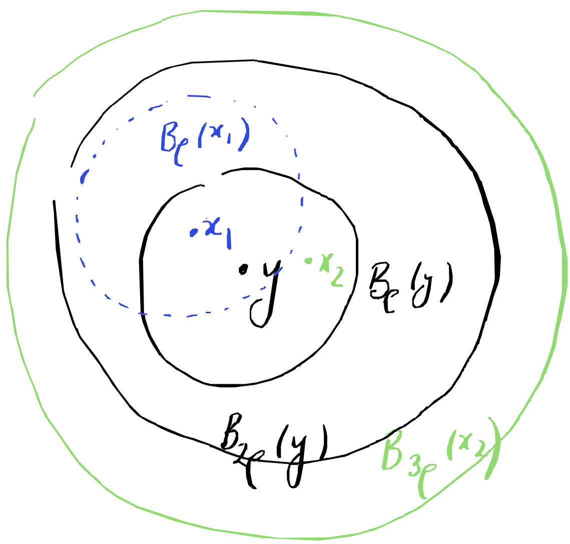
Suppose $u \in C^2(\Omega)$, $u \geq 0$, and $\Delta u = 0$ in Ω. Then if $\Omega' \subset\subset \Omega$ is any bounded subdomain, we have

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

for some $C = C(n, \Omega', \Omega)$ **but independent of u**.

Proof First, choose $y \in \Omega$ and $\rho > 0$ s.t. $\overline{B_{4\rho}(y)} \subset \Omega$, and pick $x_1, x_2 \in B_\rho(y)$.

Ω



MVP $\Rightarrow$

$B_\rho(x_1) \subset B_{2\rho}(y)$

$$u(x_1) = \frac{1}{\omega_n \rho^n}\int_{B_\rho(x_1)} u \le \frac{1}{\omega_n \rho^n}\int_{B_{2\rho}(y)} u$$

&

$$u(x_2) = \frac{1}{\omega_n (3\rho)^n}\int_{B_{3\rho}(x_2)} u \ge \frac{1}{\omega_n (3\rho)^n}\int_{B_{2\rho}(y)} u$$

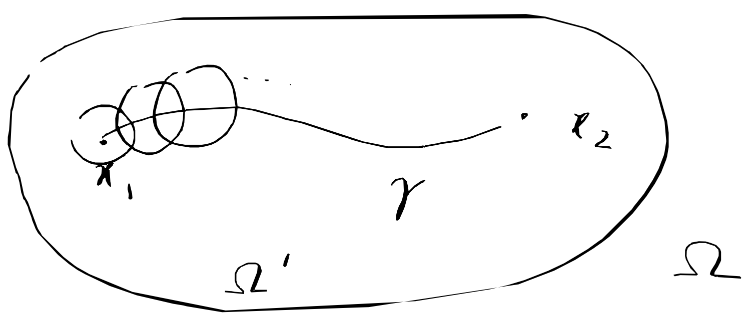
$B_{2\rho}(y) \subset B_{3\rho}(x_2)$

$$\Rightarrow \quad u(x_1) \le 3^n u(x_2) \quad \forall x_1, x_2 \in B_\rho(y).$$

So the desired inequality holds locally in balls, with constant indep. of ρ, y provided $\overline{B_{4\rho}(y)} \subset \Omega$.

So now choose $x_1, x_2 \in \overline{\Omega'} \subset \Omega$ s.t. $\sup_{\Omega'} u = u(x_1)$ & $\inf_{\Omega'} u = u(x_2)$.

By path connectedness of Ω, $\exists$ a cts path $\gamma \subset \overline{\Omega'}$ joining x_1 & x_2.



Choose $\rho > 0$ s.t. $4\rho < \operatorname{dist}(\gamma, \partial\Omega)$. Then choose $N = N(\Omega', \Omega)$ s.t. we can cover γ by N balls of radius ρ, $\gamma \subset \bigcup_{i=1}^{N} B_\rho(y_i)$ for some $y_i \in \Omega'$. Then apply the local result along γ in each ball to get

$$u(x_1) \leq \underbrace{3^n \ldots 3^n}_{N(\Omega';\Omega)} u(x_2) = 3^{nN} u(x_2).$$

$\square$

In fact can say more about harmonic functions:

<u>Theorem 1.6</u> (Derivative Estimates for Δ)

Suppose $u \in C^3(\Omega)$ is harmonic on Ω.

Then if $B_\rho(y) \subset \Omega$, then

$$|Du(y)| \leq \frac{C}{\rho} \sup_{\partial B_\rho(y)} |u|$$

for $C = C(n)$.

<u>Proof</u> $\Delta u = 0 \Rightarrow \Delta(D_i u) = 0$ in Ω,

So $D_i u$ also harmonic. By the MVP,

$$D_i u(y) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} D_i u$$

$$= \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} \nabla \cdot \begin{pmatrix} 0 \\ \vdots \\ u \\ \vdots \\ 0 \end{pmatrix}$$

divergence thm

$$= \frac{1}{\omega_n \rho^n} \int_{\partial B_\rho(y)} u(x)\, \nu_i(x)\, dx,$$

where $\nu_i(x) = \frac{x_i - y_i}{\rho}$. As $|\nu_i| \le 1$, we have

$$|D_i u(y)| \le \frac{1}{\omega_n \rho^n} \sup_{\partial B_\rho(y)} |u| \underbrace{\int_{\partial B_\rho(y)} dx}_{= n\rho^{n-1}\omega_n}$$

$$= \frac{n}{\rho} \sup_{\partial B_\rho(y)} |u|. \qquad \square$$

<u>Remark</u> The requirement that $u \in C^3(\Omega)$ may be relaxed to $u \in C^2(\Omega)$ by using a standard mollifier (exercise).

<u>Remark</u> Can apply this result repeatedly to get that for $\Omega' \subset \Omega'' \subset\subset \Omega$ and any multi-index α, by compactness of $\overline{\Omega'}$,

$$\sup_{\Omega'} |D^\alpha u| \leq C \sup_{\Omega''} |u| ,$$

whenever u is sufficiently regular, where $C = C(n, \alpha, \Omega, \Omega')$. In other words,

$$\| D^\alpha u \|_{L^\infty(\Omega')} \lesssim \| u \|_{L^\infty(\Omega'')}$$

"inequality up to a constant"

<u>Remark</u> In fact, these derivative estimates can be improved to L^1 instead of L^∞ on the RHS: by MVP, for some $y \in \overline{\Omega''} \subset \Omega$

$$\begin{aligned} \sup_{\Omega''} |u| &= |u|(y) \\ &= \left| c \int_{B_\epsilon(y)} u(x)\, dx \right| \\ &\leq C \int_{B_\epsilon(y)} |u| \end{aligned}$$

$$\lesssim C\int_\Omega |u|$$

$$\Rightarrow \quad \sup_{\Omega'} |D^\alpha u| \leq C\int_\Omega |u| .$$

In other words, $\|D^\alpha u\|_{L^\infty(\Omega')} \lesssim \|u\|_{L^1(\Omega)}$.

We can now prove:

<u>Theorem 1.7</u> (Liouville's Theorem for Δ)

If $u \in C^\infty(\mathbb{R}^n)$ is harmonic in $\mathbb{R}^n$ and grows <u>sublinearly at ∞</u>, then $u =$ const

<u>Remark</u> Here "u grows sublinearly at ∞" means $|u(x)| \lesssim C(1+|x|^\alpha)$ for some $\alpha \in (0,1)$.

<u>Proof</u> From <u>Thm 1.6</u>, $\forall y \in \mathbb{R}^n$

$$|Du(y)| \lesssim \frac{C}{e} \sup_{B_e(y)} |u|$$

$$\overset{\text{growth assumption}}{\lesssim} \frac{C}{e}\left(1+(e+|y|)^\alpha\right) .$$

Now let $\rho \to \infty$ to get $Du(y) = 0$.

As y was arbitrary, we have $Du \equiv 0$ and $u \equiv$ constant. $\square$

Last time: proved Liouville's Theorem for Δ.

One more consequence of the WMP:

<u>Theorem 1.8</u> (Uniqueness of Solutions to the Dirichlet Problem for Δ)

Suppose Ω is a bounded domain and $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} \Delta u_1 = \Delta u_2 & \text{in } \Omega \\ u_1 = u_2 & \text{on } \partial\Omega \end{cases}$$

Then $u_1 = u_2$ on $\overline{\Omega}$.

<u>Proof</u> Set $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω & $w = 0$ on $\partial\Omega$. WMP then implies $w = 0$ in $\overline{\Omega}$.

<u>Remark</u> Trivial but conceptually important; can get the same using integration by parts, but not all PDEs will be in "divergence form".

What about existence?

§1.2 Existence Theory for Harmonic Functions

Classical problem: solve the Dirichlet problem for Δ. Specifically, given $\Omega \subset \mathbb{R}^n$ bounded, and $\varphi : \overline{\Omega} \to \mathbb{R}$ continuous, we wish to find $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

We will assume for simplicity that $\partial\Omega$ is smooth and that $\varphi \in C^\infty(\overline{\Omega}; \mathbb{R})$.

Methods of approach

(1) Hilbert Space Method (cf III Analysis of PDEs).
Use the Riesz representation theorem to get a solution $u \in H^1(\Omega)$. Then deal with regularity separately. Relies on the linearity of the equation.

(II) <u>Direct Method of Calculus of Variations</u>

Rephrase $\Delta u = 0$ as a variational problem (ie $\delta \int |Du|^2 = 0$) and prove existence of solutions in $H^1(\Omega)$. Improve regularity later.

(III) <u>Perron's Method</u> Use the fact that solvability in balls implies solvability in more general domains. Based on maximum principles — see later.

<u>Remark</u> All three methods prove existence in a larger space first, then improve regularity later.

Let us look at (II) in more detail. Define

$$\mathcal{J} = \{ w \in H^1(\Omega) : w - \varphi \in H^1_0(\Omega) \},$$

ie functions which agree with φ on $\partial\Omega$.

Clearly $\varphi \in \mathcal{J}$, so $\mathcal{J} \neq \emptyset$. Set

$$\mathcal{E}[w] = \int_\Omega |Dw|^2$$

and $\beta := \inf_{w \in \mathcal{J}} \mathcal{E}[w] \geq 0$. By definition of infimum, $\exists$ a sequence $(w_j)_j \subset \mathcal{J}$ s.t. $\mathcal{E}[w_j] \longrightarrow \beta$. We wish to extract a convergent subsequence, and show that its limit is a solution. Clearly for j suff. large

$$\int_\Omega |Dw_j|^2 \leq \beta + 1 .$$

Also, by the Poincaré inequality, since $w_j - \varphi \in H^1_0(\Omega)$,

$$\int_\Omega |w_j - \varphi|^2 \leq C \int_\Omega |D(w_j - \varphi)|^2$$

using Young's ineq. $\Rightarrow$ $$\int_\Omega |w_j|^2 \leq C(\Omega, \varphi, \beta) < \infty ,$$

since $\varphi \in H^1(\Omega)$. Hence $\|w_j\|^2_{H^1(\Omega)} \leq C$ for j large, ie the sequence $(w_j)_j$ is uniformly bounded in $H^1(\Omega)$. Using

Banach - Alaoglu & Rellich - Kondrachov theorems, $\exists$ a subsequence $(w_{j_k})_k$ s.t

$$w_{j_k} \xrightarrow[\text{— Alaoglu}]{\text{Banach}} w \quad \text{in} \quad H^1(\Omega)$$

and $$w_{j_k} \xrightarrow[\text{— Kondrachov}]{\text{Rellich}} w \quad \text{in} \quad L^2(\Omega)$$

for some $w \in H^1(\Omega)$.

Quick digression! Recall Rellich - Kondrachov: if Ω bounded, $1 \leq p < n$, then

- $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$
- $W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \forall \; 1 \leq q < p^*$

ie the inclusion (identity) operator is compact, which implies convergence as above.

where $p^* = \frac{np}{n-p}$. When $p = 2$, $p^* = \frac{2n}{n-2}$

& always $q = 2 < \frac{2n}{n-2}$ iff $n > 2$.

Hence $\forall v \in H^1(\Omega)$

$$\int_\Omega Dw_{j_k} \cdot Dv \longrightarrow \int_\Omega Dw \cdot Dv$$

(as $\int_\Omega (w_{jk} - w) v \leq \|w_{jk} - w\|_{L^2} \|v\|_{L^2} \to 0$).

Also, clearly $w_{jk} - \varphi \to w - \varphi$ in $H^1(\Omega)$ as φ smooth. But $w_{jk} - \varphi \in H^1_0(\Omega)$ and $H^1_0(\Omega)$ is norm-closed, so weakly closed, hence $w - \varphi \in H^1_0(\Omega)$, ie $w \in \mathcal{J}$.

<u>Quick digression 2.</u>

<u>Lemma</u> If X is a normed vector space & $C \subset X$ convex subset, then C norm-closed $\iff$ C is weakly closed.

<u>Proof</u> $\Leftarrow$) Exercise (easy).

$\Rightarrow$) We show $X \setminus C$ is weakly open.

Let $x_0 \in X \setminus C$. By Hahn-Banach Separation, $\exists\, \phi \in X'$ such that $\phi|_C = 0$ & $\phi(x_0) \neq 0$.

Then $\{x \in X : |\phi(x)| > \frac{1}{2}|\phi(x_0)|\} \subset X \setminus C$ is a weakly open nbhd of x_0. $\square$

Finally, since $\mathcal{E}[\cdot]$ is sequentially weakly lower semicontinuous (swlsc) in $H^1(\Omega)$, we have

$$\mathcal{E}[w] \leq \liminf_{k\to\infty} \mathcal{E}[w_{j_k}] = \beta$$

and hence $\mathcal{E}[w] = \beta$.

Quick digression 3

$\mathcal{E}[w]$ swlsc in $H^1(\Omega)$ means $\forall\, u_j \rightharpoonup u$ in $H^1(\Omega)$,

$$\mathcal{E}[u] \leq \liminf_{j} \mathcal{E}[u_j].$$

To see this, note that $\forall\, v \in H^1(\Omega)$

$$\int_\Omega Du_j \cdot Dv \longrightarrow \int_\Omega Du \cdot Dv$$

So with $u = v$ we have

$$\int_\Omega Du_j \cdot Du \longrightarrow \int_\Omega |Du|^2$$

so

$$\mathcal{E}[u] = \lim_j \int_\Omega Du_j \cdot Du$$

$$= \liminf_j \int_\Omega Du_j \cdot Du$$

$$\leq \liminf_j \mathcal{E}[u_j]^{1/2} \mathcal{E}[u]^{1/2}.$$

We have found a global minimum w,

ie s.t. $\forall v \in H^1_0(\Omega)$, $w + tv \in \mathcal{g}$,

$$\mathcal{E}[w+tv] \geq \mathcal{E}[w],$$

ie the derivative at $t=0$ of $f(t) = \mathcal{E}[w+tv]$ vanishes. We have

$$f'(0) = D\mathcal{E}[w][v] = \lim_{t \to 0} \frac{\mathcal{E}[w+tv] - \mathcal{E}[w]}{t}$$

$$= \lim_{t \to 0} \left(2 \int_\Omega Dw \cdot Dv + O(t) \right)$$

$$= 0$$

$$\Rightarrow \int_\Omega Dw \cdot Dv = 0 \quad \forall v \in H^1_0(\Omega).$$

But this is exactly the weak formulation of $\Delta w = 0$, so we have proved the existence of a weak solution.

Next time: regularity (weak solutions are in fact smooth).

§ 1.3 Interior Regularity for Δ

We wish to prove more regularity for the weak solution. We have shown $\exists\, u \in L^1(\Omega)$, Ω bounded, s.t.

$$\int_\Omega u \,\Delta v = 0 \qquad \forall\, v \in C_c^\infty(\Omega).$$

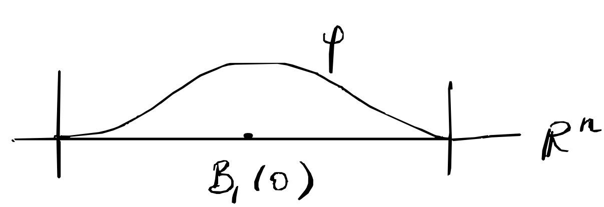
We have:

Theorem 1.9 (Weyl's Lemma)

Weakly harmonic functions are smooth. That is, for $\Omega \subset \mathbb{R}^n$ open and $u \in L^1_{loc}(\Omega)$

$$\int_\Omega u \,\Delta v = 0 \quad \forall\, v \in C_c^\infty(\Omega) \implies u \in C^\infty(\Omega) \;\&\; \Delta u = 0.$$

Proof Mollify u: take $\varphi \in C^\infty(\mathbb{R}^n)$ with $\varphi \equiv 0$ in $\mathbb{R}^n \setminus B_1(0)$, $\varphi \geq 0$, and $\int_{\mathbb{R}^n} \varphi = 1$,



where φ radially symmetric. Now for $\sigma > 0$ put

$$\varphi_\sigma(x) = \frac{1}{\sigma^n} \varphi\left(\frac{x}{\sigma}\right).$$

Then $\varphi_\sigma \in C_c^\infty(B_\sigma(0))$, $\varphi_\sigma \geq 0$, $\int_{\mathbb{R}^n} \varphi_\sigma = 1$, and finally define

$$u_\sigma(x) := (\varphi_\sigma * u)(x).$$

This is well-defined for

$$x \in \Omega_\sigma = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \sigma\},$$

u_σ is smooth in Ω_σ, and also $u_\sigma \to u$ in $L^1_{loc}(\Omega)$ (cf Thm 4.1 in Evans & Gariepy).

Then we claim that $\Delta u_\sigma = 0$ in Ω_σ.

Indeed,

$$\frac{\partial}{\partial x_i} u_\sigma(x) = \int_\Omega u(y) \frac{\partial}{\partial x_i}(\varphi_\sigma(x-y))\, dy$$

$$= -\int_\Omega u(y) \frac{\partial}{\partial y_i}(\varphi_\sigma(x-y))\, dy$$

$$\Rightarrow \quad \Delta_x u_\sigma(x) = \int_\Omega u(y)\, \Delta_y \left(\varphi_\sigma(x-y)\right) dy$$
$$= 0$$

as u weakly harmonic. Hence by derivative estimates, for $\Omega' \subset\subset \Omega$

$$\| D^\alpha u_\sigma \|_{L^\infty(\Omega')} \lesssim \| u_\sigma \|_{L^1(\Omega'_{\sigma_1})}$$

for some $\sigma_1(\Omega')$ small enough, where

$$\Omega'_{\sigma_1} = \{ x \in \Omega : \mathrm{dist}(x, \partial\Omega') < \sigma_1 \} \cup \Omega' .$$

But $u_\sigma \to u$ in $L^1_{loc}(\Omega)$, so for σ_1 small enough

$$\| u_\sigma \|_{L^1(\Omega'_{\sigma_1})} \leq \| u \|_{L^1(\Omega'_{\sigma_1})} + 1$$

and so $D^\alpha u_\sigma$ is uniformly bounded in $L^\infty(\Omega')$. Hence by Arzelà–Ascoli, as bounded derivatives $\Rightarrow$ equicontinuity, $\exists$ sequence $(\sigma_j)_{j=1}^\infty$, $\sigma_j \searrow 0$ and $\tilde{u} \in C^\infty(\Omega)$ s.t. $u_{\sigma_j} \to \tilde{u}$

in $C^k(\Omega')$ $\forall k \in \mathbb{N}$. Hence

$$\Delta \tilde{u} = \lim_j \Delta u_{\sigma_j} = 0 \quad \underbrace{\text{in } \Omega}_{\text{as } \Omega' \text{ was arbitrary}}$$

by continuity of derivatives. Since $u_{\sigma_j} \to u$ a.e. in Ω (from mollification), we must have $u = \tilde{u}$ a.e. in Ω. $\square$

Remark We do not treat boundary regularity here, but it is possible to get at least $u \in C^0(\overline{\Omega})$.

Let us now improve our $C^\infty(\partial\Omega)$ existence result to $C^0(\partial\Omega)$.

Theorem 1.10 (Existence & Uniqueness to the Dirichlet Problem for Δ with C^0 boundary data)
Suppose Ω is bounded with sufficiently smooth* boundary. Then for any $\varphi \in C^0(\partial\Omega)$

$\exists!\ u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ solving

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

<u>Remark</u> We may have $\int_\Omega |Du|^2 = \infty$ for this solution!

<u>Proof</u> Choose a sequence $(\varphi_n)_n \subset C^\infty(\mathbb{R}^n)$ s.t. $\varphi_n \to \varphi$ uniformly on $\partial\Omega$. We know that for each φ_n $\exists\ u_n \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ such that $\Delta u_n = 0$ in Ω and $u_n = \varphi_n$ on $\partial\Omega$.

Then $\forall n, m \in \mathbb{N}$, $\Delta(u_n - u_m) = 0$ in Ω, and $u_n - u_m = \varphi_n - \varphi_m$ on $\partial\Omega$. By the WMP,

$$\sup_{\Omega} |u_n - u_m| \leq \sup_{\partial\Omega} |u_n - u_m| = \|\varphi_n - \varphi_m\|_{L^\infty(\partial\Omega)} \to 0$$

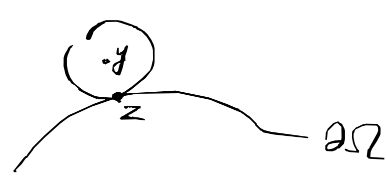
as $n, m \to \infty$. So $(u_n)_n$ is Cauchy in $C^0(\overline{\Omega})$. By completeness of $C^0(\overline{\Omega})$, we get that $\exists\, u \in C^0(\overline{\Omega})$ s.t. $u_n \longrightarrow u$ uniformly on $\overline{\Omega}$. In particular, $u = \varphi$ on $\partial\Omega$.

Further, by the interior derivative estimates for u_n, $(u_n)_n$ also converges in $C^k(\Omega')$ any k for any $\Omega' \subset\subset \Omega$. Hence we have $u \in C^\infty(\Omega)$, and $\Delta u = 0$ in Ω.

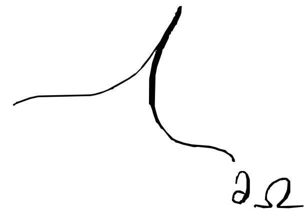
$\square$

Remarks

- If $\partial\Omega$ is C^2, then the requirement (*) of the theorem is satisfied. More generally, if Ω satisfies the "exterior sphere condition" : $\forall z \in \partial\Omega\ \exists\, B_\varepsilon(y) : \overline{B_\varepsilon(y)} \cap \overline{\Omega} = \{z\}$, then it is also satisfied.



- $\exists$ bounded domains for which the conclusion of the theorem does not hold, eg if $\partial\Omega$ has a cusp.



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§2 General 2nd Order Elliptic Operators

From now on, we write

$$Lu := a^{ij}\partial_i\partial_j u + b^i \partial_i u + cu.$$

We work on $\Omega \subset \mathbb{R}^n$ open, and here $u \in C^2(\Omega)$, $a^{ij}, b^i, c : \Omega \to \mathbb{R}$. Consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

for given $f : \Omega \to \mathbb{R}$, $\varphi : \partial\Omega \to \mathbb{R}$.

If we can write L in <u>divergence form</u>,

$$Lu = \partial_i (a^{ij} \partial_j u) + \tilde{b}^i \partial_i u + cu ,$$

then can use Hilbert space methods (cf III Analysis of PDEs). If not, however, different methods are needed. We will develop <u>"Schauder theory"</u>, whose idea is to deform the elliptic operator L into Δ using a sequence of rescalings, and which avoids Sobolev spaces.

Since $u \in C^2(\Omega)$, we wlog assume

$a^{ij} = a^{ji}$.

<u>Definition 2.1</u>

(1) L is <u>elliptic</u> in Ω if the matrix $a^{ij}(x)$ is positive definite $\forall x \in \Omega$. That is,

$$0 < \lambda(x) |\xi|^2 \leq a^{ij}(x) \xi^i \xi^j \leq \Lambda(x) |\xi|^2$$

$\forall \xi \in \mathbb{R}^n \setminus \{0\}$, where

$\lambda =$ min eigenvalue of a^{ij}

$\Lambda =$ max eigenvalue of a^{ij}

(ii) L is <u>strictly elliptic</u> in Ω if

$\exists\ \lambda_0 > 0$ s.t. $\lambda_0 < \lambda(x)\ \forall x \in \Omega$,

and

(iii) L is <u>uniformly elliptic</u> in Ω if L is elliptic and Λ/λ is uniformly bounded in Ω.

<u>Example</u>

The minimal surface equation

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0$$

has

$$a^{ij} = \left(\delta_{ij} - \frac{\nabla_i u \nabla_j u}{1+|\nabla u|^2} \right) \frac{1}{\sqrt{1+|\nabla u|^2}}.$$

This is always elliptic, but not uniformly.

In general, uniformly elliptic $\not\Rightarrow$ strictly elliptic.

Goal of this section: develop an existence & regularity theory for solutions $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, Ω bounded, f, φ continuous.

In fact, the right spaces will turn out to be $a^{ij}, b^i, c, f \in C^{0,\alpha}(\overline{\Omega})$, and we will prove $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$.

<u>Remark</u>

Since $a^{ij} \in C^{0,\alpha}(\overline{\Omega})$ and not C^1, we will not be able to write L in divergence form; $\rightsquigarrow$ no Sobolev theory.

§2.1 Basic Properties

<u>Theorem 2.2</u> (Weak Maximum Principle (WMP))

Suppose that L is elliptic and that $\sup_{\Omega} \left| \frac{b^i}{\lambda} \right| < \infty$. Suppose Ω is bounded, open, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu \geq 0$

(ie u is a <u>subsolution</u>). <u>Then</u>:

(I) if $c = 0$, then $\sup_{\Omega} u = \sup_{\partial\Omega} u$

(II) if $c \leq 0$, then $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$,

where $u^+ = \max\{u, 0\}$.

<u>Remark</u> The assumption $c \leq 0$ in Ω is crucial. For example, consider

- $n = 1$, $\Omega = (0, \pi)$, $u'' + u = 0$, $u(x) = \sin x$. Here $c \equiv 1 > 0$ and $\sup_{\Omega} u = 1$, but $\sup_{\partial\Omega} u^+ = 0$.

- $n = 2$, $\Omega = (0, \pi)^2$, $\Delta u + 2u = 0$, $u(x,y) = \sin(x)\sin(y)$. Here also $u|_{\partial\Omega} = 0$.

<u>Proof</u>

(I) ($c \equiv 0$) If $Lu > 0$ in Ω, then in fact there cannot be a local maximum in $\mathring{\Omega}$ (ie SMP holds). Indeed, if $x_0 \in \mathring{\Omega}$ is a local max., then

$$\partial_i u(x_0) = 0 \qquad \& \qquad \partial_i \partial_j u(x_0) \leq 0.$$

Since $a^{ij}(x_0) > 0$ (ellipticity), we have

$$a^{ij} \partial_i \partial_j u(x_0) = \mathrm{Tr}(a \cdot \nabla^2 u(x_0))$$

diagonalize both matrices & use cyclic property of trace $\leadsto \leq 0$.

Hence $0 < Lu(x_0) = \underbrace{a^{ij} \partial_i \partial_j u(x_0)}_{\leq 0} + \underbrace{b^i \partial_i u(x_0)}_{=0}$, which is a contradiction.

More generally, consider $Lu \geq 0$ and put $v(x) = e^{\gamma x_1}$, $\gamma > 0$ to be chosen.

Then have

$$\partial_1 v = \gamma e^{\gamma x_1}, \qquad \partial_i v = 0 \quad \forall i \neq 1,$$

$$\partial_1 \partial_1 v = \gamma^2 e^{\gamma x_1}, \qquad \partial_i \partial_j v = 0 \quad \forall (i,j) \neq (1,1)$$

So

$$Lv = e^{\gamma x_1}(\gamma^2 a'' + b'\gamma)$$

$$\overset{\text{ellipticity}}{\geq} e^{\gamma x_1}(\lambda\gamma^2 + b'\gamma)$$

$$= \lambda e^{\gamma x_1}\left(\gamma^2 + \frac{b'}{\lambda}\gamma\right) > 0 \quad \text{in } \Omega$$

choosing γ suff. large, which is possible by $\left\|\frac{b}{\lambda}\right\|_{L^\infty(\Omega)} < \infty$

Since $Lu \geq 0$, we therefore have $L(u+\varepsilon v) > 0$ $\forall \varepsilon > 0$. Applying the first case,

$$u(x) \overset{v \geq 0}{\leq} \sup_{\Omega}(u+\varepsilon v) \leq \sup_{\partial\Omega}(u+\varepsilon v)$$

$$\leq \sup_{\Omega} u + \varepsilon \sup_{\Omega} v .$$

Now take $\varepsilon \searrow 0$ to get $u(x) \leq \sup_{\partial\Omega} u$ $\forall x \in \Omega$, i.e.

$$\sup_{\Omega} \leq \sup_{\partial\Omega} u .$$

The other inequality $\sup_{\partial\Omega} u \le \sup_{\Omega} u$ is trivial.

(II) $(c \le 0)$ In this case, define

$$L_0 u = a^{ij}\partial_i\partial_j u + b^i \partial_i u,$$

and $\Omega^+ = \{x \in \Omega : u(x) > 0\}$. Since $cu|_{\Omega^+} \le 0$,

$$L_0 u = Lu - cu \ge 0 \quad \text{on } \Omega^+.$$

Note if $\Omega^+ = \phi$, then $u \le 0$ on Ω, and so $u^+ \equiv 0$, so the conclusion is trivial.

So assume $\Omega^+ \ne \phi$. Then $\partial\Omega^+ \cap \partial\Omega \ne \phi$ and $\exists\, x_0 \in \partial\Omega^+ \cap \partial\Omega$ s.t. $u(x_0) > 0$

(indeed, if not, then $u \le 0$ on $\partial\Omega^+$: since $\partial\Omega^+ \cap \partial\Omega = \phi$, so $\partial\Omega^+ \subset \mathring{\Omega}$, so $\partial\Omega^+ \subset \Omega \setminus \Omega^+$, so $u|_{\partial\Omega^+} \le 0$. But then this contradicts (1) for L_0 on Ω^+.)

Hence

$$\sup_{\Omega} u \overset{\Omega^+ \neq \phi}{=} \sup_{\Omega^+} u \overset{(1)}{=} \sup_{\partial\Omega^+} u \overset{(*)}{\leq} \sup_{\partial\Omega} u \overset{u \leq u^+}{\leq} \sup_{\partial\Omega} u^+ ,$$

where $(*)$ is true because points on $\partial\Omega^+$ are either in $\mathring{\Omega}$ (when $u = 0$), or on $\partial\Omega$. $\square$

Some corollaries

<u>Corollary 2.3</u> Let Ω be bounded, open, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Suppose L is elliptic, $\sup_{\Omega} \frac{|b|}{\lambda} < \infty$, and $c \leq 0$ in Ω.

Then

(i) if $Lu \leq 0$ in Ω, then

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-,$$

where $u^- = \min\{u, 0\}$.

(ii) if $Lu = 0$, then $\sup_{\partial\Omega} |u| = \sup_{\Omega} |u|$.

Proof Exercise

Corollary 2.4 Let L, Ω be as in Cor. 2.3. Suppose $u, v, w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$Lu \geq 0, \quad Lv = 0, \quad Lw \leq 0 \quad \text{in } \Omega$$

Then

(i) if $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$;

(ii) if $v \leq w$ on $\partial\Omega$, then $v \leq w$ in $\overline{\Omega}$.

Proof Exercise.

In order to build towards the SMP, we need the following.

Theorem 2.5 (Hopf Boundary Point Lemma)

Let $\Omega \subset \mathbb{R}^n$ be open, $y \in \partial\Omega$, and suppose Ω satisfies the interior sphere condition at y: $\exists\, R > 0$, $z \in \Omega$ s.t. $B_R(z) \subset \Omega$ & $y \in \partial B_R(z)$. Let L be uniformly elliptic in Ω with

$$\sup_{\Omega} \frac{|b|}{\lambda} + \sup_{\Omega} \frac{|c|}{\lambda} < \infty.$$

Suppose $u \in C^2(\Omega) \cap C^0(\{y\} \cup \Omega)$ satisfies $u(y) > u(x)$ $\forall\, x \in \Omega$, and $Lu \geq 0$ in Ω. Finally, assume one of the following:

(I) $c \equiv 0$ in Ω

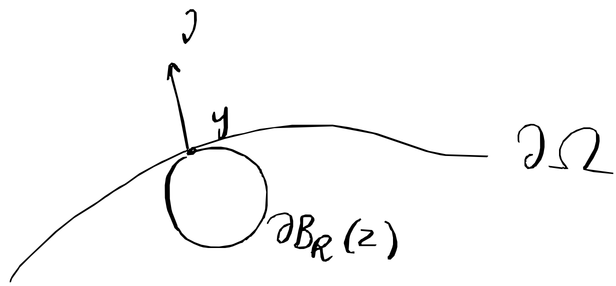
(II) $c \leq 0$ in Ω & $u(y) \geq 0$

(III) $u(y) = 0$ (& no assumption on c).

<u>Then</u> $\frac{\partial u}{\partial \nu}(y) > 0$ if it exists,

where ν is the outward unit normal

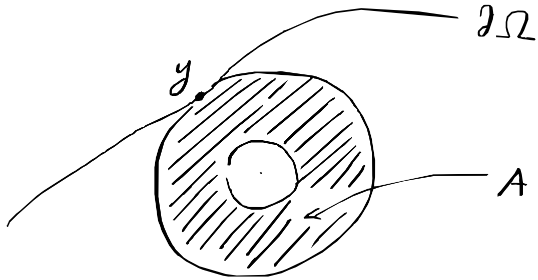
to $\partial B_R(z)$.



<u>Remark</u> $\frac{\partial u}{\partial \nu}(y) \geq 0$ is trivial by WMP;

the content is the <u>strict</u> inequality.

Proof Let $A = B_R(z) \setminus B_{R/2}(z)$.



Cases (i) + (ii): On A consider

$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2}.$$

Clearly

$$\partial_i v(x) = -2\alpha (x_i - z_i) e^{-\alpha|x-z|^2},$$

$$\partial_i \partial_j v(x) = -2\alpha \delta_{ij} e^{-\alpha|x-z|^2} + 4\alpha^2 (x_i - z_i)(x_j - z_j) e^{-\alpha|x-z|^2},$$

So on A

$$Lv = e^{-\alpha|x-z|^2}\Big(a^{ij} 4\alpha^2 (x_i - z_i)(x_j - z_j) - 2\alpha a^{ii} - 2\alpha b^i (x_i - z_i) + c\Big) - ce^{-\alpha R^2}$$

ellipticity & sgn(c)

$$\geq e^{-\alpha|x-z|^2}\Big(4\alpha^2 \lambda(x) \underbrace{|x-z|^2}_{\geq (R/2)^2 \text{ on } A} - 2\alpha n \Lambda(x)$$

$$- 2\alpha |b| |x-z| - |c| \Big)$$

$$\geq e^{-\alpha|x-z|^2} \lambda(x) \Big(\alpha^2 R^2 - 2\alpha n \sup_{\Omega} \frac{\Lambda}{\lambda}$$

$$- \alpha R \sup_{\Omega} \frac{|b|}{\lambda} - \sup_{\Omega} \frac{|c|}{\lambda} \Big)$$

> 0 for α suff. large.

Fix such an α. Then let

$$w(x) = u(x) - u(y) + \varepsilon v(x) ,$$

$\varepsilon > 0$ to be chosen. We have

$$Lw = Lu + \varepsilon Lv - c u(y) \geq 0$$

in A, by above. Also, $v|_{\partial B_R(z)} = 0$

& $u(x) \leq u(y)$ on $\overline{\Omega}$, so $w|_{\partial B_R(z)} \leq 0$.

Moreover, $u(x) < u(y)$ on $\partial B_{R/2}(z)$, so we can choose $\varepsilon > 0$ small enough s.t.

$$w\big|_{\partial B_{R/2}(z)} < 0 .$$

Hence $w\big|_{\partial A} \leq 0$ Applying WMP to w in A, we get

$$u(x) - u(y) + \varepsilon v(x) \leq 0 \quad \text{in } A .$$

So for $t<0$ small we have

$$\frac{u(y+t\nu) - u(y)}{t} \geq -\varepsilon \, \frac{v(y+t\nu) - \overbrace{v(y)}^{=0}}{t} .$$

Sending $t \nearrow 0$,

$$\frac{\partial u}{\partial \nu}(y) \geq -\varepsilon \frac{\partial v}{\partial \nu}(y)$$

$$= -\varepsilon \, \partial_i v(y) \left(\frac{y_i - x_i}{R} \right)$$

$$= 2\alpha \varepsilon R e^{-\alpha R^2} > 0 .$$

<u>Case (III)</u> : $(u(y)=0)$

Consider $\tilde{L} := L - c^+$ s.t.

$$\tilde{L}u = a^{ij}\partial_i\partial_j u + b^i\partial_i u + \underbrace{(c-c^+)}_{\leq 0}u.$$

Then

$$\tilde{L}u = \underbrace{Lu}_{\geq 0} - \underbrace{c^+u}_{\leq c^+u(y)=0} \geq 0,$$

so apply previous case to $\tilde{L}$. $\square$

With the Hopf Boundary Point Lemma, we are ready to prove the SMP.

<u>Theorem 2.6</u> (Strong Maximum Principle (SMP))

Suppose $\Omega \subset \mathbb{R}^n$ is a domain, not necessarily bounded, s.t. $\partial\Omega \neq \emptyset$ satisfies the interior sphere condition $\forall y \in \partial\Omega$. Let L be uniformly elliptic with $\sup_\Omega \left(\frac{|b|}{\lambda} + \frac{|c|}{\lambda}\right) < \infty$, $u \in C^2(\Omega)$,

$M = \sup_\Omega u < \infty$, and

$$Lu \geq 0 \quad \text{in } \Omega.$$

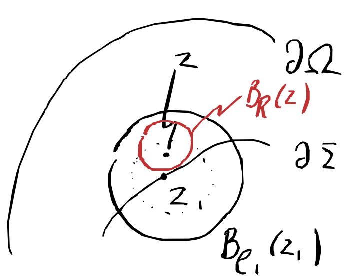
Then

(I) if $c = 0$ & $u(y) = M$ for some $y \in \Omega$, then $u \equiv M$ in Ω,

(II) if $c \leq 0$, $M \geq 0$ and $u(y) = M$ for some $y \in \Omega$, then $u \equiv M$ in Ω,

(III) if $M = 0$ and $u(y) = M = 0$ for some $y \in \Omega$, then $u \equiv 0$ in Ω.

Proof Let $\Sigma = \{ x \in \Omega : u(x) = M \}$. By continuity, Σ is closed in Ω. Suppose $\Omega \setminus \Sigma \neq \phi$. Pick $z \in \Omega \setminus \Sigma$ s.t. $\text{dist}(z, \partial\Omega) > \text{dist}(z, \partial\Sigma)$. How?



1. First pick $z_1 \in \partial\Sigma \cap \Omega$
2. Then pick $\rho_1 > 0$ s.t. $B_{\rho_1}(z_1) \subset \Omega$
3. Then just pick any $z \in B_{\rho_1/2}(z_1) \setminus \Sigma$

Then let $R = \sup \{ \rho : B_\rho(z) \subset \Omega \setminus \Sigma \}$.

By choice of z, $\overline{B_R(z)}$ will intersect $\partial\Sigma$ but not $\partial\Omega$. So $\exists\ y \in \partial B_R(z) \cap \Sigma$. Since $Du(y) = 0$, this contradicts the Hopf Lemma.

So $\Omega \setminus \Sigma = \phi$, ie $\Omega = \Sigma$.

The three cases follow directly from Hopf. $\square$

Some corollaries:

<u>Corollary 2.7</u> (Comparison Principle)

Let $L = a^{ij}\partial_i\partial_j + b^i\partial_i + c$ be uniformly elliptic in a domain $\Omega \subset \mathbb{R}^n$, with $\sup_\Omega \left(\frac{|b|}{\lambda} + \frac{|c|}{\lambda}\right) < \infty$. Suppose $u, v \in C^2(\Omega)$ satisfy

$$Lu \geq Lv \qquad \text{and} \qquad u \leq v$$

in Ω. Then $u = v$ on Ω, <u>or</u> $u < v$ everywhere in Ω.

Proof Have $L(u-v) \geq 0$ in Ω & $u-v \leq 0$ in Ω; if $\exists\ x_0 \in \Omega$ s.t. $u(x_0) = v(x_0)$, then SMP (iii) $\Rightarrow$ $u \equiv v$ in Ω. If not, then $u \not\equiv v$ in Ω, and so $u < v$ in Ω. $\square$

Corollary 2.8 (Uniqueness for the Neumann Problem)
Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $\partial\Omega$ satisfies the interior sphere condition at each point. Suppose L is uniformly elliptic with $\sup_{\Omega}\left(\frac{|b|}{\lambda} + \frac{|c|}{\lambda}\right) < \infty$, and $c \leq 0$.
Then if $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ s.t.

$$\begin{cases} Lu_i = f & \text{in } \Omega \\ \dfrac{\partial u_i}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

($i \in \{1,2\}$) for some $f: \Omega \to \mathbb{R}$, $g: \partial\Omega \to \mathbb{R}$, then $u_1 = u_2 + c$ for some constant c.

Proof The difference $u = u_1 - u_2$ satisfies

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega . \end{cases}$$

Let $M = \sup_{\overline{\Omega}} u$. WLOG $M \geq 0$ (otherwise consider $-u$). By SMP, if $u \not\equiv M$ on $\overline{\Omega}$, then $\exists\, y \in \partial\Omega$ such that $u(y) = M$ and $u(x) < u(y)$ $\forall\, x \in \Omega$. By the Hopf Lemma, we get $\frac{\partial u}{\partial \nu}(y) \neq 0$, which is a contradiction. $\square$

Remark This says that the Neumann problem with zero data has solutions which are constants, $L(M) = 0$. But $L(M) = Mc(x) = 0$ $\forall\, x \in \Omega$, so if $c \not\equiv 0$ this implies $M = 0$, so solutions are unique. This constant only non-zero when $c \equiv 0$.

Remark on the proof of SMP: it is not needed for $\partial\Sigma$ to satisfy the ISC at every point. Recall we construct $B_R(z) \subset \Omega \setminus \Sigma$ s.t. $\exists$ $y \in \partial B_R(z) \cap \Sigma$ at which $Du(y) = 0$. Then apply Hopf e.g. in $B_R(z)$ (in particular, $\partial\Sigma$ satisfies ISC at y, but we have no control over where y is).
Moreover, the condition that $\partial\Omega$ satisfy ISC is unnecessary.

Theorem 2.9 (Maximum Principle A Priori Estimate (MPAPE))

Suppose $\Omega \subset \mathbb{R}^n$ bounded domain, L elliptic, $c \leq 0$, $\beta = \frac{|b|}{\lambda} \in L^\infty(\Omega)$, and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ & $f : \Omega \to \mathbb{R}$.
Then

(I) if $Lu \geq f$, then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \sup_\Omega \left(\frac{|f|}{\lambda}\right).$$

(II) If $Lu = f$, then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C \sup_\Omega \left(\frac{|f|}{\lambda}\right),$$

where $C = C(\beta, \operatorname{diam}(\Omega))$.

Proof of MPAPE

Put $d = \operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|$. As Ω bounded, we have

$$\Omega \subset \{x : a \leq x_1 \leq a+d\}$$

for some $a \in \mathbb{R}$. WLOG $a=0$. Idea: construct subsolution & use WMP. Let

$$V(x) = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f|}{\lambda},$$

α to be chosen. We compute

$$(a^{ij}\partial_i\partial_j + b^i\partial_i)e^{\alpha x_1} = e^{\alpha x_1}(a^{11}\alpha^2 + b^1\alpha)$$

$$\underset{\text{by ellipticity}}{\overset{a^{11}\geq\lambda}{\geq}} e^{\alpha x_1}\lambda\left(\alpha^2 + \frac{b^1}{\lambda}\alpha\right)$$

$$\overset{\frac{b^1}{\lambda}\geq-\beta}{\geq} e^{\alpha x_1}\lambda(\alpha^2 - \beta\alpha)$$

$$\underset{\alpha=\beta+1}{\overset{\text{taking}}{\geq}} \lambda$$

Hence

$$Lv = (a^{ij}\partial_i\partial_j + b^i\partial_i)\left(-e^{\alpha x_1}\sup\frac{|f|}{\lambda}\right) + cv$$

$$\leq cv - \lambda \sup_{\Omega}\frac{|f|}{\lambda}$$

since
$c \leq 0$, $v \geq 0$

$$\leq -\lambda \sup_{\Omega}\frac{|f|}{\lambda}$$

Then

(1) if $Lu \geq f$,

$$L(u-v) \geq f + \lambda \sup_{\Omega}\frac{|f|}{\lambda}$$

$$= \lambda\left(\frac{f}{\lambda} + \sup_{\Omega}\frac{|f|}{\lambda}\right)$$

$$\geq 0.$$

Since $u \leq u^+$, $u|_{\partial\Omega} \leq v|_{\partial\Omega}$ (from def. of v), so by WMP, $u \leq v$ in Ω. So

$$\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} \frac{|f|}{\lambda},$$

where $C = \sup_{\Omega} \left(e^{(\beta+1)d} - e^{(\beta+1)x_1} \right)$.

(ii) If $Lu = f$, apply (i) to $-u$ as well & combine $\square$

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§ 2.2 Hölder Spaces

Fix $\Omega \subset \mathbb{R}^n$ open and $\alpha \in (0,1]$.

Definition 2.10

We say that $u : \Omega \to \mathbb{R}$ is uniformly Hölder continuous with exponent α, or uniformly α-Hölder continuous, if

$$[u]_{\alpha,\Omega} := \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}} < \infty$$

We call $[u]_{\alpha, \Omega}$ the <u>Hölder semi-norm</u>

<u>Remark</u> if $\alpha = 1$, u is <u>uniformly Lipschitz</u>.

If $\alpha > 1$, then u would be constant by MVT.

<u>Definition 2.11</u>

We say that u is <u>locally α-Hölder continuous</u> in Ω if $\forall\, K \subset\subset \Omega$, $u|_K : K \to \mathbb{R}$ is uniformly α-Hölder continuous.

Let $k \in \mathbb{N} \cup \{\infty\}$. Recall for a multi-index $\beta \in \mathbb{N}^n$, $|\beta| = \sum_i \beta_i$ and

$$C^k(\Omega) := \{ u : \Omega \to \mathbb{R} : D^\beta u \text{ exists } \& \text{ is continuous } \forall \beta \text{ s.t. } |\beta| \leq k \}$$

Definition 2.12

We define the <u>Hölder spaces</u>

$$C^{k,\alpha}(\Omega) := \{ u \in C^k(\Omega) : D^\beta u \text{ is locally } \alpha\text{-Hölder continuous } \forall \beta \text{ s.t. } |\beta| = k \}$$

<u>and</u>

$$C^{k,\alpha}(\overline{\Omega}) := \{ \text{—, —} \ \underline{\text{uniformly}} \ \alpha\text{-Hölder} \ \text{—, —} \}.$$

If $\alpha \in (0,1)$, we write

$$C^{\alpha}(\Omega) := C^{0,\alpha}(\Omega),$$

$$C^{\alpha}(\overline{\Omega}) := C^{0,\alpha}(\overline{\Omega}),$$

and for $k \in \mathbb{N} \cup \{\infty\}$

$$C^{k,0}(\Omega) := C^{k}(\Omega),$$

$$C^{k,0}(\overline{\Omega}) := C^{k}(\overline{\Omega}).$$

<u>Remark</u> Note that $C^{k+1}(\Omega) \neq C^{k,1}(\Omega)$, e.g. $C^1 \neq C^{0,1}$, as Lipschitz $\not\Longrightarrow$ continuously differentiable. However note Lipschitz $\Rightarrow$ differentiable a.e.

Also define

$$C_0^{k,\alpha}(\Omega) \equiv C_c^{k,\alpha}(\Omega)$$
$$:= \{ u \in C^{k,\alpha}(\Omega) : \operatorname{supp}(u) \subset \Omega \text{ compact} \},$$

where $\operatorname{supp}(u) = \overline{\{ x \in \Omega : u(x) \neq 0 \}}$.

Next we define norms & semi-norms on Hölder spaces. For $k \in \mathbb{N}$, $u \in C^k(\overline{\Omega})$ set

$$[u]_{k;\Omega} \equiv |D^k u|_{0;\Omega}$$
$$:= \sup_{|\beta|=k} |D^\beta u|_{0;\Omega}$$
$$= \sup_{|\beta|=k} \left(\sup_{x \in \Omega} |D^\beta u(x)| \right).$$

For $u \in C^{k,\alpha}(\overline{\Omega})$, set

$$[u]_{k,\alpha;\Omega} \equiv [D^k u]_{\alpha;\Omega}$$
$$:= \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega} .$$

Note these are <u>semi</u>-norms. To get norms, put

$$\|u\|_{C^k(\overline{\Omega})} \equiv |u|_{k;\Omega}$$
$$\equiv |u|_{k,0;\Omega}$$
$$:= \sum_{j=0}^{k} |D^j u|_{0;\Omega}$$

and

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \equiv |u|_{k,\alpha;\Omega}$$
$$:= |u|_{k;\Omega} + [D^k u]_{\alpha;\Omega} .$$

With these norms C^k, $C^{k,\alpha}$ become <u>Banach spaces</u>.

We will need to understand the compactness properties of these spaces.

Theorem 2.13 (Arzelà-Ascoli for Hölder Spaces)

Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, $\alpha \in (0,1]$.

If $(u_j)_j \subset C^{k,\alpha}(\Omega)$ satisfies

$$\sup_j |u_j|_{k,\alpha;\Omega'} < \infty$$

$\forall \Omega' \subset\subset \Omega$, then $\exists$ $u \in C^{k,\alpha}(\Omega)$ and a subsequence $(u_{j'})_{j'}$ s.t.

$$u_{j'} \longrightarrow u \quad \text{in } \underline{\underline{C^k(\overline{\Omega}')}}$$

$\forall \Omega' \subset\subset \Omega$.

Remark There is no claim about convergence in $C^{k,\alpha}(\overline{\Omega}')$.

Proof Example Sheet 2.

We need two more ingredients before we can start Schauder theory.

Ingredient 1 Interpolation. If we have Banach spaces $X \subset\subset Y \subset Z$, we can bound the norm in Y by X & Z norms. Interpolation is the interchange of sizes of X-norm & Z-norm. For us: $X = C^{k,\alpha}(\bar{\Omega})$, $Y = C^k(\bar{\Omega})$, $Z = C^0(\bar{\Omega})$.

Theorem 2.14 (Interpolation Inequality for Hölder Spaces)

Let $\varepsilon > 0$, $l \in \mathbb{N}$, $\alpha \in (0,1]$. Then $\exists$ $C = C(n, l, \alpha, \varepsilon) \in (0, \infty)$ s.t. if $u \in C^{l,\alpha}(\overline{B_R(x_0)})$, then

$$R^k |D^k u|_{0; B_R(x_0)} \leq \varepsilon R^{l+\alpha} [D^l u]_{\alpha; B_R(x_0)} + C |u|_{0; B_R(x_0)}$$

$\forall\ 0 \leq k \leq l$.

Sketch proof (Details in Example Sheet 2)

By rescaling and considering $v(x) = u(x_0 + Rx)$, suffices to prove for $R=1$ & $x_0 = 0$. Then argue by contradiction & Arzelà–Ascoli. $\square$

Ingredient 2. The Absorbing Lemma.

Let $B_R(x) \subset \mathbb{R}^n$ be fixed & S a non-negative sub-additive function on the collection of sub-balls of $B_R(x)$, ie if

$$B_\rho(y) \subset \bigcup_{j=1}^{N} B_{\rho_j}(y_j) \subset B_R(x),$$

then

$$S(B_\rho(y)) \leq \sum_{j=1}^{N} S(B_{\rho_j}(y_j)).$$

Theorem 2.15 (Simon's Absorbing Lemma)

Let $\lambda \in [0, \infty)$, $\vartheta \in (0,1)$. Then $\exists$ $\delta = \delta(n, \lambda, \vartheta) \in (0,1)$ such that the following is true.

Suppose for all balls $B_\rho(y) \subset B_R(x)$ we have

$$\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta \rho^\lambda S(B_\rho(y)) + \gamma$$

for some fixed $\gamma > 0$. Then

$$R^\lambda S(B_{\theta R}(x)) \leq C\gamma,$$

$C = C(n, \theta, \lambda)$.

Proof of Simon's Absorbing Lemma.

Put

$$Q = \sup_{B_\rho(y) \subset B_R(x)} \rho^\lambda S(B_{\theta\rho}(y)).$$

By sub-additivity of S, we have

$$Q \leq R^\lambda S(B_R(x)) < \infty.$$

Recall we have

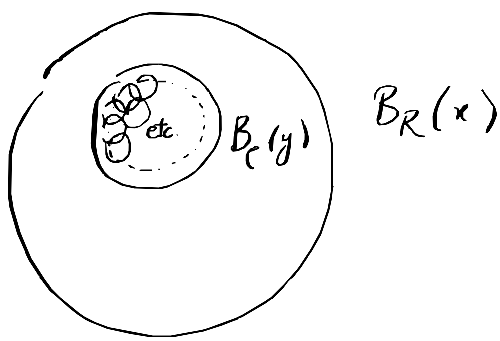
$$\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta \rho^\lambda S(B_\rho(y)) + \gamma. \quad (*)$$

Applying this with $\rho \longrightarrow \theta\rho$, we have that for $B_\rho(y) \subset B_R(x)$

$$(\theta\rho)^\lambda S(B_{\theta^2\rho}(y)) \leq \delta\,(\theta\rho)^\lambda S(B_{\theta\rho}(y)) + \gamma$$

$$\leq \delta\,\theta^\lambda Q + \gamma.$$

Fix any $B_\rho(y) \subset B_R(x)$. Cover $B_{\theta\rho}(y)$ by balls $\{B_{(1-\theta)\theta^2\rho}(z_j)\}_{j=1}^N$, $N \leq \underbrace{C(\theta, n)}_{\text{does not depend on } \rho \text{ or } y!}$, $z_j \in B_{\theta\rho}(y)$.



How? Choose a maximal pairwise disjoint collection of balls

$$\left\{ B_{\frac{(1-\theta)\theta^2\rho}{2}}(z_j) \right\}_{j=1}^N$$

with $z_j \in B_{\theta\rho}(y)$. We claim that these z_j's work. Indeed, if not, then

$$\exists\, z \in B_{\theta_\rho}(y) \setminus \bigcup_{j=1}^{N} B_{(1-\theta)\theta^2_\rho}(z_j).$$

But then we would have

$$d(z, z_j) \geq (1-\theta)\theta^2_\rho \qquad \forall j,$$

so

$$B_{\frac{(1-\theta)\theta^2_\rho}{2}}(z) \cap B_{\frac{(1-\theta)\theta^2_\rho}{2}}(z_j) = \phi,$$

which contradicts maximality.

Bound on N: note

$$\bigcup_{j=1}^{N} B_{\frac{(1-\theta)\theta^2_\rho}{2}}(z_j) \subset B_{\frac{(1-\theta)\theta^2_\rho}{2} + \theta_\rho}(y)$$

(simply from radii). This gives a volume bound: LHS are disjoint, so

$$N \omega_n \left(\frac{(1-\theta)\theta^2_\rho}{2}\right)^n \leq \omega_n \left(\frac{(1-\theta)\theta^2_\rho}{2} + \theta_\rho\right)^n$$

$$\Rightarrow \quad N \leq \frac{\left(\frac{(1-\theta)\theta}{2} + 1\right)^n}{\left(\frac{(1-\theta)\theta}{2}\right)^n}.$$

By sub-additivity, we have

$$\rho^{\lambda} S(B_{\theta\rho}(y)) \leq \rho^{\lambda} \sum_{j=1}^{N} S\left(B_{(1-\theta)\theta^2\rho}(z_j)\right)$$

(*) with $\rho \rightsquigarrow (1-\theta)\theta\rho$

$$\leq ((1-\theta)\theta)^{-\lambda} \sum_{j=1}^{N} \Big(\delta \underbrace{((1-\theta)\theta\rho)^{\lambda} S(B_{(1-\theta)\theta\rho}(z_j))}_{\leq Q} + \gamma \Big)$$

$$\leq \delta((1-\theta)\theta)^{-\lambda} N Q + N\gamma((1-\theta)\theta)^{-\lambda}.$$

Now take the sup over all $B_\rho(y) \subset B_R(x)$:

$$Q \leq \delta C_1 Q + C_2 \gamma,$$

where C_1, C_2 depend only on n, θ & λ.

Choose $\delta = \frac{1}{2C_1}$; then

$$Q \leq 2C_2\gamma. \qquad \square$$

§3 Schauder Theory

§3.1 Interior Schauder Estimates

We will first prove interior estimates in the unit ball, and then extend to general domains afterwards.

The main point is: if coefficients of L are α-Hölder continuous, then any $C^{2,\alpha}$ solution of $Lu=f$ can be bounded in $C^{2,\alpha}$ on a smaller ball by $|u|_0$ and a Hölder norm of f.

Theorem 3.1 (Unit Scale Interior Schauder Estimate)

Let $\alpha \in (0,1)$, $\beta > 0$. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(B_1(0))$, with

$$|a^{ij}|_{0,\alpha;B_1(0)} + |b^i|_{0,\alpha;B_1(0)} + |c|_{0,\alpha;B_1(0)} \leq \beta.$$

Assume L strictly elliptic. Then if $u \in C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$ and $f \in C^{0,\alpha}(B_1(0))$ satisfy

$Lu = f$ in $B_1(0)$, then

$$\boxed{|u|_{2,\alpha;B_{1/2}(0)} \leq C\left(|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)}\right)}$$

for some constant $C = C(n, \alpha, \beta, \lambda) < \infty$.

Remarks

- Remarkable result: $|u|_{0,B_1(0)}$ & f control two derivatives of u!
- Can never take $\alpha = 0$ or $\alpha = 1$ in these estimates (see Example Sheet 1 — Thm false).
- Strict ellipticity gives a lower bound for λ, & bound on $|a^{ij}|_{0,\alpha;B_1(0)}$ gives an upper bound on Λ. So Λ/λ is bounded, so here we also have uniform ellipticity.
- Will in fact strengthen this to

$$|u|_{2,\alpha;B_\theta(0)} \leq C_\theta\left(|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)}\right)$$

for any $\theta \in (0,1)$.

- The Schauder estimate gives a compactness property for the space of solutions to $Lu = f$:

If $(u_k)_k \subset C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$ solve $Lu_k = f$ in $B_1(0)$ and

$$\gamma := \sup_k \sup_{B_1(0)} |u_k| < \infty,$$

then estimate $\Longrightarrow$

$$|u_k|_{2,\alpha;B_\theta(0)} \leq C(\gamma, n, \theta, \alpha, \beta, \lambda, f).$$

By AA, $\exists$ a subsequence $(u_{k'})_{k'}$ & $u \in C^{2,\alpha}(B_1(0))$ such that $u_{k'} \longrightarrow u$ in $C^2(B_\theta(0))$ (any $\theta \in (0,1)$). Passing to the limit pointwise in $Lu_k = f$ then shows $Lu = f$.

- There are no assumptions or conclusions about the C^2 norm up to the boundary.

Proof of the Unit Scale Interior Schauder Estimate

Write $B_r := B_r(0)$. Working in a slightly smaller ball, can WLOG assume $|u|_{2;B_1} < \infty$.

Three steps:

- reduction
- contradiction
- simplified PDE

Step 1 Reduction.

Claim It suffices to prove the following: for any given $\delta \in (0,1)$ $\exists\, C > 0$ s.t.

$$[D^2 u]_{\alpha; B_{1/2}} \leq \delta [D^2 u]_{\alpha; B_1} + C\left(|u|_{2;B_1} + |f|_{0,\alpha;B_1}\right) \quad (3.1)$$

Proof of claim

Suppose $Lu = f \Rightarrow u$ satisfies (3.1). By Hölder interpolation inequality, then

$$[D^2 u]_{\alpha; B_{1/2}} \leq 2\delta [D^2 u]_{\alpha; B_1} + C\left(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\right) \quad (3.2)$$

Given any sub-ball $B_\rho(y) \subset B_1(0)$, put $\tilde{u}(x) = u(y+\rho x)$. Then $\tilde{u}$ satisfies in $B_1(0)$

$$\tilde{a}^{ij}\partial^2_{ij}\tilde{u} + \tilde{b}^i\partial_i\tilde{u} + \tilde{c}\tilde{u} = \tilde{f}, \qquad (*)$$

where

$$\tilde{a}^{ij}(x) = a^{ij}(y+\rho x), \quad \tilde{b}^i(x) = \rho b^i(y+\rho x),$$

$$\tilde{c}(x) = \rho^2 c(y+\rho x), \quad \tilde{f}(x) = \rho^2 f(y+\rho x).$$

Further,

$$|\tilde{a}^{ij}|_{0,\alpha;B_1} + |\tilde{b}^i|_{0,\alpha;B_1} + |\tilde{c}|_{0,\alpha;B_1}$$

$$\leq |a^{ij}|_{0;B_\rho(y)} + \rho^\alpha[a^{ij}]_{\alpha;B_\rho(y)}$$

$$+ \rho|b^i|_{0;B_\rho(y)} + \rho^{1+\alpha}[b^i]_{\alpha;B_\rho(y)}$$

$+$ similarly for c

$$\leq \beta \qquad \text{as } \rho \leq 1$$

Since $\tilde{a}^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$, the PDE $(*)$ is still strictly elliptic. So by assumption, (3.2) holds for $\tilde{u}$ & $\tilde{f}$ (call it $(\widetilde{3.2})$), which, written in terms of u, gives

$$\rho^{2+\alpha}[D^2u]_{\alpha; B_{\rho/2}(y)} \leq 2\delta\rho^{2+\alpha}[D^2u]_{\alpha; B_\rho(y)}$$

$$+ C\left(|u|_{0,B_\rho(y)} + \rho^2|f|_{0,B_\rho(y)} + \rho^{2+\alpha}[f]_{\alpha;B_\rho(y)}\right)$$

$$\leq 2\delta\rho^{2+\alpha}[D^2u]_{\alpha; B_\rho(y)}$$

$$+ \underbrace{C\left(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\right)}_{=:\gamma, \text{ indep of } \rho \text{ or } y.}$$

Hence by the absorbing lemma, after choosing δ suitably,

$$[D^2u]_{\alpha; B_{1/2}} \leq C\left(|u|_{0;B_1} + |f|_{0,\alpha;B_1}\right)$$

where C depends on $n, \alpha, \lambda, \beta$. This is the conclusion of the theorem, after using interpolation again.

Step 2 Contradiction

Negation of (3.1) is: Suppose $\exists\, \delta$ s.t. $\forall\, k \in \mathbb{N}$ $\exists\, a^{ij}_k, b^i_k, c_k$ with

$$|a^{ij}_k|_{0,\alpha;B_1} + |b^i_k|_{0,\alpha;B_1} + |c_k|_{0,\alpha;B_1} \leq \beta,$$

β indep. of k, & $a^{ij}_k \xi^i \xi^j \geq \lambda |\xi|^2$, & $u_k \in C^{2,\alpha}(B_1) \cap C^0(\overline{B_1})$ solving $L_k u_k = f_k \in C^{0,\alpha}(B_1)$,

but

$$[D^2u_k]_{\alpha; B_{1/2}} > \delta [D^2u_k]_{\alpha; B_1} + k\left(|u_k|_{2;B_1} + |f_k|_{0,\alpha;B_1}\right) \quad (3.3)$$

Now, by definition of $[D^2 u_k]_{\alpha; B_{1/2}}$ and by passing to a subsequence, we can assume $\exists$ $x_k, y_k \in B_{1/2}$ and ~~fixed~~ $\ell, m \in \{1, \dots, n\}$ s.t.

Can find x_k, y_k s.t. this is true for each k & ℓ_k, m_k; but only finitely many ℓ, m

$$\frac{|D^2_{\ell m} u_k(x_k) - D^2_{\ell m} u_k(y_k)|}{|x_k - y_k|^\alpha} > \frac{1}{2} [D^2 u_k]_{\alpha; B_{1/2}}$$

$\forall k$. Put $\rho_k = |x_k - y_k|$. Then

$$\frac{1}{2}[D^2 u_k]_{\alpha; B_{1/2}} < \frac{|D^2_{\ell m} u_k(x_k)| + |D^2_{\ell m} u_k(y_k)|}{\rho_k^\alpha}$$

$$\leq \frac{2|u_k|_{2; B_1}}{\rho_k^\alpha}$$

$$\overset{(3.3)}{\leq} \frac{2[D^2 u_k]_{\alpha; B_{1/2}}}{k \rho_k^\alpha},$$

So in particular $\rho_k^\alpha < \frac{4}{k} \longrightarrow 0$.

Next, rescale u_k appropriately and take the limit; we set

$$\tilde{u}_k(x) := \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha}\,[D^2 u_k]_{\alpha;B_1}},$$

where $q_k(x) = u_k(x_k) + \rho_k x \cdot Du_k(x_k) + \frac{\rho_k^2}{2} x^i x^j D^2_{ij} u_k(x_k)$

is the 2-jet of u_k. Note x has nothing to do with x_k. Now clearly

$$\tilde{u}_k(0) = 0, \quad D\tilde{u}_k(0) = 0, \quad D^2\tilde{u}_k(0) = 0.$$

Also, u_k defined on $B_1 \implies \tilde{u}_k$ defined on

$B_{1/\rho_k}\left(-\frac{x_k}{\rho_k}\right) \underset{\text{as } x_k \in B_{1/2}}{\supset} B_{\frac{1}{2\rho_k}}(0)$. So by direct

calculation

$$[D^2\tilde{u}_k]_{\alpha;\,B_{\frac{1}{2\rho_k}}(0)} \leq 1.$$

Hence, using (3.3) to bound $|\tilde{u}_k|_{2;B_R}$ for any $R > 1$,

$$|\tilde{u}_k|_{2,\alpha;\, B_R(0)} \leq C R^{2+\alpha}.$$

Hence by AA, up to a subsequence, $\exists$ $v \in C^{2,\alpha}(\mathbb{R}^n)$ s.t. $\tilde{u}_k \longrightarrow v$ in $C^2(B_R(0))$ $\forall R > 0$, and s.t.

$$[D^2 v]_{\alpha;\, \mathbb{R}^n} \leq 1$$

(note same v for every R, directly from AA).

Step 3 PDE for v.

First, note $w_k(x) = u_k(x_k + \rho_k x)$ satisfies

$$\tilde{L}_k w_k = \tilde{f}_k \quad \text{in} \quad B_{\frac{1}{2\rho_k}}(0),$$

where

$$\tilde{L}_k = \tilde{a}^{ij}_k D^2_{ij} + \tilde{b}^i_k D_i + \tilde{c}_k,$$

with $\tilde{a}^{ij}$ as before with

$\rho \longrightarrow \rho_k$, $y \longrightarrow x_k$,

ie eg $\tilde{a}^{ij}_k = a^{ij}_k(x_k + \rho_k x)$, etc.

Then, since

$$w_k(x) = \rho_k^{2+\alpha}[D^2 u_k]_{\alpha; B_1} \tilde{u}_k(x) + q_k(x),$$

we have

$$\tilde{L}_k \tilde{u}_k = g_k = \frac{\tilde{f}_k - \tilde{L}_k q_k}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha; B_1}}.$$

Will find that in the limit $\tilde{L}_k$ tends uniformly to a constant coefficient operator, and also $g_k \longrightarrow$ constant. Have

$$[\tilde{a}^{ij}_k]_{\alpha; B_{\frac{1}{2\rho_k}}(0)} \overset{\text{scaling \& inclusion}}{\leq} \rho_k^{\alpha}[a^{ij}_k]_{\alpha; B_1} \leq \rho_k^{\alpha} \beta \longrightarrow 0$$

$$\& \quad |\tilde{a}^{ij}_k|_{0; B_{\frac{1}{2\rho_k}}(0)} \leq |a^{ij}_k|_{0; B_1} \leq \beta,$$

since $\alpha > 0$ & $\rho_k \to 0$. By AA, get $\tilde{a}^{ij}_k \longrightarrow \tilde{a}^{ij} \in C^{0,\alpha}(\mathbb{R}^n)$ locally uniformly, with the limit s.t. $[\tilde{a}^{ij}]_{\alpha; \mathbb{R}^n} = 0 \quad \forall \alpha > 0$.

Hence $\tilde{a}^{ij} =$ const. Also,

$$|\tilde{b}^i_k|_{0; B_{1/2\rho_k}(0)} \leq \rho_k \beta \longrightarrow 0,$$

$$|\tilde{c}_k|_{0; B_{1/2\rho_k}(0)} \leq \rho_k^2 \beta \longrightarrow 0,$$

So $\tilde{b}^i_k \to 0$ & $\tilde{c}_k \longrightarrow 0$ locally uniformly on $\mathbb{R}^n$.

Furthermore:

<u>Claim</u> $g_k \to 0$ locally uniformly.

<u>Proof</u> Example Sheet 2. $\square$

So taking the limit in $\tilde{L}_k u_k = g_k$, we obtain

$$(3.4) \qquad \tilde{a}^{ij} D^2_{ij} v = 0 \qquad \text{on } \mathbb{R}^n$$

constants

Also in the limit $\tilde{a}^{ij} \xi^i \xi^j \geqslant \lambda |\xi|^2$ (hence <u>strict</u> ellipticity important here).

Continuing Step 3 of proof of Unit Scale Interior Schauder Estimate we derived

$$(3.4) \qquad \tilde{a}^{ij} D^2_{ij} v = 0$$

(strictly elliptic). The eigenvalues of $A = (\tilde{a}^{ij})$ are positive, so we can diagonalize to:

$$PAP^T = Q = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

$\lambda_i \geq \lambda > 0 \quad \forall i$. Let $w(x) = v(P^T x)$. Then

$$D^2 v(x) = P^T D^2 w(Px) P$$

So (3.4) becomes

$$\begin{aligned} 0 &= \operatorname{tr}(A D^2 v) \\ &= \operatorname{tr}(Q D^2 w(Px)) \\ &= \sum_{i=1}^{n} \lambda_i D^2_{ii} w(Px) \end{aligned}$$

So

$$\sum_{i=1}^{n} \lambda_i \, D^2_{ii} w(x) = 0 .$$

Rescaling,

$$\tilde{w}(x) := w(\sqrt{\lambda_1}\, x_1, \ldots, \sqrt{\lambda_n}\, x_n),$$

we have

$$\Delta \tilde{w} = 0 \qquad \text{on } \mathbb{R}^n$$

&

$$[D^2 \tilde{w}]_{\alpha ; \mathbb{R}^n} < \infty$$

$$\Longrightarrow \tilde{w} \in C^\infty(\mathbb{R}^n) .$$

In particular, $\Delta (D^2_{ij} \tilde{w}) = 0$ on $\mathbb{R}^n$.
But by Hölder continuity we know

$$|D^2_{ij} \tilde{w}(x)| \leq |D^2_{ij} \tilde{w}(0)| + [D^2 \tilde{w}]_{\alpha ; \mathbb{R}^n} |x|^\alpha$$

and Liouville's Thm $(\alpha < 1)$ $\Rightarrow$ $D^2_{ij} \tilde{w} = \text{const}.$
Hence $D^2_{ij} v = \text{const.}$ $\forall\, i, j$. ~~But~~

$$D^2 v(0) = 0 \qquad \left(\text{from } D^2 \tilde{u}_k(0) = 0 \right),$$

so $D^2 v \equiv 0$.

<u>On the other hand</u>, consider $\zeta_k = \dfrac{y_k - x_k}{\rho_k}$;

$|\zeta_k| = 1$, and

$$u_k(x_k + \rho_k \zeta_k) = u_k(y_k).$$

So

$$|D^2_{\ell m} \tilde{u}_k(\zeta_k)| = \left| \frac{\rho_k^2 \left(D^2_{\ell m} u_k(y_k) - D^2_{\ell m} u_k(x_k) \right)}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1}} \right|$$

by choice of x_k, y_k from earlier

$$\geq \frac{1}{2} \frac{[D^2 u_k]_{\alpha; B_{1/2}}}{[D^2 u_k]_{\alpha; B_1}}$$

(3.3)

$$\geq \frac{\delta}{2}.$$

As $\zeta_k \in \mathbb{S}^{n-1}$ $\forall k$, can find convergent subsequence $\zeta_k \to \zeta$; then in the limit

$$|D^2_{\ell m} v(\zeta)| \geq \frac{\delta}{2}$$

which contradicts $D^2_{\ell m} v \equiv 0$. $\#$ $\square$

We now give some corollaries.

<u>Corollary 3.2</u> (Scale Invariant Interior Schauder Estimate)

Suppose $B_R(x_0) \subset \mathbb{R}^n$ & $a^{ij}, b^i, c \in C^{0,\alpha}(B_R(x_0))$ are s.t. $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for some $\lambda > 0$ $\forall x \in B_R(x_0)$ $\forall \xi \in \mathbb{R}^n$. Suppose

$$|a^{ij}|_{0;B_R(x_0)} + R^\alpha [a^{ij}]_{\alpha;B_R(x_0)}$$
$$+ R\left(|b^i|_{0;B_R(x_0)} + R^\alpha [b^i]_{\alpha;B_R(x_0)}\right)$$
$$+ R^2\left(|c|_{0;B_R(x_0)} + R^\alpha [c]_{\alpha;B_R(x_0)}\right) \leq \beta$$

for some $\beta > 0$. Suppose $u \in C^{2,\alpha}(B_R(x_0)) \cap C^0(\overline{B_R(x_0)})$ satisfies $Lu = f$ in $B_R(x_0)$ for some $f \in C^{0,\alpha}(B_R(x_0))$. Then

$$|u|'_{2,\alpha;B_{\frac{R}{2}}(x_0)} \le C\left(|u|_{0;B_R(x_0)} + R^2|f|_{0;B_R(x_0)} + R^{2+\alpha}[f]_{\alpha;B_R(x_0)}\right),$$

where

$$|u|'_{k,\alpha;B_\rho(y)} := \sum_{j=0}^{k} \rho^j |D^j u|_{0;B_\rho(y)} + \rho^{k+\alpha}[D^k u]_{\alpha;B_\rho(y)}$$

and $C = C(n, \lambda, a, \beta)$, indep. of u & R.

<u>Proof</u> Apply Theorem 3.1 with $x \longrightarrow x_0 + Rx$. □

We now extend interior estimates to arbitrary domains.

<u>Corollary 3.3</u> (Interior Schauder Estimates in General Domains)

Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ open & bounded,
& suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$, where

$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta$$

with $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2$, $\lambda > 0$, $\forall x \in \Omega\ \forall \xi \in \mathbb{R}^n$.

Suppose $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ solves $Lu = f \in C^{0,\alpha}(\Omega)$.

Then $\forall\ \tilde{\Omega} \subset\subset \Omega$

$$|u|_{2,\alpha;\tilde{\Omega}} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right),$$

where $\quad C = C(n, \alpha, \beta, \lambda, \operatorname{dist}(\tilde{\Omega}, \partial\Omega))$.

<u>Proof</u> Let $d = \operatorname{dist}(\tilde{\Omega}, \partial\Omega)$

$$= \sup\{r > 0 : (\tilde{\Omega})_r \subset \Omega\},$$

where $(\tilde{\Omega})_r = \bigcup_{x \in \tilde{\Omega}} B_r(x)$ is r-nbhd of $\tilde{\Omega}$.

For any $x \in \tilde{\Omega}$ we have $B_d(x) \subset \Omega$, so

$$|a^{ij}|_{0,\alpha;B_d(x)} + d\,|b^i|'_{0,\alpha;B_d(x)} + d^2|c|'_{0,\alpha;B_d(x)} \leq C(d)\beta.$$

Then by Corollary 3.2, get

$$|u|_{0;B_{d/2}(x)} + d|Du|_{0;B_{d/2}(x)} + d^2|D^2u|_{0;B_{d/2}(x)}$$

$$+ d^{2+\alpha}[D^2u]_{\alpha;B_{d/2}(x)}$$

$$\leq C\left(|u|_{0;B_d(x)} + d^2|f|_{0;B_d(x)} + d^{2+\alpha}[f]_{\alpha;B_d(x)}\right)$$

$$\leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right) \qquad (3.5)$$

$C = C(n, \lambda, \alpha, \beta, d)$

In particular, $\forall x \in \tilde{\Omega}$

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right).$$

So

$$|u|_{2;\tilde{\Omega}} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right). \qquad (3.6)$$

But also by (3.5)

$$\sup_{\substack{x,y\in\tilde{\Omega}\\ |x-y|<\frac{d}{2}}} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right).$$

On the other hand, if $|x-y| \geq \frac{d}{2}$, then

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq \left(\frac{d}{2}\right)^{-\alpha} \left(|D^2u(x)| + |D^2u(y)| \right)$$

$$\leq \left(\frac{d}{2}\right)^{-\alpha} |u|_{2,\tilde{\Omega}}$$

$$\leq C\left(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega} \right).$$

Hence

$$[D^2u]_{\alpha,\tilde{\Omega}} \leq C\left(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega} \right), \quad (3.7)$$

and combine (3.6) & (3.7). $\square$

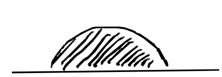
§ 3.2 Boundary Schauder Estimates

Write

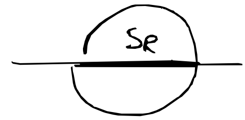
$$\mathbb{R}^n_+ = \{ (x', x^n) : x' \in \mathbb{R}^{n-1}, \ x^n > 0 \},$$

$$\mathbb{R}^n_- = \{ (x', x^n) : x' \in \mathbb{R}^{n-1}, \ x^n < 0 \},$$

$$B^{\pm}_R(y) := B_R(y) \cap \mathbb{R}^n_{\pm}$$



$B_R^{\pm} := B_R^{\pm}(0)$,

$S_R(y) := B_R(y) \cap \{x_n = 0\}$. 

$S_R := S_R(0)$

<u>Theorem 3.4</u> (Boundary Schauder Estimate in Unit Ball)

Suppose $\alpha \in (0,1)$, $a^{ij}, b^i, c \in C^{0,\alpha}(B_1^+)$ &

$|a^{ij}|_{0,\alpha;B_1^+} + |b^i|_{0,\alpha;B_1^+} + |c|_{0,\alpha;B_1^+} \le \beta$,

$a^{ij}(x)\xi^i\xi^j \ge \lambda|\xi|^2 \;\; \forall x \in B_1^+ \;\; \forall \xi \in \mathbb{R}^n$. Suppose $u \in C^{2,\alpha}(B_1^+)$ solves

$$\begin{cases} Lu = f \in C^{0,\alpha}(B_1^+) & \text{in } B_1^+ \\ u = \varphi \in C^{2,\alpha}(B_1^+) & \text{on } S_1 \end{cases}$$

<u>Then</u>

$$|u|_{2,\alpha;B_{1/2}^+} \le C\left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+} + |\varphi|_{2,\alpha;B_1^+}\right)$$

for some $C = C(n, \lambda, \alpha, \beta)$.

Proof of Theorem 3.4

By considering $v = u - \varphi$, suffices to prove for $\varphi \equiv 0$ (because $L\varphi \in C^{0,\alpha}(B_1^+)$).

We proceed as in Thm 3.1.

1. Reduction step exactly the same.
2. & 3. Need to be slightly modified.

Claim $\forall \delta > 0 \ \exists \ C = C(n, \lambda, \alpha, \beta, \delta)$ s.t.

$$[D^2 u]_{\alpha; B_{1/2}^+} \le \delta [D^2 u]_{\alpha; B_1^+} + C\left(|u|_{2; B_1^+} + |f|_{0,\alpha; B_1^+}\right).$$

Proof of Claim Argue by contradiction. If not true, then (as before) passing to a subsequence we have $\exists \ x_k, y_k \in B_{1/2}^+$, $u_k \in C^{2,\alpha}(B_1^+)$ s.t. $L_k u_k = f_k \in C^{0,\alpha}(B_1^+)$, &

$$\begin{cases} \dfrac{|D^2_{\ell m} u_k(x_k) - D^2_{\ell m} u_k(y_k)|}{|x_k - y_k|^\alpha} > \dfrac{1}{2}[D^2 u_k]_{\alpha; B_{1/2}^+} \\ [D^2 u_k]_{\alpha; B_{1/2}^+} > \delta [D^2 u_k]_{\alpha; B_1^+} + k\left(|u_k|_{2; B_1^+} + |f_k|_{0,\alpha; B_1^+}\right) \end{cases}$$

Then, as before,

$$\rho_k := |x_k - y_k| \longrightarrow 0$$

as $k \to \infty$. But now we have <u>two cases</u>:

(1) <u>either</u> $\limsup_{k \to \infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \infty$,

(2) <u>or</u> $\limsup_{k \to \infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \mu < \infty$.

<u>Case 1</u> there $\forall R > 0$ & k suff. large

$$\frac{1}{2} \geq \operatorname{dist}(x_k, S_1) \geq R\rho_k,$$

So, as $x_k \in B_{1/2}^+$, we have

$$B_{R\rho_k}(x_k) \subset B_1^+.$$

Set (as before)

$$\tilde{u}_k(x) = \frac{u_k(x_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1^+}},$$

where

$$q_k(x) = u_k(x_k) + \rho_k x^i D_i u_k(x_k) + \frac{\rho_k^2}{2} x^i x^j D^2_{ij} u_k(x_k).$$

Then $\tilde{u}_k$ defined in $B_R(0)$ and

$$|\tilde{u}_k|_{2,\alpha;B_R(0)} \leq C(R).$$

Using AA, the proof goes through as in Thm 3.1 (exercise).

Case 2: $\limsup_{k \to \infty} \frac{\operatorname{dist}(x_k, S_1)}{\rho_k} = \mu < \infty.$

Let $z_k := \operatorname{proj}_{\{x^n = 0\}}(x_k)$, ie

$$z_k = (x_k^1, \dots, x_k^{n-1}, 0).$$

As usual, look at $C^{2,\alpha}$ norm: define

$$\tilde{u}_k(x) := \frac{u_k(z_k + \rho_k x) - q_k(x)}{\rho_k^{2+\alpha} [D^2 u_k]_{\alpha; B_1^+}},$$

where $q_k(x) = u_k(z_k) + \rho_k x^i D_i u_k(z_k) + \frac{\rho_k^2}{2} x^i x^j D^2_{ij} u_k(z_k)$.

In particular, as before, for any $R > 0$

$$[D^2 \tilde{u}_k]_{\alpha; B_R^+(0)} \leq 1,$$

and

$$|\tilde{u}_k|_{2,\alpha; B_R^+(0)} \leq C(R)$$

for suff. large k, <u>and</u> $\tilde{u}_k|_{S_R} = 0$ <u>since</u>:

$$u|_{S_1} = 0, \quad D_i u|_{S_1} = 0 \quad \forall\, i \neq n,$$

so in fact

$$q_k(x) = \rho_k D_n u_k(z_k) x^n + \frac{\rho_k^2}{2} D^2_{nj} u_k(z_k) x^n x^j,$$

so on $\{x^n = 0\}$ $q_k(x) = 0$.

Set

$$\xi_k = \frac{x_k - z_k}{\rho_k}, \qquad \eta_k = \frac{y_k - z_k}{\rho_k}.$$

Then for k suff. large

$$|\xi_k| \leq 2\mu,$$

and

$$|\eta_k| \leq \frac{|x_k - y_k| + |x_k - z_k|}{\rho_k} \leq 1 + 2\mu .$$

So both sequences are bounded (& lie in compact subsets of $\mathbb{R}^n$), so $\exists$ convergent subsequences:

$$\xi_k \longrightarrow \xi , \quad \eta_k \longrightarrow \eta .$$

Then differentiating $\tilde{u}_k$ & plugging in, get

$$D^2_{\ell m} \tilde{u}_k(\xi_k) = \frac{D^2_{\ell m} u_k(x_k) - D^2_{\ell m} u_k(z_k)}{\rho_k^\alpha [D^2 u_k]_{\alpha; B_1^+}}$$

&

$$D^2_{\ell m} \tilde{u}_k(\eta_k) = \frac{D^2_{\ell m} u_k(y_k) - D^2_{\ell m} u_k(z_k)}{\rho_k^\alpha [D^2 u_k]_{\alpha; B_1^+}}$$

So

$$|D^2_{\ell m} \tilde{u}_k(\xi_k) - D^2_{\ell m} \tilde{u}_k(\eta_k)| =$$

$$= \frac{\left| D^2_{\ell m} u_k(x_k) - D^2_{\ell m} u_k(y_k) \right|}{\rho_k^\alpha \left[D^2 u_k \right]_{\alpha; B_1^+}}$$

$$\geq \frac{\frac{1}{2} \left[D^2 u_k \right]_{\alpha; B_{1/2}^+}}{\left[D^2 u_k \right]_{\alpha; B_1^+}}$$

$$\geq \frac{\delta}{2} \qquad (*)$$

Then by AA, we get that $\exists\, v \in C^{2,\alpha}(\mathbb{R}^n_+ \cup \{x^n = 0\})$ s.t. (up to a subsequence) $\tilde{u}_k \longrightarrow v$ in C^2 on compact subsets of $\mathbb{R}^n_+ \cup \{x^n = 0\}$.

As before, we find v satisfies

$$\tilde{a}^{ij} D^2_{ij} v = 0 \qquad \text{elliptic},$$

$\tilde{a}^{ij}$ constants, & also $v\big|_{\{x^n=0\}} = 0$.

Then, as before, by rotation & scaling of constants, we get that we have

$w \in C^2(\overline{H})$, $\quad H = \{x^n > 0\}$, with

$$\begin{cases} \Delta w = 0 & \text{in } H \\ w|_{\partial H} = 0 \end{cases}$$

By making an odd reflection across ∂H (see below) for w, we can extend w to a harmonic function $\tilde{w}$ on $\mathbb{R}^n$, with

$$[D^2 \tilde{w}]_{\alpha; \mathbb{R}^n} < \infty$$

But then this implies that $D^2 \tilde{w}$ is harmonic & sublinear, so by Liouville's Theorem $D^2 \tilde{w} = \text{constant}$.

But this then contradicts $(*)$ (after taking $(*)$ to the limit: $D^2_{\ell m} v(\xi) \neq D^2_{\ell m} v(\eta)$), & this completes the proof of the claim. $\square$

To finish the proof of the theorem,

by interpolation and scaling (just as in Thm 3.1), we get for any ball $B_\rho(y) \subset B_1$ with $y \in \{x^n = 0\}$

$$\rho^{2+\alpha} [D^2 u]_{\alpha; B_{\rho/2}(y)^+} \le \delta \rho^{2+\alpha} [D^2 u]_{\alpha; B_\rho(y)^+} + C\left(|u|_{0; B_1^+} + |f|_{0,\alpha; B_1^+}\right).$$

Also, by the interior Schauder estimate, for any ball $\overline{B_\rho(y)} \subset B_1^+$ we have

$$\rho^{2+\alpha} [D^2 u]_{\alpha; B_{\rho/2}(y)} \le C\left(|u|_{0; B_1^+} + |f|_{0,\alpha; B_1^+}\right).$$

The conclusion then follows from the boundary absorbing lemma (see below).

$\square$

New results used in this proof:

Proposition 3.5 (Reflection Principle for Harmonic Functions)

Let Ω^+ be an open subset of $\mathbb{R}^n_+$, and let $T = \partial\Omega^+ \cap \{x^n = 0\}$. Let Ω^- be the reflection of Ω^+ across $\{x^n=0\}$, i.e.

$$\Omega^- = \{(x^1, \dots, -x^n) : (x^1, \dots, x^n) \in \Omega^+\}.$$

Let $v \in C^2(\Omega^+) \cap C^0(\Omega^+ \cup T)$. Then let $\bar{v}$ be the odd reflection of v in T, i.e.

$$\bar{v} : \Omega^+ \cup T \cup \Omega^- \longrightarrow \mathbb{R}$$

$$\bar{v}(x^1, \dots, x^n) = \begin{cases} v(x^1, \dots, x^n), & (x^1, \dots, x^n) \in \Omega^+ \cup T \\ -v(x^1, \dots, x^n), & (x^1, \dots, x^n) \in \Omega^-. \end{cases}$$

Then if $\Delta v = 0$ in Ω^+ & $v|_T = 0$, then $\bar{v} \in C^2((\Omega^+ \cup T \cup \Omega^-)^\circ)$ and $\Delta \bar{v} = 0$ in $(\Omega^+ \cup T \cup \Omega^-)^\circ$.

Proof Example Sheet 2 (uses MVP). $\square$

Remark This is of course trivial if $T = \emptyset$, as then Ω^+ & Ω^- disjoint. The important part is C^2 across T.

Proposition 3.6 (Absorbing Lemma, Boundary Version)

Given $\theta \in (0,1)$, $\mu \in \mathbb{R}$, $R > 0$ $\exists\, \delta = \delta(n, \theta, \mu)$ and $C = C(n, \theta, \mu)$ s.t. for

$$\mathcal{B} = \{ B_\rho(y) : B_\rho(y) \subset B_R^+(0) \},$$

$$\mathcal{B}^+ = \{ B_\rho^+(y) : y^n = 0 ,\ B_\rho(y)^+ \subset B_R(0)^+ \}$$

and

$$S : \mathcal{B} \cup \mathcal{B}^+ \longrightarrow \mathbb{R}$$

a sub-additive non-negative function satisfying

$$\rho^\mu S(B^+_{\theta\rho}(y)) \leq \delta \rho^\mu S(B_\rho^+(y)) + \gamma$$

for $B_\rho^+(y) \in \mathcal{B}^+$, and

$$\rho^\mu S(B_{\theta\rho}(y)) \leq \gamma$$

for $B_\rho(y) \in \mathcal{B}$, then

$$R^{\mu} S(B^+_{\theta R}(0)) \le C\gamma.$$

Proof Example Sheet 2. $\square$

<u>Remark</u> There was a typo in the proof of the Absorbing Lemma in Lecture 8:

Immediately before the conclusion of the proof I wrote the estimate

$$e^{\lambda} S(B_{\theta_c}(y)) \le e^{\lambda} \sum_{j=1}^{N} S(B_{(1-\theta)\theta_c^2}(z_j))$$

$$\le ((1-\theta)\theta)^{-\lambda} \sum_{j=1}^{N} \Big(\delta \underbrace{((1-\theta)\theta_c)^{\lambda} S(B_{(1-\theta)\theta_c}(z_j))}_{\le Q\theta^{\lambda}} + \gamma \Big)$$

$$\le \delta((1-\theta)\theta)^{-\lambda} NQ\theta^{\lambda} + N\gamma((1-\theta)\theta)^{-\lambda},$$

with the terms in red missing there.

This makes no difference to the rest of the proof.

We now extend the boundary Schauder estimate to general domains.

Shorthand: write "hypothesis (H)" to mean

"Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain & $\alpha \in (0,1)$. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$ are s.t.

$$|a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta$$

and suppose that $\exists \lambda > 0$ s.t.

$$a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall x \in \Omega \ \forall \xi \in \mathbb{R}^n.$$

As always,

$$Lu = a^{ij}D^2_{ij}u + b^iD_iu + cu."$$

<u>Theorem 3.7</u> (Boundary Schauder Estimate in General Domains)

Suppose hypothesis (H) holds. Then $\exists\ \varepsilon = \varepsilon(\Omega) > 0$ s.t. if $u \in C^{2,\alpha}(\overline{\Omega})$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$ solve

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Then $\forall\, x \in \partial\Omega$ we have

$$|u|_{2,\alpha;B_\varepsilon(x)\cap\Omega} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$

<u>Remark</u> Need Ω to be $C^{2,\alpha}$ to have any chance of u being $C^{2,\alpha}$ on $\partial\Omega$.

<u>Proof</u> For any $z \in \partial\Omega$, by definition of Ω being $C^{2,\alpha}$, we know $\exists R > 0$ &

$$\psi : B_R(z) \longrightarrow D \subset \mathbb{R}^n$$

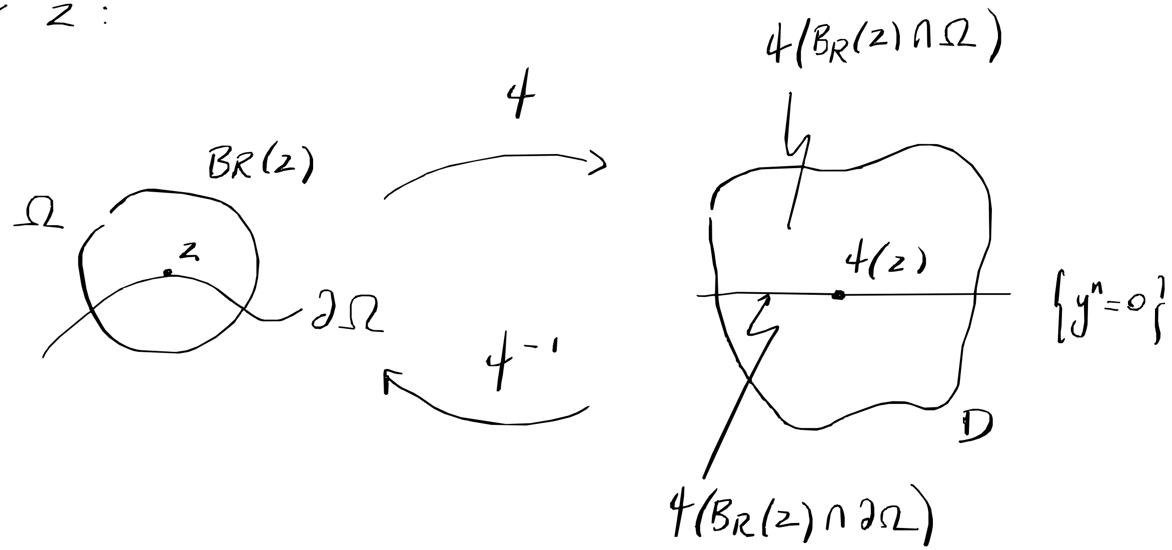
a $C^{2,\alpha}$ - diffeomorphism s.t

$$\psi\left(B_R(z)\cap\Omega\right) \subset \{y^n > 0\}$$

&

$$\psi\left(B_R(z)\cap\partial\Omega\right) \subset \{y^n = 0\},$$

ie ψ "rectifies" or "straightens out" $\partial\Omega$ near z:



Let $x = (x^1, \ldots, x^n)$ be coordinates for Ω

and let $y = (y^1, \ldots, y^n)$ be coordinates on D.

Let

$$\tilde{u}(y) = u(\psi^{-1}(y)),$$

ie $\tilde{u} = u \circ \psi^{-1}$ the pullback of u along ψ^{-1}.

Then $$\tilde{u}\Big|_{\{y^n=0\}\cap D} = (\varphi \circ \psi^{-1})\Big|_{\{y^n=0\}\cap D} =: \tilde{\varphi}.$$

Note that $\tilde{\varphi}$ is still $C^{2,\alpha}$. To apply Theorem 3.4, need to find PDE satisfied by $\tilde{u}$ and show it satisfies hypotheses.

Note $u(x) = \tilde{u}(\psi(x))$, so

$$\partial_{x_i} u = \partial_{y_k} u \frac{\partial \psi^k}{\partial x^i}, \quad \text{and}$$

$$\partial^2_{x_i x_j} u = \partial^2_{y_\ell y_k} \tilde{u}(\psi(x)) \frac{\partial \psi^k}{\partial x^i} \frac{\partial \psi^\ell}{\partial x^j} + \partial_{y_k} \tilde{u}(\psi(x)) \frac{\partial^2 \psi^k}{\partial x^i \partial x^j}.$$

Hence can find coefficients of PDE satisfied by $\tilde{u}$ explicitly

$$\begin{cases} A^{\ell k} \partial^2_{y_\ell y_k} \tilde{u} + B^\ell \partial_{y_\ell} \tilde{u} + C \tilde{u} = \tilde{f} & \text{on } D \\ \tilde{u} = \tilde{\varphi} & \text{on } D \cap \{y^n = 0\}, \end{cases}$$

where eg

$$A^{\ell k} = a^{ij} \frac{\partial \psi^k}{\partial x^i} \frac{\partial \psi^\ell}{\partial x^j},$$

$$\tilde{f} = f \circ \psi^{-1}, \quad \text{etc.}$$

(exercise: write down B^ℓ & C).

Now want to rescale. choose $\sigma > 0$ s.t. $B_\sigma(\psi(z)) \subset D$ (D open). We want to apply Thm 3.9 with $\bar{u}(y) = \tilde{u}(\psi(z) + \sigma y)$ in place of u; we will obtain

$$(\dagger) \quad |\tilde{u}|'_{2,\alpha; B^+_{\sigma/2}(\psi(z))} \leq C\Big(|\tilde{u}|_{0; B^+_\sigma(\psi(z))} + \sigma^2 |\tilde{f}|'_{0,\alpha; B^+_\sigma(\psi(z))} + |\tilde{\varphi}|'_{2,\alpha; B^+_\sigma(\psi(z))}\Big)$$

for some $C = C(n, \lambda, \alpha, \beta, \psi)$.

To do this, we need to check assumptions hold:

$$(*) \quad |A^{\ell k}|'_{0,\alpha; B^+_\sigma(\psi(z))} + |B^\ell|'_{0,\alpha; B^+_\sigma(\psi(z))} + |C|'_{0,\alpha; B^+_\sigma(\psi(z))} \leq \mu\beta$$

for some $\mu = \mu(\psi)$, and we also need to

Check ellipticity of A^{lk}. For this note

$$A^{lk}(y)\,\xi^l\xi^k = a^{ij}\,\partial_i(\xi\cdot\psi)\,\partial_j(\xi\cdot\psi)\Big|_x$$

a_{ij} elliptic

$$\geq \lambda\,|D(\xi\cdot\psi)|^2\Big|_{\psi^{-1}(y)}$$

$$\geq \lambda\, c(\psi)\,|\xi|^2,$$

where the last inequality is true because ψ a diffeomorphism

$$\xi\cdot y = \xi\cdot\psi(\psi^{-1}(y)) \implies$$

$$\xi = D(\xi\cdot\psi)\Big|_{\psi^{-1}(y)}\cdot D\psi^{-1}(y) \implies$$

$$|\xi| \leq |D(\xi\cdot\psi)|\Big|_{\psi^{-1}(y)}\ \underbrace{\|D\psi^{-1}\|_0}_{c(\psi)^{-1/2}} > 0$$

Can check $(*)$ similarly (exercise). So transforming (RHS) of $(\dagger)$ back to $\tilde{f}\to f$, $\tilde{\varphi}\to\varphi$, $\tilde{u}\to u$, we have

$$|\tilde{u}|_{2,\alpha;B^+_{\sigma/2}(\psi(z))} \leq C\Big(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\Big),$$

where $C = C(n, \lambda, \alpha, \beta, \psi, \sigma)$.

Now pick $r = r(z) > 0$ s.t. $B_r(z) \subset \psi^{-1}\big(B_{\sigma/2}(\psi(z))\big)$. Then from above

$$(\dagger\dagger) \quad |u|_{2,\alpha;B_r(z)\cap\Omega} \leq C\Big(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\Big),$$

where C has dependences as before. All this was done for a fixed boundary point z, so here in fact write $\psi = \psi_z$, $\sigma = \sigma_z$, $C = C_z$. Eq. $(\dagger\dagger)$ is the result we want for a ball, so we finish with a compactness argument.

Clearly

$$\partial\Omega \subset \bigcup_{z\in\partial\Omega} B_{r(z)/2}(z).$$

By compactness of $\partial\Omega$, $\exists$ a finite subcover, so

$$\partial\Omega \subset \bigcup_{j=1}^{N} B_{r(z_j)/2}(z_j).$$

Then let $\displaystyle \varepsilon = \min_{1 \le j \le N} \left\{ \frac{r(z_j)}{2} \right\}$ &

$\displaystyle C = \max_{1 \le j \le N} \{ C_{z_j} \}$. Then for any $x \in \partial\Omega$

$$|u|_{2,\alpha;B_\varepsilon(x)\cap\Omega} \le C \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega} \right).$$

$\square$

§ 3.3 Global Schauder Estimates

We can now combine everything we have done to get global Schauder estimates.

Theorem 3.8 (Global Schauder Estimates)

Suppose hypothesis (H) holds. Suppose also that Ω is a $C^{2,\alpha}$ domain. Then, if $u \in C^{2,\alpha}(\Omega)$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\Omega)$ satisfy

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

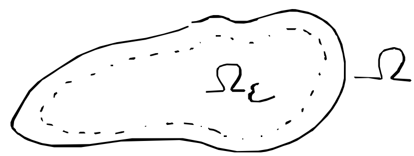
then

$$|u|_{2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right),$$

where $C = C(n, \lambda, \alpha, \beta, \Omega)$.

Proof Let $\varepsilon = \varepsilon(\Omega)$ be as in Thm 3.7.

Then let



$$\Omega_\varepsilon = \left\{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \frac{\varepsilon}{4} \right\}.$$

Then by Corollary 3.3, we have

$$|u|_{2,\alpha;\Omega_\varepsilon} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} \right),$$

$C = C(n, \lambda, \alpha, \beta, \Omega)$. Then note that

$$\Omega \setminus \Omega_\varepsilon \subset \bigcup_{x \in \partial\Omega} B_{\varepsilon/4}(x),$$

therefore $\forall x \in \Omega$:

- either $x \in \Omega_\varepsilon$, when by interior estimate

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} \right),$$

- or $x \in \Omega \setminus \Omega_\varepsilon$, when $x \in B_{\varepsilon/4}(y)$ for some $y \in \partial\Omega$, so by Thm 3.7

$$|u(x)| + |Du(x)| + |D^2u(x)| \leq |u|_{2,\alpha;B_\varepsilon(y)}$$

$$\leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$

Combining, we have

$$|u|_{2;\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$

To finish, just need to bound $[D^2u]_{\alpha;\Omega}$.

Let $x, y \in \Omega$. Then

- <u>either</u> $|x-y| < \frac{\varepsilon}{4}$, when we have two cases:
  - if either $\text{dist}(x, \partial\Omega) < \frac{\varepsilon}{4}$ or $\text{dist}(y, \partial\Omega) < \frac{\varepsilon}{4}$, then $x, y \in B_{\varepsilon/2}(z)$ for some $z \in \partial\Omega$. Then by Thm 3.7

$$\frac{|D^2_{ij}u(x) - D^2_{ij}u(y)|}{|x-y|^\alpha} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$

  - otherwise both $x, y \in \Omega_\varepsilon$, in which case the interior estimate applies.

In both cases we are done.

• or $|x-y| \geq \frac{\varepsilon}{4}$. Then

$$\frac{|D^2_{ij}u(x) - D^2_{ij}u(y)|}{|x-y|^\alpha} \leq \left(\frac{\varepsilon}{4}\right)^{-\alpha} 2|u|_{2;\Omega}$$

by interior estimate

$$\leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right).$$

Taking the largest of the C's, we are done. $\square$

---

§4 Solvability of the Dirichlet Problem

New set of hypotheses in this section. Write

(H) $\Omega \subset \mathbb{R}^n$ bounded domain. Suppose a^{ij}, b^i, $c \in C^{0,\alpha}(\Omega)$, and that $\exists\, \lambda > 0$ s.t. $a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \;\forall x \in \Omega \;\forall \xi \in \mathbb{R}^n$.

$\alpha \in (0,1)$

and denote

$$Lu = a^{ij}D^2_{ij}u + b^iD_iu + cu$$

(ie L strictly elliptic)

We will prove that if additionally $c \leq 0$ in Ω, then (under mild regularity assumptions on $\partial\Omega$), the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

is uniquely solvable in $C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ for any $f \in C^{0,\alpha}(\Omega)$

We first prove the following reduction.

Theorem 4.1 Suppose (H) holds and assume that Ω is a $C^{2,\alpha}$ domain, and $c \leq 0$ in Ω.

Then

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \text{ has a solution } u \in C^{2,\alpha}(\Omega) \ \forall f \in C^{0,\alpha}(\Omega) \ \forall \varphi \in C^{2,\alpha}(\Omega)$$

$$\Updownarrow$$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \text{ has a solution } u \in C^{2,\alpha}(\Omega) \ \forall f \in C^{0,\alpha}(\Omega) \ \forall \varphi \in C^{2,\alpha}(\Omega)$$

Remark Since $c \leq 0$, the solution (if it exists, for either problem) is unique.

Remark We will prove Thm 4.1 using a continuity method, by continuously deforming the operator L into Δ.

Proof By considering $u - \varphi \in C^{2,\alpha}(\Omega)$, it suffices to establish the equivalence in the case $\varphi \equiv 0$. Then consider for $t \in [0,1]$ the 1-parameter family of operators

$$L_t : C^{2,\alpha}_0(\bar{\Omega}) \longrightarrow C^{0,\alpha}(\Omega)$$

$$\begin{aligned} L_t u &:= tLu + (1-t)\Delta u \\ &= a^{ij}_t D^2_{ij} u + b^i_t D_i u + c_t u, \end{aligned}$$

where

$$\begin{aligned} a^{ij}_t &= t a^{ij} + (1-t)\delta_{ij}, \\ b^i_t &= t b^i, \\ c_t &= tc. \end{aligned}$$

Note that clearly $L_0 = \Delta$ & $L_1 = L$.

We now have

$$|a^{ij}_t|_{0,\alpha;\Omega} + |b^i_t|_{0,\alpha;\Omega} + |c_t|_{0,\alpha;\Omega} \leq \max\{1, \beta\},$$

where $\beta := |a^{ij}|_{0,\alpha;\Omega} + |b^i|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega}$.

Also clearly $a^{ij}_t(x)\xi^i\xi^j \geq \min\{1,\lambda\}|\xi|^2$ $\forall\, t \in [0,1]$. Hence we can apply the global Schauder estimate (Thm 3.8) to get

$$|u|_{2,\alpha;\Omega} \leq C_1\left(|u|_{0;\Omega} + |L_t u|_{0,\alpha;\Omega}\right)$$

$\forall\, u \in C^{2,\alpha}_0(\overline{\Omega})$, where $C_1 = C_1(n, \alpha, \lambda, \beta, \Omega)$.

Next, by the MPAPE (Thm 2.9)

$$|u|_{0;\Omega} \leq C_2 |L_t u|_{0,\alpha;\Omega}$$

since $u \in C^{2,\alpha}_0(\overline{\Omega})$, where $C_2 = C_2(n, \lambda, \Omega)$.

Combining these, we get

$$(4.1) \qquad |u|_{2,\alpha;\Omega} \le C\,|L_t u|_{0,\alpha;\Omega}$$

$\forall t \in [0,1]$ $\forall u \in C^{2,\alpha}_0(\bar{\Omega})$, where $C = C(n,\alpha,\lambda,\beta,\Omega)$.
In particular, note that (4.1) $\Rightarrow$ L_t injective on $C^{2,\alpha}_0(\bar{\Omega})$.

However, the solvability of $L_t u = f$ in $C^{2,\alpha}_0(\bar{\Omega})$ for arbitrary $f \in C^{0,\alpha}(\Omega)$ is asking for surjectivity of L_t.

So fix $f \in C^{0,\alpha}(\Omega)$. We will show that if L_t is surjective at some t, then it is surjective at all $t \in [0,1]$. Suppose L_s is surjective (and hence bijective, by (4.1)) for some $s \in [0,1]$. Then by (4.1)

$$(4.2) \qquad |L_s^{-1} g|_{2,\alpha;\Omega} \le C\,|g|_{0,\alpha;\Omega}$$

$\forall g \in C^{0,\alpha}(\Omega)$.

Then for any $t \in [0,1]$,

$$L_t u = f \iff L_s u + (L_t - L_s)u = f$$
$$\iff u = L_s^{-1} f + L_s^{-1}(L_s - L_t)u,$$

by linearity of L_s^{-1}. Hence to find such a solution u to $L_t u = f$, we are looking for a fixed point of the map

$$T : C_0^{2,\alpha}(\overline{\Omega}) \longrightarrow C_0^{2,\alpha}(\overline{\Omega})$$
$$T(u) = L_s^{-1} f + L_s^{-1}((L_s - L_t)u)$$

We will use the contraction mapping theorem to show that T has a fixed point. We have

Claim T is a contraction mapping for t sufficiently close to s.

Proof For any $u, v \in C_0^{2,\alpha}(\overline{\Omega})$ we have

$$T(u) - T(v) = L_s^{-1}((L_s - L_t)(u-v))$$

using form of L_t
$$= (s-t) L_s^{-1}((L - \Delta)(u-v))$$

Hence using (4.2),

$$|T(u)-T(v)|_{2,\alpha;\Omega} \leq |s-t|\, C\, |(L-\Delta)(u-v)|_{0,\alpha;\Omega}$$
$$\leq |s-t|\, \tilde{C}\, |u-v|_{2,\alpha;\Omega},$$

using the fact that L & Δ are bounded operators from $C^{2,\alpha}(\Omega)$ into $C^{0,\alpha}(\Omega)$, where $\tilde{C} = \tilde{C}(n,\alpha,\lambda,\beta,\Omega)$. Hence if $|s-t| < \frac{1}{2\tilde{C}}$ (e.g.), then T is a contraction. $\square$

Therefore by the contraction mapping theorem, T has a fixed point for such t, and hence $L_t u = f$ is solvable.

Now observe that since $\frac{1}{2\tilde{C}}$ does not depend on s, we can break up the interval $[0,1]$ into subintervals of length $\frac{1}{2\tilde{C}}$ and propagate the above claim to obtain that $L_t u = f$ is solvable $\forall\, t \in [0,1]$ if $\exists$ $s \in [0,1]$ s.t. $L_s u = f$ is solvable.

Therefore the result follows by assuming that $L_t u = f$ is solvable at either $t=0$ or $t=1$. $\square$

<u>Remark</u> The proof shows equivalence of solvability $\forall\, t \in [0,1]$, not just L & Δ.

<u>Remark</u> The proof is a continuity method: we show that the set $A = \{t \in [0,1] : L_t \text{ is a bijection}\}$ is both open and closed, and non-empty (by assumption), hence is the full interval $[0,1]$

So we have reduced the existence for the Dirichlet problem in $C^{2,\alpha}(\overline{\Omega})$

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

to the existence for the Dirichlet problem

$$(*) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases},$$

$f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Note Δ is simpler than L as it has (i) constant coefficients and (ii) is in divergence form

Next prove the existence of solutions to $(*)$ in balls, and then extend to more general domains using "Perron's method".

Proposition 4.2 Let $B = B_R(y) \subset \mathbb{R}^n$. If $f \in C^\infty(\overline{B})$ and $\varphi \in C^\infty(\overline{B})$, then there exists a unique solution $u \in C^\infty(\overline{B})$ to $(*)$ with $\Omega = B$.

Proof This follows, e.g., from the Riesz representation theorem (see Part III Analysis of PDEs) $\square$

Proposition 4.3 Let B be as in Prop. 4.2. If $f \in C^{0,\alpha}(\overline{B})$ and $\varphi \in C^0(\overline{B})$, then $\exists!$ solution $u \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ to $(*)$ with $\Omega = B$. Moreover, if $\varphi \in C^{2,\alpha}(\overline{B})$, then $u \in C^{2,\alpha}(\overline{B})$.

Proof Extend f to $\tilde{f} \in C_c^{0,\alpha}(\mathbb{R}^n)$, φ to $\tilde{\varphi} \in C_c^0(\mathbb{R}^n)$, and mollify both $\tilde{f}$ and $\tilde{\varphi}$ as follows. Choose $\eta \in C_c^\infty(\mathbb{R}^n)$ s.t. $\eta|_{\mathbb{R}^n \setminus B_1(0)} = 0$, $\eta \geq 0$, $\int_{\mathbb{R}^n} \eta = 1$, and put $\eta_\sigma(x) = \sigma^{-n}\eta\left(\frac{x}{\sigma}\right)$ (this is a standard mollifier).

For a sequence $(\sigma_k)_k$, $\sigma_k \searrow 0$, put

$$f_k(x) := \int_{\mathbb{R}^n} \tilde{f}(y)\, \eta_{\sigma_k}(x-y)\, dy = (\tilde{f} * \eta_{\sigma_k})(x),$$

$$\varphi_k(x) := \int_{\mathbb{R}^n} \tilde{\varphi}(y)\, \eta_{\sigma_k}(x-y)\, dy = (\tilde{\varphi} * \eta_{\sigma_k})(x)$$

Have $f_k, \varphi_k \in C^\infty(\mathbb{R}^n)$, and by Prop 4.2 $\exists !\ u_k \in C^\infty(B)$ s.t.

$$\begin{cases} \Delta u_k = f_k & \text{in } B \\ u_k = \varphi_k & \text{on } \partial B \end{cases}$$

We know $f_k \to f$ uniformly on $\overline{B}$, $\varphi_k \to \varphi$ uniformly on ∂B, and

$$\sup_B |f_k| \le \sup_{\mathbb{R}^n} |\tilde{f}| \quad \forall k,$$

hence

$$|f_k(x) - f_k(z)| \le \int_{\mathbb{R}^n} |\tilde{f}(x-y) - \tilde{f}(z-y)|\, \eta_{\sigma_k}(y)\, dy$$

$$\leq [\tilde{f}]_{0,\alpha} |x-z|^{\alpha}$$

$$\Rightarrow \quad |f_k|_{0,\alpha} \leq |\tilde{f}|_{0,\alpha} .$$

Similarly, we have $\sup |\varphi_k| \leq \sup_{\mathbb{R}^n} |\tilde{\varphi}|$

Then as

$$\begin{cases} \Delta(u_k - u_\ell) = f_k - f_\ell \\ (u_k - u_\ell)|_{\partial B} = \varphi_k - \varphi_\ell \end{cases},$$

by MPAPE (Thm 2.9)

$$\sup_{\overline{B}} |u_k - u_\ell| \leq \sup_{\partial B} |\varphi_k - \varphi_\ell| + C \sup_{\overline{B}} |f_k - f_\ell| .$$

Hence $(u_k)_k$ is uniformly Cauchy in $\overline{B}$ and so $\exists\, u \in C^0(\overline{B})$ st $u_k \to u$ in $L^\infty(\overline{B})$ (ie uniformly, & as continuity preserved under convergence in L^∞). Moreover, $u|_{\partial B} = \varphi$ by taking a limit of $u_k|_{\partial B}$.

It remains to improve the regularity of u

using Schauder estimates. By interior Schauder estimates (Corollary 3.3), for any sub-ball $\tilde{B} \subset\subset B$

$$|u_k|_{2,\alpha,\tilde{B}} \leq C\left(\sup_B |u_k| + |f_k|_{0,\alpha;B}\right)$$

MPAPE

$$\leq C\left(\sup_{\partial B} |\varphi_k| + |f_k|_{0,\alpha;\mathbb{R}^n}\right)$$

$$\leq C\left(\sup_{\partial B} |\tilde{\varphi}| + |\tilde{f}|_{0,\alpha;\mathbb{R}^n}\right), \quad \Big\}\text{ indep of } k$$

where $C = C(\tilde{B}, B)$. By Arzela – Ascoli, up to a subsequence, $u_k \xrightarrow{C^2_{loc}(B)} u \in C^{2,\alpha}(B)$.
By passing to the limit in the equation, this solves $\Delta u = f$ in B.

For the "moreover" part, if $\varphi \in C^{2,\alpha}(\overline{B})$, the same argument works, except now we may use the global Schauder estimate for the conclusion. $\square$

§ 4.1 Perron's method

Idea: solvability in balls $\Rightarrow$ solvability in general domains. Based on the maximum principle.

Let us make the following observations.

Observation 1

with $c \leq 0$

Fix $f \in C^{0,\alpha}(\overline{\Omega})$ and let $u \in C^2(\Omega)$.

Then u is a <u>subsolution</u> of $Lu = f$ (ie $Lu \geq f$ in Ω) $\Leftrightarrow$ $\forall B \Subset \Omega$ we have $u \leq u_B$, where $u_B \in C^2(B) \cap C^0(\overline{B})$ is the unique solution to

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B \end{cases}.$$

This follows from the WMP (<u>exercise</u>: check this equivalence). Note such a u_B exists by previous discussion.

This equivalence provides a way of generalizing subsolutions as merely continuous functions.

Observation 2

Let $f \in C^{0,\alpha}(\bar\Omega)$ and $\varphi \in C^0(\partial\Omega)$.
Set

again $c \leq 0$

$$S_\varphi := \{ v \in C^2(\Omega) \cap C^0(\bar\Omega) : Lv \geq f \text{ in } \Omega \ \& \ v \leq \varphi \text{ on } \partial\Omega \}.$$

Then if $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ solves

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

then in fact $u(x) = \sup_{v \in S_\varphi} v(x)$ (ie "solution is the supremum over subsolutions").

(exercise: check this)

[To show this, note that $u \in S_\varphi$ & $u \geq v$ $\forall v \in S_\varphi$: $\begin{cases} L(u-v) \leq 0 \\ (u-v) \geq 0 \end{cases} \underset{WMP}{\Longrightarrow} v \leq u$]

Now modify our assumptions to include "$c \leq 0$" in hypothesis (H).

<u>Definition 4.4</u>

Let $f \in C^{0,\alpha}(\overline{\Omega})$. Then we say $u \in C^0(\Omega)$ is a <u>sub (super) solution</u> to $Lu = f$ in Ω if, for all open balls B, $\overline{B} \subset \Omega$, we have $u \leq u_B$ ($u \geq u_B$) in B, where u_B is the unique function in $C^{2,\alpha}(B) \cap C^0(\overline{B})$ solving

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B \end{cases}.$$

<u>Remark</u> Since according to this generalization sub (super) solutions need only be C^0, taking maxima or minima over sub (super) solutions will give another sub (super) solution. On the other hand, taking maxima or minima over C^2 functions in general destroys C^2 regularity.

Lemma 4.5

(I) Let $u, v \in C^0(\overline{\Omega})$ be sub- and super-solutions respectively to $L\tilde{u} = f$ in Ω. Then if $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.

(II) If $u_1, u_2 \in C^0(\Omega)$ are subsolutions to $L\tilde{u} = f$ in Ω, then $v := \max\{u_1, u_2\}$ is also a subsolution.

(III) Let $u \in C^0(\Omega)$ be a subsolution to $L\tilde{u} = f$ in Ω. Let B be a ball, $\overline{B} \subset \Omega$, and let $u_B \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ be the unique solution to

$$\begin{cases} Lu_B = f & \text{in } B \\ u_B = u & \text{on } \partial B. \end{cases}$$

Then define the <u>L-lift of u in B</u> by

$$U := \begin{cases} u_B & \text{on } B \\ u & \text{on } \Omega \setminus B \end{cases}$$

Then $U \in C^0(\Omega)$, and U is a subsolution to $L\tilde{u} = f$ in Ω.

Proof In example sheet 3.

Remark Property (I) is true for C^2 subsolutions, but properties (II) & (III) are useful facts & only true for C^0 subsolutions.

Fix $\varphi \in C^0(\partial\Omega)$ and define

$$\mathcal{S}_\varphi = \{ v \in C^0(\overline{\Omega}) \ v \text{ is a subsolution to } Lu = f \text{ in } \Omega \text{ s.t. } v \leq \varphi \text{ on } \partial\Omega \}.$$

We then have the following

Theorem 4.6 Let (H) hold (now with $c \leq 0$), $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\partial\Omega)$. Set

$$u = \sup_{v \in \mathcal{S}_\varphi} v$$

Then $u \in C^{2,\alpha}(\Omega)$, and $Lu = f$ in Ω.

Proof

Claim 1 $\mathcal{S}_\varphi \neq \phi$.

Proof of Claim 1

Recall that (H) entails that Ω is bounded, so $\exists\, y_1$ s.t. $\Omega \subset \{x \in \mathbb{R}^n : x_1 \geq y_1\}$.
Put $y = (y_1, \dots, y_n)$ and $d := \sup_{x \in \Omega} |x_1 - y_1|$
and define for $x \in \overline{\Omega}$

$$s(x) := -\sup_{\partial\Omega} |\varphi| - \big(\underbrace{e^{\gamma d} - e^{\gamma(x_1 - y_1)}}_{\geq 0}\big) \sup_{\Omega} |f|.$$

Then clearly $s \leq \varphi$ on $\partial\Omega$ and since $s \in C^2(\overline{\Omega})$ (in fact $C^\infty(\overline{\Omega})$)

$$Ls = a^{11} \gamma^2 e^{\gamma(x_1 - y_1)} \sup_{\Omega} |f| + b^1 \gamma e^{\gamma(x_1 - y_1)} \sup_{\Omega} |f| + \underbrace{cs}_{\geq 0}$$

$$\geq \sup_{\Omega} |f| \left(a^{11} \gamma + b^1\right) e^{\gamma(x_1 - y_1)} \gamma$$

$$\geq \sup_{\Omega} |f| \left(\lambda \gamma + b^1\right) \gamma e^{\gamma(x_1 - y_1)}$$

$\geq f$ if γ is chosen sufficiently large

Hence $s \in \mathcal{G}_\varphi$. $\square$

Continuing the proof of Thm. 4.6, have that $s(x) = -\sup_{\partial\Omega}|\varphi| - (e^{\gamma d} - e^{\gamma(x_1 - y_1)})\sup_{\Omega}|f|$ $\in \mathcal{G}_\varphi \neq \phi$. Moreover, $s_1 = -s$ satisfies $Ls_1 \leq f$ in Ω & $s_1 \geq \varphi$ on $\partial\Omega$. So by Lemma 4.5 (i), $v \leq s_1$ $\forall\, v \in \mathcal{G}_\varphi$ and hence $u = \sup_{v \in \mathcal{G}_\varphi} v \leq s_1$ in Ω, ie u is well-defined.

Now fix $z \in \Omega$ & a ball $\overline{B_R(z)} \subset \Omega$. Then by definition of u, $\exists$ a sequence $v_j \subset \mathcal{G}_\varphi$ s.t. $v_j(z) \longrightarrow u(z)$. By replacing v_j with $\max\{v_j, s\}$, we may assume wlog that $\sup_{\Omega}|v_j| \leq \mu$ for some $\mu \in \mathbb{R}$ $\forall j$.

Let V_j be the L-lift of v_j in $B = B_R(z)$.

Then we know that $V_j \in C^{2,\alpha}(B) \cap C^0(\overline{B})$,

with

$$\begin{cases} LV_j = f \text{ in } B, \\ V_j = v_j \text{ on } \partial B, \end{cases}$$

and $V_j = v_j$ in $\Omega \setminus B$. By Lemma 4.5(III),

$V_j \in \mathcal{S}_\varphi$, and hence $V_j \leq u$ in Ω.

Let $w_j = V_j|_{B_R(z)}$. For w_j, by the

Interior Schauder Estimate

$$|w_j|_{2,\alpha;B_{R/2}(z)} \lesssim |w_j|_{0,B_R(z)} + |f|_{0,\alpha;B_R(z)}$$

MPAPE for w_j

$$\lesssim |v_j|_{0;\partial B_R(z)} + |f|_{0,\alpha;\Omega}$$

$$\lesssim \mu + |f|_{0,\alpha;\Omega}.$$

Hence by Arzela-Ascoli, up to a subsequence,

$\exists\, V \in C^{2,\alpha}(\overline{B_{R/2}(z)})$ such that $V_j \longrightarrow V$

in $C^2(\overline{B_{R/2}(z)})$, and $LV = f$ in $B_{R/2}(z)$.

We know $V \le u$ in $B_{R/2}(z)$ (as each $V_j \le u$)

But also the v_j's are subsolutions and V_j solves the problem on $B_R(z)$, we have

$$v_j \le V_j \quad \text{in} \quad B_R(z).$$

In particular, $v_j(z) \le V_j(z)$, so taking the limit shows $u(z) \le V(z)$, and hence $u(z) = V(z)$. We can extend this:

Claim $u = V$ in $B_{R/16}(z)$

Once this is established, then we are done, since $z \in \Omega$ was arbitrary, and by covering Ω with balls, $u = V \in C^{2,\alpha}(\Omega)$ and $Lu = f$ in Ω.

Proof Suppose not. Then $\exists\, z_1 \in B_{R/16}(z)$ s.t. $V(z_1) < u(z_1)$, as we know in general that $V \le u$ in $B_{R/2}(z)$. As u defined as a supremum, $\exists\, \tilde{w} \in \mathcal{S}_\varphi$ s.t. $V(z_1) < \tilde{w}(z_1) \le u(z_1)$.

So let $\tilde{v}_j = \max\{\tilde{w}, V_j\}$. Then note $\tilde{v}_j \in \mathcal{S}_\varphi$, and let $\tilde{V}_j$ be the L-lift of $\tilde{v}_j$ in $B_{R/4}(z_1)$. As before, passing to a subsequence, the interior Schauder estimates show that $\exists\ \tilde{V} \in C^{2,\alpha}(B_{R/8}(z_1))$ s.t. $\tilde{V}_j \to \tilde{V}$ in $C^2(B_{R/8}(z_1))$.

Hence $V(z_1) < \tilde{W}(z_1) \leq \tilde{v}_j(z_1) \leq \tilde{V}_j(z_1)$, so we have $V(z_1) < \tilde{V}(z_1)$. But as $V_j \leq \tilde{V}_j$ in $B_{R/4}(z_1)$, we obtain $V \leq \tilde{V}$ in $B_{R/8}(z_1)$. But since $V(z) = u(z)$, u being the supremum, we in fact have $V(z) = \tilde{V}(z)$ $(z \in B_{R/8}(z_1))$. Since $LV = f$ & $L\tilde{V} = f$ in $B_{R/8}(z_1)$, we have $L(V - \tilde{V}) = 0$ in $B_{R/8}(z_1)$ & so by the SMP & using $V(z) = \tilde{V}(z)$, we get $V \equiv \tilde{V}$ in $B_{R/8}(z_1)$, which contradicts $V(z_1) < \tilde{V}(z_1)$.

$\square$

This completes the proof of the theorem. $\square$

Thm 4.6 constructs the "Perron" solution u in the interior of Ω, but for now we have no information on its boundary values. We now move to showing that if $\partial\Omega$ satisfies the exterior sphere condition everywhere, then the Perron solution has the regularity $u \in C^{\circ}(\overline{\Omega})$, with $u = \varphi$ on $\partial\Omega$.

We will need the concept of barriers.

Definition 4.7

Assume the standard hypotheses (H), and let $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$. Fix $x_0 \in \partial\Omega$. Then:

(I) A sequence of functions $w_i^+ : \overline{\Omega} \to \mathbb{R}$ is said to be an upper barrier at x_0 to (L, f, φ) if:

- $w_i^+ \in C^0(\overline{\Omega})$ and w_i^+ is a supersolution to $Lu = f$ in Ω, with $w_i^+ \geq \varphi$ on $\partial\Omega$, and
- $w_i^+(x_0) \to \varphi(x_0)$ as $i \to \infty$.

(II) A sequence $w_i^- : \overline{\Omega} \to \mathbb{R}$ is said to be a lower barrier at x_0 to (L, f, φ) if:

- $w_i^- \in C^0(\overline{\Omega})$ and w_i^- is a subsolution to $Lu = f$ in Ω, with $w_i^- \leq \varphi$ on $\partial\Omega$, and
- $w_i^-(x_0) \to \varphi(x_0)$ as $i \to \infty$

We then have:

<u>Proposition 4.8</u> Suppose (H) holds, and $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^0(\partial\Omega)$ and $x_0 \in \partial\Omega$. Suppose $\exists$ upper and lower barriers at x_0 to (L, f, φ). Then the Perron solution, given by Thm 4.6, satisfies $u(x) \longrightarrow \varphi(x_0)$ as $x \to x_0$ $(x \in \Omega)$.

<u>Proof</u> From the definition of the Perron solution u, we clearly have $u(x) \geq w_i^-(x)$ $\forall x \in \Omega$ $\forall i$, where w_i^- are the lower barriers.

Next, note that for any $v \in \mathcal{J}_\varphi$ we have $w_i^+(x) \geq v(x)$ $\forall x \in \Omega$ (as supersolutions lie above subsolutions if they are so at the boundary). So we get

$$u(x) := \sup_{v \in \mathcal{J}_\varphi} v(x) \leq w_i^+(x) \qquad \forall i \, \forall x \in \Omega.$$

Hence $w_i^-(x) \leq u(x) \leq w_i^+(x) \quad \forall x \in \Omega$,

$\forall i$. Now given $\varepsilon > 0$, choose i s.t.

$$\begin{cases} 0 \leq w_i^+(x_0) - \varphi(x_0) \leq \dfrac{\varepsilon}{2}, \\ 0 \leq \varphi(x_0) - w_i^-(x_0) \leq \dfrac{\varepsilon}{2}. \end{cases}$$

Then, by continuity, we can choose $\delta > 0$ s.t.
$|w_i^\pm(x) - w_i^\pm(x_0)| < \frac{\varepsilon}{2} \quad \forall x \in B_\delta(x_0) \cap \overline{\Omega}$.
By the triangle inequality, it then follows that $|u(x) - \varphi(x)| \leq \varepsilon$ if $x \in B_\delta(x) \cap \Omega$. $\square$

<u>Proposition 4.9</u>

Suppose (H) holds Let $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$, and $x_0 \in \partial\Omega$. Then if $\exists\, w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ s.t.

(i) $Lw \leq -1$ in Ω, and

(ii) $w(x_0) = 0$ and $w(x) > 0 \quad \forall x \in \partial\Omega \setminus \{x_0\}$, then $\exists$ lower & upper barriers at x_0 relative to (L, f, φ).

In fact, for any sequence $\varepsilon_i \searrow 0$ $\exists$ constants $k_i > 0$ s.t.

$$w_i^{\pm}(x) = \varphi(x_0) \pm \varepsilon_i \pm k_i w(x)$$

are upper & lower barriers at x_0.

Proof By continuity, for any $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $\varphi(x) \leq \varphi(x_0) + \varepsilon$ $\forall x \in \partial\Omega \cap B_\delta(x_0)$.

As $\varphi \in L^\infty(\partial\Omega)$, and w positive and continuous on the compact set $\partial\Omega \setminus B_\delta(x_0)$, we can choose $l_\varepsilon > 0$ s.t. $\forall x \in \partial\Omega \setminus B_\delta(x_0)$

$$l_\varepsilon w(x) \geq \varphi(x) - \varphi(x_0) - \varepsilon$$

(since w attains a positive minimum). Hence

$$\varphi(x_0) + \varepsilon + l_\varepsilon w(x) \geq \varphi(x) \quad \forall x \in \partial\Omega,$$

since on $B_\delta(x_0)$ this is true by choice of δ. So let (i.e. increase l_ε if necessary)

$$k_\varepsilon = \max\left\{ l_\varepsilon, \sup_{x \in \Omega} |f(x) - c(x)\varphi(x_0)| \right\},$$

then if we set $w_\varepsilon = \varphi(x_0) + \varepsilon + k_\varepsilon w$, thus satisfies $w_\varepsilon(x) \geq \varphi(x) \ \forall x \in \partial\Omega$

and

$$Lw_\varepsilon = c(\varepsilon + \varphi(x_0)) + k_\varepsilon Lw$$

$$\leq -k_\varepsilon + c(\varepsilon + \varphi(x_0))$$

$$(c \leq 0) \leq -k_\varepsilon + c(x)\varphi(x_0)$$

$$\leq -\sup_{x\in\Omega} |f(x) - c(x)\varphi(x_0)| + c(x)\varphi(x_0)$$

$$\leq f.$$

Hence $w_\varepsilon(x)$ is a supersolution & satisfies $w_\varepsilon(x_0) = \varphi(x_0) + \varepsilon$, so for any sequence $\varepsilon_i \searrow 0$ $\{w_i^+ := w_{\varepsilon_i}\}_i$ is an upper barrier.

Similarly, $w_i^- = \varphi(x_0) - \varepsilon_i - k_{\varepsilon_i} w(x)$ is a lower barrier (exercise: check this). $\square$

Hence existence of w as in Prop. 4.9 implies the existence of barriers, which in turn implies correct boundary behaviour of the Perron solution.

When does such a w exist?

Proposition 4.10

If Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$, then upper and lower barriers at x_0 relative to (L, f, φ) exist. In particular, if $\partial\Omega$ is C^2, then the Perron solution agrees with φ on $\partial\Omega$, and hence is a genuine solution.

Proof Enough to exhibit the existence of w as in Prop. 4.9. By the exterior sphere condition, $\exists\, R > 0$ s.t. $\overline{B_R(y)} \cap \overline{\Omega} = \{x_0\}$. For constants $\mu, \sigma > 0$ to be chosen s.t. $\forall\, x \in \Omega$

$$w(x) := \mu\left(R^{-\sigma} - |x-y|^{-\sigma}\right).$$

Then

$$
\begin{aligned}
Lw &= c(x)\left(\mu R^{-\sigma} - \mu |x-y|^{-\sigma}\right) \\
&+ \mu \sigma b^i (x_i - y_i) |x-y|^{-\sigma-2} \\
&+ \mu \sigma a_{ii} |x-y|^{-\sigma-2} \\
&- \mu \sigma (\sigma+2) a^{ij} (x_i - y_i)(x_j - y_j) |x-y|^{-\sigma-4}
\end{aligned}
$$

$$
\begin{aligned}
\leq \mu |x-y|^{-\sigma-2} \Big(&- c(x) |x-y|^2 \\
&+ \sigma \|b\|_{L^\infty} |x-y| + \sigma \|\mathrm{tr}\, a\|_{L^\infty} \\
&- \sigma(\sigma+2)\lambda \Big).
\end{aligned}
$$

exercise: check

As $\lambda > 0$ and $\mathrm{diam}(\Omega) < \infty$, the quadratic-in-$\sigma$-term dominates, so choose μ, σ large enough so that $Lw \leq -1$ on Ω. Moreover, by construction $w(x_0) = 0$ & $w(x) > 0$ on $\partial\Omega \setminus \{x_0\}$, and we are done. $\square$

We can now therefore obtain:

<u>Theorem 4.11</u> Let (H) hold, and $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^0(\partial\Omega)$. Then $\exists!\ u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ solving

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

whenever $\partial\Omega$ satisfies the exterior sphere condition at every point.

<u>Proof</u> Combine Prop. 4.9, Prop. 4.10, and Thm. 4.6. $\square$

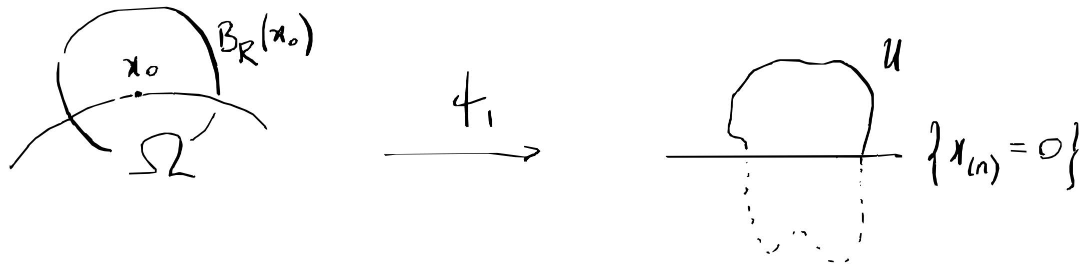
Finally, it remains to show that if $\varphi \in C^{2,\alpha}$ as well, then u extends as $C^{2,\alpha}$ up to the boundary.

<u>Theorem 4.12</u> Let (H) hold and Ω be a bounded $C^{2,\alpha}$ domain. Then for any $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$ $\exists!$ $u \in C^{2,\alpha}(\overline{\Omega})$ s.t.

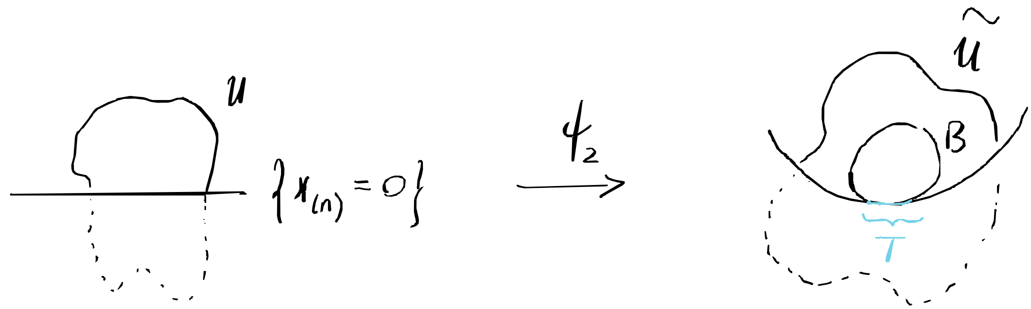
$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega . \end{cases}$$

<u>Proof</u> Let $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ be the unique solution given by <u>Thm 4.11</u>, only need to show $C^{2,\alpha}$ regularity on $\partial\Omega$.

By definition of Ω being $C^{2,\alpha}$, given $x_0 \in \partial\Omega$ $\exists$ $\psi_1 : B_R(x_0) \longrightarrow \mathcal{U}$ a $C^{2,\alpha}$ diffeomorphism, where $\mathcal{U} \subset \mathbb{R}^n$, and $\psi_1(B_R(x_0) \cap \Omega) = \mathcal{U} \cap \{x_{(n)} > 0\}$.



Now compose ψ_1 with a diffeomorphism ψ_2 which deforms a part of $\{x_{(n)}=0\}$ onto a boundary of a ball B:



Denote $\psi := \psi_2 \circ \psi_1$. Hence $\exists$ a $C^{2,\alpha}$ diffeo. $\psi : B_R(x_0) \longrightarrow \tilde{\mathcal{U}}$ s.t. $\exists$ a ball B, $\overline{B} \subset \psi(\overline{\Omega} \cap B_R(x_0))$, and $T := \psi(\partial\Omega \cap B_\varrho(x_0)) \subset \partial B$ for some $\varrho > 0$.

Let $\tilde{x} := \psi(x)$ be the coordinates on $\tilde{\mathcal{U}}$. Then $Lu(x) = f(x)$ transforms to a strictly elliptic equation $\tilde{L}\tilde{u}(y) = \tilde{f}(y)$ in $\tilde{\mathcal{U}}$, with $\tilde{u}(\tilde{x}) = \tilde{\varphi}(\tilde{x})$ on T.

Now solve the Dirichlet problem

$$\begin{cases} \tilde{L} v = \tilde{f} & \text{in } B \\ v = \tilde{u} & \text{on } \partial B \end{cases}$$

— this is possible as B a ball, and we obtain $v \in C^{2,\alpha}(B) \cap C^0(\overline{B})$, and we know $v \in C^{2,\alpha}(B \cup T)$, since $\tilde{u} = \tilde{\varphi} \in C^{2,\alpha}(T)$ (Prop. 4.3).

But $\tilde{L}(v - \tilde{u}) = 0$ in B and $v - \tilde{u} = 0$ on ∂B, & $v - \tilde{u} \in C^0(\overline{B})$, so by the maximum principle $v \equiv \tilde{u}$ in $\overline{B}$. Hence $\tilde{u} \in C^{2,\alpha}(B \cup T)$. Transforming back along ψ^{-1}, we have

$$u \in C^{2,\alpha}(\psi^{-1}(\overline{B}) \cap B_\rho(x_0)),$$

and since x_0 was arbitrary, $u \in C^{2,\alpha}(\overline{\Omega})$. $\square$

§4.2 Fredholm Alternative for Elliptic PDEs

Let V be a normed vector space, $T: V \to V$ bounded, linear, compact map.

Recall T compact $\Leftrightarrow$ if $(x_j)_j \subset V$ a bounded sequence, then $(T(x_j))_j$ has a convergent subsequence.

Then the general Fredholm Alternative states:

(i) either $x + T(x) = 0$ has a non-zero solution,

(ii) or $\forall y \in V$, $x + T(x) = y$ has a unique solution $x \in V$.

For a proof, see Gilbarg & Trudinger, §5.

Let us apply this to our setting.

Theorem 4.13 (Fredholm Alternative)

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, $\alpha \in (0,1)$. Let $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ and

L be a strictly elliptic operator in Ω.

Then :

(I) <u>either</u> the homogeneous problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-zero solution in $C^{2,\alpha}(\overline{\Omega})$,

(II) <u>or</u> for any given $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

has a unique solution in $C^{2,\alpha}(\overline{\Omega})$.

<u>Remarks</u>

- Note that there is no assumption on the sign of c here. For existence & uniqueness we only need that the homogeneous problem has only the trivial solution.

• If uniqueness holds, then existence also holds; indeed, if u_1, u_2 satisfy $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$, then $w = u_1 - u_2$ solves the homogeneous problem. Uniqueness gives $w = 0$, whence FA implies existence.

<u>Proof</u> Enough to establish the case $\varphi = 0$, by replacing u with $u - \varphi$ and f with $f - L\varphi \in C^{0,\alpha}(\overline{\Omega})$.

Let σ be a constant s.t. $\sigma \geq \sup_{\Omega} c$ and consider the map

$$L_\sigma : C_0^{2,\alpha}(\overline{\Omega}) \longrightarrow C^{0,\alpha}(\overline{\Omega})$$

work in $C_0^{2,\alpha}(\overline{\Omega})$ as our BCs are zero

$$L_\sigma u := Lu - \sigma u$$

Then by <u>Thm 4.12</u>, we know that L_σ is a bijection ($c_\sigma = c - \sigma \leq 0$). So by the global Schauder estimate and MPAPE,

$$|u|_{2,\alpha;\overline{\Omega}} \lesssim |L_\sigma u|_{0,\alpha;\overline{\Omega}}$$

$\forall\ u \in C^{2,\alpha}_0(\overline{\Omega})$. Equivalently, since L_σ bijective,

$$|L_\sigma^{-1}(f)|_{2,\alpha;\overline{\Omega}} \lesssim |f|_{0,\alpha;\overline{\Omega}}$$

$\forall f \in C^{0,\alpha}(\overline{\Omega})$, ie L^{-1}_σ is a bounded linear map. Now the inclusion $\iota : C^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ is a compact map (eg by Arzela-Ascoli), so

$$T_\sigma = \iota \circ L_\sigma^{-1} : C^{0,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega})$$

is a compact linear map. Hence so is σT_σ. Applying abstract Fredholm alternative to σT_σ, we have:

(1) <u>either</u> $u + \sigma T_\sigma(u) = 0$ has a non-zero solution $u \in C^{0,\alpha}(\overline{\Omega})$,

(ii) <u>or</u> for any $f \in C^{0,\alpha}(\bar{\Omega})$ $\exists!$ $u \in C^{0,\alpha}(\bar{\Omega})$ s.t. $u + \sigma T_\sigma(u) = L_\sigma^{-1}(f)$.

Observe that $u, f \in C^{0,\alpha}(\bar{\Omega}) \implies$ $L_\sigma^{-1}(u), L_\sigma^{-1}(f) \in C_0^{2,\alpha}(\bar{\Omega})$. But then in both cases $u \in C_0^{2,\alpha}(\bar{\Omega})$, since either (i) $u = -\sigma \iota(L_\sigma^{-1}(u))$, or (ii) $u = -\sigma \iota(L_\sigma^{-1}(u)) + L_\sigma^{-1}(f)$. Hence u has required regularity. It remains to, in each case, apply $L_\sigma = L - \sigma$ to both sides (can do this as $u \in C_0^{2,\alpha}(\bar{\Omega})$) to obtain

(i) <u>either</u> $Lu = 0$ has a non-zero solution $u \in C_0^{2,\alpha}(\bar{\Omega})$,

(ii) <u>or</u> for any $f \in C^{0,\alpha}(\bar{\Omega})$ $\exists! u \in C_0^{2,\alpha}(\bar{\Omega})$ solving $Lu = f$ in Ω.

$\square$

<u>Remark</u> Note that $K = \{ u \in C^{2,\alpha}_0(\overline{\Omega}) : Lu = 0$ in $\Omega \}$ is a <u>finite-dimensional</u> subspace of $C^{2,\alpha}_0(\overline{\Omega})$ when equipped with the $C^{0,\alpha}(\overline{\Omega})$ norm. Indeed, $Lu = 0 \iff u = -\sigma L_\sigma^{-1}(u)$, which gives (by the estimate in the proof)

$$|u|_{2,\alpha;\overline{\Omega}} = |\sigma| \, |L_\sigma^{-1}(u)|_{2,\alpha;\overline{\Omega}}$$

$$\lesssim |u|_{0,\alpha;\overline{\Omega}},$$

ie the unit ball in $C^{0,\alpha}$ gives a uniform bound on the $C^{2,\alpha}$ norms. By Arzela-Ascoli, the unit ball in $(K, \|\cdot\|_{0,\alpha;\overline{\Omega}})$ is therefore compact, which implies $(K, \|\cdot\|_{0,\alpha;\overline{\Omega}})$ is finite-dimensional.

---

§ 4.3 Higher Regularity of Solutions

Lemma 4.14 Let $\Omega \subset \mathbb{R}^n$ be an open domain, $\alpha \in (0,1)$, and $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$. Suppose that L is strictly elliptic in Ω. Suppose $f \in C^{0,\alpha}(\Omega)$ and $u \in C^2(\Omega)$ solve $Lu = f$ in Ω. Then in fact $u \in C^{2,\alpha}(\Omega)$, and

$$|u|_{2,\alpha;\tilde{\Omega}} \le C\left(|u|_{0,\Omega'} + |f|_{0,\alpha;\Omega'}\right)$$

$\forall\ \tilde{\Omega} \subset\subset \Omega' \subset\subset \Omega$, where $C = C(n, \alpha, \Lambda, \text{dist}(\tilde{\Omega}, \partial\Omega'))$.

Proof Let $L_1 u = a^{ij} D_i D_j u + b^i D_i u = f_1$, $f_1 = f - cu \in C^{0,\alpha}(\Omega)$. Fix $B = B_\rho(y) \subset\subset \Omega$. Then by Thm 4.11 for $L_1 u = f_1$ in B, $\exists!$ $v \in C^{2,\alpha}(B) \cap C^0(\overline{B})$ s.t. $L_1 v = f_1$ in B and $v = u$ on ∂B. But then $L_1(v-u) = 0$ in B and $v - u = 0$ on ∂B, & $v - u \in C^0(\overline{B})$. So by the WMP $v = u$ in B. Hence $u \in C^{2,\alpha}(\Omega)$. The estimate follows from the interior Schauder theory. $\square$

We can now keep cranking up the regularity.

<u>Theorem 4.15</u> (Higher Interior Regularity)
Suppose $\Omega \subset \mathbb{R}^n$ is open and $\alpha \in (0,1)$. Let $a^{ij}, b^i, c \in C^{k,\alpha}(\Omega)$ for some $k \geq 0$. Suppose L is strictly elliptic in Ω, and $f \in C^{k,\alpha}(\Omega)$, and that $u \in C^2(\Omega)$ solves $Lu = f$ in Ω.

Then $u \in C^{k+2,\alpha}(\Omega)$, and

$$|u|_{k+2,\alpha;\Omega'} \leq C\left(|u|_{0;\Omega_1} + |f|_{k,\alpha;\Omega_1}\right)$$

$\forall\ \Omega' \subset\subset \Omega_1 \subset\subset \Omega$, where $C = C(n, k, \alpha, \lambda, L, \text{dist}(\Omega', \partial\Omega_1))$.

<u>Proof</u> We prove this by induction on k; Lemma 4.14 establishes the $k=0$ case.

Step 1 Establish the $k=1$ case, ie if $a^{ij}, b^i, c, f \in C^{1,\alpha}(\Omega)$ and $u \in C^{2,\alpha}(\Omega)$, then $u \in C^{3,\alpha}(\Omega)$ and

$$|u|_{3,\alpha;\Omega'} \leq C\left(|u|_{0;\Omega_1} + |f|_{1,\alpha;\Omega_1}\right).$$

Note that we can wlog assume $u \in C^{2,\alpha}$, by the $k=0$ case.

To see this, let $\Omega'' \subset\subset \Omega'$ and fix a direction $l \in \{1, \dots, n\}$. Then for $0 < |h| < \mathrm{dist}(\Omega'', \Omega_1)$ let $\delta_{l,h}$ be the difference quotient operator, i.e.

$$\delta_{l,h}\, g(x) = \frac{g(x+he_l) - g(x)}{h}$$

for $g : \Omega_1 \to \mathbb{R}$. We have

$$L(\delta_{l,h} u)(x) = \frac{1}{h}\Big[Lu(x+he_l) - \overbrace{Lu(x)}^{=f(x)}\Big]$$

$$= \frac{1}{h}\Big[a^{ij}(x) D^2_{ij} u(x+he_l) + b^i(x) D_i u(x+he_l) + c(x)u(x+he_l) - f(x)\Big]$$

$$= \frac{1}{h}\Big[\big(a^{ij}(x) - a^{ij}(x+he_\ell)\big) D^2_{ij} u(x+he_\ell)$$

adding & subtracting $f(x+he_\ell) = Lu(x+he_\ell)$

$$+ \big(b^i(x) - b^i(x+he_\ell)\big) D_i u(x+he_\ell)$$

$$+ \big(c(x) - c(x+he_\ell)\big) u(x+he_\ell)$$

$$+ f(x+he_\ell) - f(x) \Big]$$

$$= -(\delta_{\ell,h} a^{ij})(x) D^2_{ij} u(x+he_\ell)$$

$$- (\delta_{\ell,h} b^i)(x) D_i u(x+he_\ell)$$

$$- (\delta_{\ell,h} c)(x) u(x+he_\ell) \quad + \quad \delta_{\ell,h} f(x)$$

$$=: g_{\ell,h}(x).$$

Now as f is differentiable, we have

$$(\delta_{\ell,h} f)(x) = \frac{1}{h}\big(f(x+he_\ell) - f(x)\big) = \frac{1}{h}\int_0^1 \frac{d}{dt}\big(f(x+the_\ell)\big)dt$$

$$= \int_0^1 (D_\ell f)(x+the_\ell)\,dt,$$

So $\sup_{\Omega'} |\delta_{\ell,h} f| \leq \sup_{\Omega''} |Df|$. Moreover, for

$x \neq y$ we have

$$(\delta_{\ell,h} f)(x) - (\delta_{\ell,h} f)(y) = \int_0^1 \Big[(D_\ell f)(x + th e_\ell) - (D_\ell f)(y + th e_\ell) \Big] dt$$

which implies $[\delta_{e,h} f]_{\alpha;\Omega''} \leq [D_\ell f]_{\alpha;\Omega'}$.

We similarly obtain the same for other coefficients, e.g. $[\delta_{e,h} a^{ij}]_{\alpha;\Omega''} \leq [D_\ell a^{ij}]_{\alpha;\Omega'}$.

Hence by the Interior Schauder estimates, we get:

$$(*) \quad |\delta_{\ell,h} u|_{2,\alpha;\Omega''} \lesssim |g_{\ell,h}|_{0,\alpha;\Omega_1} + |\delta_{\ell,h} u|_{0;\Omega_1}$$

using our bounds above $\longrightarrow$ $$\lesssim |u|_{2,\alpha;\Omega_1} + |f|_{1,\alpha;\Omega_1}.$$

Hence we have a uniform bound on $|\delta_{\ell,h} u|_{2,\alpha;\Omega''}$ independent of h, so by Arzelà–Ascoli we can find $(h_j)_j \searrow 0$ such that $\delta_{\ell,h_j} u \longrightarrow w$, with the convergence in $C^2(\overline{\Omega''})$, where $w \in C^{2,\alpha}(\overline{\Omega''})$

But we know $\delta_{l,h_j} u \longrightarrow D_l u$, so we have $D_l u = w \in C^{2,\alpha}(\Omega'')$, ie $u \in C^{3,\alpha}(\Omega'')$.

By $(*)$ and Lemma 4.14, we conclude the $k=1$ case

Step 2 Prove the $k \geq 2$ case. Now we want to prove that if $a^{ij}, b^i, c, f \in C^{k,\alpha}(\Omega)$ and $u \in C^{k+1,\alpha}(\Omega)$, then infact $u \in C^{k+2,\alpha}(\Omega)$, with the required estimate.

To see this, let γ be any multi-index with $|\gamma| = k-1$. Then from the product rule

$$
\begin{aligned}
L(D^\gamma u) = D^\gamma f - \sum_{\beta < \gamma} \frac{\gamma!}{\beta!(\gamma-\beta)!} \Big(& D^{\gamma-\beta} a^{ij} \cdot D^\beta D^2_{ij} u \\
& + D^{\gamma-\beta} b^i \cdot D^\beta D_i u \\
& + D^{\gamma-\beta} c \cdot D^\beta u \Big),
\end{aligned}
$$

where $\beta < \gamma$ means $\beta_i \leq \gamma_i \ \forall i$ and $\beta_i < \gamma_i$ for some i. Then one can check that the regularity assumptions on this hold, ie the RHS is in $C^{1,\alpha}(\Omega)$, and we can apply Step 1 to obtain $D^\gamma u \in C^{3,\alpha}(\Omega)$, ie $u \in C^{k+2,\alpha}(\Omega)$, with the required estimate.

Finally:

Step 3 Prove the theorem with the estimate, which follows by inducting on k. $\square$

Remark For a global regularity result see §6 of Gilbarg & Trudinger.

<u>Remark</u> We can apply Thm 4.15 to certain nonlinear problems, e.g. the minimal surface equation. Suppose $u \in C^2(\Omega)$ solves

$$D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0$$

in Ω; then in fact $u \in C^\infty(\Omega)$.

Indeed, we have

$$\underbrace{\left(\delta_{ij} - \frac{D_i u \, D_j u}{1 + |Du|^2} \right)}_{a^{ij}} D^2_{ij} u = 0$$

Hence if $u \in C^2(\Omega)$, then $a^{ij} \in C^1(\Omega) \subset C^{0,\alpha}(\Omega)$, so Lemma 4.14 $\Rightarrow$ $u \in C^{2,\alpha}(\Omega)$

But then $Du \in C^{1,\alpha}(\Omega)$, so in fact $a^{ij} \in C^{1,\alpha}(\Omega)$, so Thm 4.15 $\Rightarrow$ $u \in C^{3,\alpha}(\Omega)$

But then $a^{ij} \in C^{2,\alpha}(\Omega)$, etc, so by induction $u \in C^\infty(\Omega)$. $\square$

§ 5 Quasilinear Elliptic Theory and the De Giorgi – Nash – Moser Theory

Let $\alpha \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ a bounded $C^{2,\alpha}$ domain. Let $a^{ij}, b \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ with coordinates $(x, z, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ and suppose $a^{ij}(x, z, p) > 0$ in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.

We now consider the quasilinear operator

$$Qu = a^{ij}(x, u, Du)\, D^2_{ij} u + b(x, u, Du).$$

The Dirichlet problem is then, given $\varphi \in C^{2,\alpha}(\overline{\Omega})$, find a solution $u \in C^{2,\alpha}(\overline{\Omega})$ such that

$$(5.1) \qquad \begin{cases} Qu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

We have the following abstract theorem.

Theorem 5.1 (Leray - Schauder Fixed Point Theorem / Schaefer's Theorem)

Let X be a Banach space and $T: X \to X$ a continuous, compact, possibly nonlinear, map. Suppose $\exists$ a constant $M > 0$ s.t. whenever $x = \sigma T(x)$ for some $\sigma \in [0,1]$, then $\|x\| < M$ (M indep. of σ).

Then $\exists\, x_0 \in X$ s.t. $x_0 = T(x_0)$.

Proof Gilbarg & Trudinger §11 $\square$

We apply Theorem 5.1 to establish the solvability of (5.1) in $C^{2,\alpha}(\overline{\Omega})$. First set $X = C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$ to be determined. Define

$$T : X \longrightarrow X$$

$$T(v) = u,$$

where

$$(*) \quad \text{linear} \begin{cases} a^{ij}(x, v, Dv)\, D^2_{ij} u + b(x, v, Dv) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

That is, we are solving the linear eq.

$$a^{ij} D^2_{ij} u = F .$$

Note that

$$x \mapsto a^{ij}(x, v(x), Dv(x)) , \quad x \mapsto b(x, v(x), Dv(x)) ,$$

are in $C^{0,\alpha\beta}(\overline{\Omega})$ (cf composition of Hölder functions), so by our linear theory $\exists!\ u \in C^{2,\alpha\beta}(\overline{\Omega}) \subset C^{1,\beta}(\overline{\Omega})$ satisfying $(*)$, so T is well-defined. Clearly solving (5.1) $\Leftrightarrow$ finding a fixed point for T.

Can check using Schauder estimates (exercise) that T is a continuous and compact map. Also, if $v \in C^{1,\beta}(\overline{\Omega})$ has $v = \sigma T(v)$ for some $\sigma \in [0,1]$, then $u = \frac{v}{\sigma}$ solves $(*)$, or equivalently

$$(5.1)_\sigma \begin{cases} a^{ij}(x, v, Dv) D^2_{ij} v + \sigma b(x, v, Dv) = 0 & \text{in } \Omega \\ v = \sigma \varphi & \text{on } \partial\Omega \end{cases}$$

Note that this also implies $v \in C^{2,\alpha}(\overline{\Omega})$:

indeed, we saw $v \in C^{1,\beta}(\overline{\Omega}) \Rightarrow T(v) \in C^{2,\alpha\beta}(\overline{\Omega})$, and hence $v = \sigma T(v) \in C^{2,\alpha\beta}(\overline{\Omega})$, so in fact the coefficients of $(*)$ are $C^{0,\alpha}(\overline{\Omega})$. $[Dv \in C^{1,\alpha\beta}(\overline{\Omega}) \subset C^{0,1}(\overline{\Omega})]$. So Schauder theory then implies the solution is in fact $C^{2,\alpha}(\overline{\Omega})$.

By Leray-Schauder Fixed Point Theorem, solvability of (5.1) will follow if we can show:

"$\exists$ a constant $M = M(\varphi, a^{ij}, b) > 0$ & $\beta > 0$ s.t. whenever $v \in C^{1,\beta}(\overline{\Omega})$ solves $(5.1)_\sigma$ for some $\sigma \in [0,1]$, we have $|v|_{1,\beta;\overline{\Omega}} \leq M$."

<u>Remark</u> We can assume, by above Schauder arguments, that in fact $v \in C^{2,\alpha}(\overline{\Omega})$.

<u>Remark</u> This theory was developed in mid 20th century to understand the minimal surface equation. De Giorgi & Nash independently solved the existence problem, and Moser later introduced new ideas to do the same. Here we look at Moser's techniques & the De Giorgi – Nash – Moser theory for equations of variational type.

Consider an operator Q which arises from E-L eqns of

$$\mathcal{F}[u] = \int_\Omega F(x, u, Du)\, dx .$$

For simplicity, assume $F(x, z, p) \equiv F(p)$, $p \in \mathbb{R}^n$. The Euler-Lagrange equation in its weak form becomes

$$\int_\Omega F_{p_i}(Du)\, D_i \varphi = 0 \quad \forall \varphi \in C_c^1(\Omega).$$

In divergence form this is $D_i\left(F_{p_i}(Du)\right) = 0$,

or in non-divergence form $F_{p_i} F_{p_j}(Du)\, D^2_{ij} u = 0$.

To get a bound on $|u|_{1,\beta}$, need a bound on Du. Recall we assume $u \in C^{2,\alpha}$, so find an equation for Du. Idea: differentiate the E-L equation. In the weak formulation, replace φ by $D_k \varphi$ $(k \in \{1, \dots, n\})$, $\varphi \in C^2_c(\overline{\Omega})$:

$$\int_\Omega F_{p_i}(Du)\, D_i(D_k \varphi) = 0$$

and integrate by parts wrt x_k to get

$$\int_\Omega F_{p_i p_j}(Du)\, D_j(D_k u)\, D_i \varphi = 0 .$$

Setting $w := D_k u \in C^{1,\alpha}(\overline{\Omega})$, we have

$$\int_\Omega F_{p_i p_j}(Du)\, D_j w \cdot D_i \varphi = 0$$

this is a divergence form eqn for w

ie

$$D_i\left(F_{p_i p_j}(Du)\, D_j w \right) = 0 \quad \text{weakly in } \Omega.$$

Now if eg F is C^∞ & convex, then this is elliptic with coefficients in $C^{1,\alpha}$. But coefficients depend on u, so usual Schauder estimates would give dependence on u.

Need a slightly deeper theory.

Aim : uniform $C^{0,\beta}(\overline{\Omega})$ estimate for w, indep of u, to control $F_{p_i p_j}(Du)$, ie coefficients in above PDE. To apply DGNM theory (to be developed), we seek an estimate of the form

$$\sup_{\overline{\Omega}} |Du| \leq K = K(\varphi).$$

Form of result to be proven is therefore:

$a^{ij} \in L^\infty_{loc}(\Omega)$ strictly elliptic & $w \in H^1_{loc}(\Omega)$ solves divergence-form equation $D_i(a^{ij} D_j w) = 0$ weakly in Ω

$$\Big\Downarrow \text{DGNM}$$

$\exists\ \beta = \beta(\lambda, \Omega', K) \in (0,1)$, where

$K = \max_{i,j} \|a^{ij}\|_{L^\infty(\Omega')}$, s.t. $w \in C^{0,\beta}(\Omega')$

with an appropriate estimate. This β will then be used to apply Leray-Schauder Fixed Point Thm in $X = C^{1,\beta}(\overline{\Omega})$.

§5.1 De Giorgi - Nash-Moser Theory

Now we consider a linear divergence-form operator

$$Lu = D_i(a^{ij} D_j u)$$

on an open domain $\Omega \subset \mathbb{R}^n$.

<u>Remark</u> DGNM theory can be extended to L with lower order terms, but in general this requires non-trivial technical changes to include $b^i D_i u$. Zeroth order terms are easier.

Here we call

$$(H) \begin{cases} a^{ij} \in L^\infty(\Omega) \\ a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{ a.e. } x \in \Omega \\ \sum_{i,j} \|a^{ij}\|^2_{L^\infty(\Omega)} \leq \Lambda . \end{cases}$$

Definition 5.1

We say $u \in H^1(\Omega)$ is a weak sub- (super-) solution to $Lu = 0$ in Ω if

$$\int_\Omega a^{ij} D_i u \, D_j v \leq 0 \quad (\geq 0)$$

$\forall \; v \in H_0^1(\Omega)$ with $v \geq 0$.

Remark Clearly $u \in H^1(\Omega)$ is a weak solution to $Lu = 0$ if and only if it is a weak sub- & super-solution.

We will prove the following.

Theorem 5.2 (Local boundedness of subsolutions)

Let (H) hold. Then if $u \in H^1(\Omega)$ is a weak subsolution to $Lu = 0$ in Ω, then for each $B_R(y) \subset\subset \Omega$ and $p > 1$ we have

$$\sup_{B_{R/2}(y)} u \leq C R^{-\frac{n}{p}} \| u^+ \|_{L^p(B_R(y))},$$

where $C = C(n, \lambda, \Lambda, p)$, $u^+ = \max\{u, 0\}$.

Remark The power of R here guarantees scale-invariance of the constant C.

Theorem 5.3 (Weak Harnack Inequality for non-negative super-solutions)
Let (H) hold. Then if u is a non-negative weak supersolution to $Lu = 0$ in Ω, then for any $B_R(y) \subset\subset \Omega$ and $p \in [1, \frac{n}{n-2})$

$$R^{-\frac{n}{p}} \|u\|_{L^p(B_R(y))} \leq C \inf_{B_{R/2}(y)} u,$$

$C = C(n, \lambda, \Lambda, p)$.

Remark Theorem 5.3 will in fact be a consequence of Theorem 5.2. We will wish to combine Thms 5.2 & 5.3 to get a Harnack inequality for weak solutions of $Lu = 0$, by using Thm 5.2 $\forall\ p > 1$.

Thm 5.2 is not hard for $p=2$, & then easy to extend to $p \geq 2$ using the $p=2$ case & Hölder's inequality. The tricky part is obtaining the cases $p \in (1,2)$.

Corollary 5.4 (Harnack Inequality for non-negative weak solutions)

Let (H) hold. Suppose that $u \in H^1(\Omega)$ is a non-negative weak solution to $Lu = 0$ in Ω. Then for any subdomain $\tilde{\Omega} \subset\subset \Omega$ we have

$$\sup_{\tilde{\Omega}} u \leq C \inf_{\tilde{\Omega}} u$$

for $C = C(n, \lambda, \Lambda, \tilde{\Omega}, \Omega)$.

Proof By Thm 5.2 & 5.3, if $B_{2R}(y) \subset\subset \Omega$, then choosing any $p \in (1, \frac{n}{n-2})$ we have

$$\sup_{B_R(y)} u \leq CR^{-\frac{n}{p}} \|u^+\|_{L^p(B_{2R}(y))} = CR^{-\frac{n}{p}} \|u\|_{L^p(B_{2R}(y))} \leq C \inf_{B_R(y)} u$$

where $C = C(n, \lambda, \Lambda)$. Then complete the proof in the same way as for Harnack for harmonic functions (cover $\tilde{\Omega}$ with balls) $\square$

With 5.2, 5.3 (still unproven) & 5.4, we can then obtain the Hölder continuity of weak solutions as follows.

<u>Theorem 5.5</u> (Hölder Continuity of Weak Solutions)

Let (H) hold. Then if $H^1(\Omega)$ is a weak solution to $Lu = 0$ in Ω, then for any $B_R(x_0) \subset\subset \Omega$ have:

(I) $\forall R \leq R_0$

$$\operatorname*{osc}_{B_R(x_0)} u \leq C\left(\frac{R}{R_0}\right)^{\mu} \operatorname*{osc}_{B_{R_0}(x_0)} u$$

for some $\mu = \mu(n, \lambda, \Lambda) \in (0, 1]$ &

$C = C(n, \lambda, \Lambda) > 0$.

(II) $u \in C^{0,\mu}(\Omega)$, with the estimate

$$R_0^{\mu}\,[u]_{0,\mu;\,B_{R_0/4}(x_0)} \leq C \sup_{B_{R_0}(x_0)} |u| \,.$$

Remarks

- Note that there is no sign assumption on u.
- Recall the oscillation of u is defined to be

$$\operatorname*{osc}_{B_R(x)} u := \sup_{B_R(x)} u - \inf_{B_R(x)} u$$

(so of course related to Hölder continuity).

Proof of Theorem 5.5

First observe that (I) $\Rightarrow$ (II). Indeed, for any $x, y \in B_{R_0/4}(x_0)$ set $d = \frac{5}{4}|x-y| \leq \frac{5}{4}\,2\left(\frac{R_0}{4}\right) = \frac{5}{8}R_0$.

Then $y \in B_d(x) \subset B_{R_0}(x_0)$ and so

$$\frac{|u(x)-u(y)|}{|x-y|^{\mu}} \leq \frac{\operatorname*{osc}_{B_d(x)} u}{\left(\frac{4}{5}d\right)^{\mu}}$$

(1) for
$R \rightsquigarrow d$
$R_0 \rightsquigarrow \frac{5}{8} R_0$
$x_0 \rightsquigarrow x$

$$\leq \frac{C\left(\frac{d}{\frac{5}{8}R_0}\right)^{\mu} \operatorname*{osc}_{B_{\frac{5}{8}R_0}(x)} u}{\left(\frac{4}{5} d\right)^{\mu}}$$

$$= C R_0^{-\mu} \operatorname*{osc}_{B_{\frac{5}{8}R_0}(x)} u$$

as
$B_{\frac{5}{8}R_0}(x) \subset B_{R_0}(x_0)$

$$\leq C R_0^{-\mu} \operatorname*{osc}_{B_{R_0}(x_0)} u .$$

Thus taking the supremum over all $x \neq y$ in $B_{R_0/4}(x_0)$ we obtain

$$[u]_{0,\mu ; B_{R_0/4}(x_0)} \leq C R_0^{-\mu} \operatorname*{osc}_{B_{R_0}(x_0)} u ,$$

which then gives the estimate since $\operatorname*{osc}_{\Omega} u \leq 2 \sup_{\Omega} |u|$.

To prove (1), consider the cases of large R/R_0 & small R/R_0 separately. So for e.g.

$R \geq \frac{1}{4} R_0$ we have

$$\operatorname*{osc}_{B_R(x_0)} u \leq \operatorname*{osc}_{B_{R_0}(x_0)} u \qquad \text{as } R \leq R_0$$

$$\leq 4^\mu \left(\frac{R}{R_0}\right)^\mu \operatorname*{osc}_{B_{R_0}(x_0)} u, \qquad \text{trivially as } \mu > 0 \ \& \ \frac{4R}{R_0} \geq 1$$

which is what we wanted with $C = 4^\mu$.

Now assume $R \leq \frac{1}{4} R_0$ & set

$$M_1 := \sup_{B_R(x_0)} u, \qquad m_1 := \inf_{B_R(x_0)} u,$$

$$M_4 := \sup_{B_{4R}(x_0)} u, \qquad m_4 := \inf_{B_{4R}(x_0)} u.$$

Then both $M_4 - u$ & $u - m_4$ are non-negative in $B_{4R}(x_0)$ & satisfy (as L has no zeroth order term)

$$L(M_4 - u) = 0 = L(u - m_4) \quad \text{in } B_{4R}(x_0).$$

So by Theorem 5.3 with $p=1$,

$$R^{-n}\int_{B_{2R}(x_0)} (M_4-u) \leq C \inf_{B_R(x_0)} (M_4-u)$$
$$= C(M_4-M_1)$$

and

$$R^{-n}\int_{B_{2R}(x_0)} (u-m_4) \leq C \inf_{B_R(x_0)} (u-m_4)$$
$$= C(m_1-m_4).$$

Add these to get

$$M_4-m_4 \leq C(M_4-m_4-(M_1-m_1))$$

or

$$(M_1-m_1) \leq \gamma (M_4-m_4),$$

where $\gamma = \frac{C-1}{C}$ (note WLOG $C>1$ as we can always increase C in Thm 5.3). Hence we have

$$\operatorname*{osc}_{B_R(x_0)} u \leq \gamma \operatorname*{osc}_{B_{4R}(x_0)} u.$$

Take $R = \frac{1}{4}R_0$. Iterating this inequality, we get for any $m \geq 1$:

$$(*) \quad \operatorname*{osc}_{B_{4^{-(m+1)}R_0}(x_0)} u \leq \gamma \operatorname*{osc}_{B_{4^{-m}R_0}(x_0)} u \leq \dots \leq \gamma^m \operatorname*{osc}_{B_{R_0/4}(x_0)} u .$$

To complete the proof for $R \leq \frac{R_0}{4}$ it remains to interpolate the inequality $(*)$ between the $4^{-m}R_0$ powers.

Take any $R < \frac{R_0}{4}$. The intervals $\left(\frac{R_0}{4^2}, \frac{R_0}{4}\right]$, $\left(\frac{R_0}{4^3}, \frac{R_0}{4^2}\right]$, … partition $\left(0, \frac{R_0}{4}\right]$, so we can find $m \geq 1$ with

$$\frac{R_0}{4^{m+1}} < R \leq \frac{R_0}{4^m}.$$

In particular this gives $m+1 \geq \frac{\log(R_0/R)}{\log 4}$, & so if we set

$$\mu := \frac{\log(\gamma)}{\log(1/4)} > 0 \qquad (\gamma \in (0,1))$$

& take $C \geq \frac{4}{3}$ ($\Leftrightarrow \gamma \geq \frac{1}{4}$), we have $\mu \leq 1$.

Then

$$\gamma^{m+1} \leq \left(\frac{R}{R_0}\right)^{\mu}$$

and since $B_R(x_0) \subset B_{R_0/4^m}(x_0)$ we have

$$\operatorname*{osc}_{B_R(x_0)} u \leq \operatorname*{osc}_{B_{R_0/4^m}(x_0)} u$$

$$\leq \gamma^m \operatorname*{osc}_{B_{R_0}(x_0)} u$$

$$\leq \frac{1}{\gamma}\left(\frac{R}{R_0}\right)^{\mu} \operatorname*{osc}_{B_{R_0}(x_0)} u .$$

Hence combining this with the $R \geq R_0/4$ case gives (1), and completes the proof. □

Remark In the proof of Thm 5.5 we used that $\sup u$ bounded. This will be justified by Thm 5.2.

Recall

Theorem 5.2 (H) & $u \in H^1(\Omega)$ solves $Lu = 0$ in Ω, then

$$\sup_{B_{R/2}(y)} u \leq C R^{-\frac{n}{p}} \|u^+\|_{L^p(B_R(y))}$$

Proof of Theorem 5.2

First note that u^+ is also a weak subsolution (ES4). So we can assume WLOG that $u \geq 0$ a.e., since $\sup_{B_{R/2}(y)} u \leq \sup_{B_{R/2}(y)} u^+$. We will build a test function using the fact that u has a sign.

Assume further that $u \geq \varepsilon > 0$ a.e. for some $\varepsilon > 0$. Once we establish the result for such a u, we can take the limit $\varepsilon \to 0$ to obtain the result for general $u \geq 0$.

By scaling & translation, we can also assume

that $y=0$ & $R=1$. Idea: obtain uniform bounds for $\|u\|_{L^p}$ $(p \gg 1)$ & take $p \to \infty$.
This is done using <u>Moser iterations</u>, and essentially by using "$v = u^\beta \eta$" as a test function (β large & $\eta \in C_c^1(B)$, $B = B_1(0)$).

<u>Define</u> (for <u>any</u> $\beta > 0$)

$$v_k := \min\{u^\beta, ku\}$$

to stop u^β getting too large

Note $v_k \to u^\beta$ pointwise as $k \to \infty$, since $u \geq \varepsilon$.

<u>Claim</u> $v_k \in H^1(B)$ with

$$Dv_k = \begin{cases} \beta u^{\beta-1} Du & \text{in } \Omega_k \\ k Du & \text{in } B \setminus \Omega_k, \end{cases}$$

where $\Omega_k = \{x \in B : u^\beta(x) \leq k u(x)\}$.

<u>Proof</u> Note that $v_k = u \min\{u^{\beta-1}, k\} = u\, w_k^{\beta-1}$
$= u\, g(w_k)$, where $w_k = \min\{u, k^{\frac{1}{\beta-1}}\}$ and
$g(t) = t^{\beta-1}$, $t \in [\varepsilon, k^{1/(\beta-1)}]$. Since g &

g' are bounded in $[\varepsilon, k^{1/(\beta-1)}]$, it follows that $g(w_k) \in H^1(B)$ & $u\, g(w_k) \in H^1(B)$ (standard facts about Sobolev spaces). $\square$

Now let $\eta \in C_c^1(B)$, $\eta \geq 0$ and put $v = v_k \eta^2$ as a non-negative test function in the definition of a subsolution. Then

$$\int_B a^{ij} D_i u \, D_j (v_k \eta^2) \leq 0$$

which gives

$$\beta \int_{\Omega_k} a^{ij} D_i u \left(u^{\beta-1} D_j u\right) \eta^2$$

$$+ k \int_{B \setminus \Omega_k} (a^{ij} D_i u \, D_j u) \eta^2$$

$$\leq -2 \int_B (a^{ij} D_i u) v_k \eta D_j \eta .$$

Using ellipticity on the LHS & bound on a^{ij}

on RHS gives

$$\lambda \beta \int_{\Omega_k} |Du|^2 u^{\beta-1} \eta^2 + \lambda k \int_{B \setminus \Omega_k} |Du|^2 \eta^2$$

$$\lesssim 2\Lambda \int_{\Omega_k} |Du| |D\eta| \eta u^\beta + 2\Lambda k \int_{B\setminus\Omega_k} |Du| |D\eta| \eta u .$$

Now use Young's inequality to split the RHS & absorb into LHS:

$$\beta \int_{\Omega_k} |Du|^2 u^{\beta-1} \eta^2 + k \int_{B\setminus\Omega_k} |Du|^2 \eta^2$$

$$\lesssim \frac{2\Lambda}{\lambda} \left(\int_{\Omega_k} |Du| \eta u^{\frac{\beta-1}{2}} \cdot u^{\frac{\beta+1}{2}} |D\eta| + k \int_{B\setminus\Omega_k} |Du| \eta |D\eta| u \right)$$

$$\lesssim \frac{\beta}{2} \int_{\Omega_k} |Du|^2 \eta^2 u^{\beta-1} + \frac{C}{\beta} \int_{\Omega_k} u^{\beta+1} |D\eta|^2$$

$$+ \frac{k}{2} \int_{B\setminus\Omega_k} |Du|^2 \eta^2 + Ck \int_{B\setminus\Omega_k} |D\eta|^2 u^2$$

to get

$$\frac{\beta}{2}\int_{\Omega_k}|Du|^2u^{\beta-1}\eta^2+\frac{k}{2}\int_{B\setminus\Omega_k}|Du|^2\eta^2$$

$$\leq \frac{C}{\beta}\int_{\Omega_k}u^{\beta+1}|D\eta|^2+Ck\int_{B\setminus\Omega_k}|D\eta|^2u^2.$$

To be continued.

Continuing the proof of Theorem 5.2. We are proving $\sup |u| \leq C(p) \|u\|_{L^p}$.

We obtained

$$\frac{\beta}{2}\int_{\Omega_k} |Du|^2 u^{\beta-1}\eta^2 \leq$$

$$\frac{C}{\beta}\int_{\Omega_k} u^{\beta+1}|D\eta|^2 + \overbrace{Ck\int_{B\setminus\Omega_k} |D\eta|^2 u^2}^{(*)}.$$

Now we want to send $k \to \infty$. For $(*)$ we know that on $B\setminus\Omega_k$, $ku \leq u^\beta$, so

$$\frac{\beta}{2}\int_{\Omega_k} |Du|^2 u^{\beta-1}\eta^2 \leq \frac{C}{\beta}\int_{\Omega_k} u^{\beta+1}|D\eta|^2 + C\int_{B\setminus\Omega_k} |D\eta|^2 u^{\beta+1}.$$

Now since Ω_k is increasing in k & $\bigcup_{k>0}\Omega_k = B$, we have $\mathbb{1}_{\Omega_k} \to \mathbb{1}_B$ pointwise, so by Fatou's Lemma for the LHS we will have

$$\int_B |Du|^2 u^{\beta-1} \eta^2 = \int_B \liminf_j \left(\mathbb{1}_{\Omega_j} |Du|^2 u^{\beta-1} \eta^2 \right)$$

$$\overset{\text{Fatou}}{\leq} \liminf_j \int_{\Omega_j} |Du|^2 u^{\beta-1} \eta^2 .$$

Also, provided

$$\int_B u^{\beta+1} |D\eta|^2 < \infty , \qquad (\dagger)$$

the Dominated Convergence Theorem gives for the RHS

$$\int_{\Omega_j} u^{\beta+1} |D\eta|^2 \to \int_B u^{\beta+1} |D\eta|^2$$

$$\& \quad \int_{B \setminus \Omega_j} u^{\beta+1} |D\eta|^2 \longrightarrow 0 ,$$

ie in the limit as $k \to \infty$

$$\int_B |Du|^2 u^{\beta-1} \eta^2 \leq \frac{C}{\beta^2} \int_B u^{\beta+1} |D\eta|^2$$

some $C = C(\Lambda/\lambda)$. The condition $(\dagger)$ will be a "bootstrap" assumption.

The Bootstrap

Call $\alpha = \beta + 1$. We have, if $\int_B u^\alpha |D\eta|^2 < \infty$,

$$\int_B |Du|^2 u^{\alpha-2} \eta^2 \leq \frac{C}{(\alpha-1)^2} \int_B u^\alpha |D\eta|^2 . \qquad (*)$$

Now, observe that

$(a+b)^2 \leq 2a^2 + 2b^2$

$$\int_B |D(u^{\alpha/2}\eta)|^2 \leq \int \left(\frac{\alpha^2}{2} u^{\alpha-2} |Du|^2 \eta^2 + 2u^\alpha |D\eta|^2 \right)$$

$$\overset{(*)}{\leq} \frac{C\alpha^2}{(\alpha-1)^2} \int_B u^\alpha |D\eta|^2 .$$

Now let

$$\sigma = \begin{cases} \frac{n}{n-2} & \text{if } n \geq 3 \\ >1 \text{ arbitrary} & \text{if } n = 2 . \end{cases}$$

Then the Sobolev Embedding Theorems give

$$\| u^{\alpha/2} \eta \|_{L^{2\sigma}(\mathbb{R}^n)} \lesssim \| D(u^{\alpha/2}\eta) \|_{L^2(\mathbb{R}^n)} ,$$

and since η has support in B, this gives

$$\left(\int_B (u^{\alpha/2}\eta)^{2\sigma} \right)^{\frac{1}{\sigma}} \leq \frac{C\alpha^2}{(\alpha-1)^2} \int_B u^\alpha |D\eta|^2$$

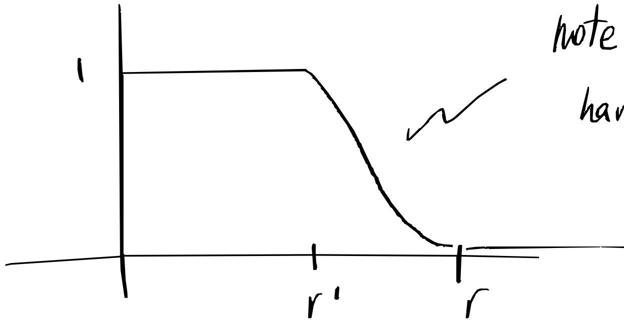
or

$$(**) \quad \left(\int_B u^{\alpha\sigma} \eta^{2\sigma} \right)^{\frac{1}{\alpha\sigma}} \leq \left(\frac{C\alpha^2}{(\alpha-1)^2} \right)^{\frac{1}{\alpha}} \left(\int_B u^\alpha |D\eta|^2 \right)^{\frac{1}{\alpha}}.$$

Note here $\alpha\sigma > \alpha$ since $\sigma > 1$, so we have increased the power on the LHS. We wish to iterate this inequality; but since η is supported on a subset of B, we need to choose the test functions carefully to avoid shrinkage to a point.

For any $r' < r < 1$ choose η s.t.

$\eta|_{B_{r'}} \equiv 1$, $\eta|_{B \setminus B_r} \equiv 0$, & $|D\eta| \leq \frac{2}{r-r'}$.



note we must have $|D\eta| > \frac{1}{r-r'}$

Then $(**)$ gives

$$\left(\int_{B_{r'}} u^{\alpha\sigma} \right)^{\frac{1}{\alpha\sigma}} \leq$$

$$\left(\frac{C\alpha^2}{(\alpha-1)^2} \right)^{1/\alpha} \left(\frac{2}{r-r'} \right)^{2/\alpha} \left(\int_{B_r} u^{\alpha} \right)^{\frac{1}{\alpha}}$$

whenever $\int_{B_r} u^\alpha < \infty$.

We will choose radii r_j such that when we iterate this,

- the constants on the RHS do not blow up
- the support of the integral on the LHS does not collapse to zero.

Put $r_j = \frac{1}{2} + \frac{1}{2^{j+1}}$, $j \in \{1, 2, \dots\}$, and use $r' = r_j$, $r = r_{j-1}$. We had $\beta > 0$ arbitrary, so $\alpha = \beta + 1 > 1$ is arbitrary. Take $\alpha = p\sigma^{j-1}$, $p > 1$. We then get

$$\left(\int_{B_{r_j}} u^{p\sigma^j} \right)^{\frac{1}{p\sigma^j}} \leq C(p)^{\frac{1}{p\sigma^{j-1}}} \, 2^{\frac{2(j+2)}{p\sigma^{j-1}}} \tag{‡‡}$$

$$r_{j-1} - r_j = 2^{-(j+1)} \qquad \times \left(\int_{B_{r_{j-1}}} u^{p\sigma^{j-1}} \right)^{\frac{1}{p\sigma^{j-1}}},$$

where $C(p)$ depends only on p since $\frac{\alpha}{\alpha-1}$ is decreasing in j, so bounded by $\left(\frac{\alpha}{\alpha-1}\right)(j=1)$ $= \frac{p}{p-1}$.

Now iterate (‡‡): dropping terms outside $B_{1/2}$ on LHS, for any j get

$$\left(\int_{B_{1/2}} u^{p\sigma^j} \right)^{\frac{1}{p\sigma^j}} \leq C(p)^{\overbrace{\sum_{\ell=1}^{\infty} \frac{1}{p\sigma^{\ell-1}}}^{<\infty}} \cdot 2^{\frac{2}{p} \overbrace{\sum_{\ell=1}^{\infty} \frac{\ell+2}{\sigma^{\ell-1}}}^{<\infty \text{ as } \sigma > 1}} \left(\int_B u^p \right)^{\frac{1}{p}}$$

Now the RHS is indep. of j, so sending $j \to \infty$ $(p\sigma^j \to \infty)$ we get

$$\sup_{B_{1/2}} |u| \leq C(p) \left(\int_B u^p \right)^{1/p}. \qquad \square$$

Remark The technique of choosing test functions & their supports in this way to ensure supp (LHS) converges to $B_{1/2}$ is called Moser's iterations.

Remark Note that very weak regularity assumptions were needed on a^{ij}, only that $a^{ij} \in L^\infty(\Omega)$.

We now turn to the proof of Theorem 5.3.

We will need the following, which we won't prove.

Lemma 5.6 (John-Nirenberg)

Suppose $u \in W^{1,1}(B)$, $B = B_1(0)$, and that $\exists M$ s.t.

$$\frac{1}{\rho^{n-1}} \int_{B_\rho(y)} |Du| \leq M \quad \forall\ B_\rho(y) \subset B.$$

<u>Then</u>

$$\int_B e^{\frac{p_0}{M}|u-u_B|} \leq C,$$

where $u_B = \frac{1}{|B|}\int_B u$, $p_0 = p_0(n)$, &

$C = C(n)$.

<u>Remark</u> We will not prove this; the lemma may be seen as showing that although $W^{1,1}$ functions are not L^∞, they are "exponentially integrable".

Recall Thm 5.3 states under assumptions (H)

if u non-neg. weak supersolution to $Lu = 0$ in Ω, then $\forall B_R(y) \subset\subset \Omega$ & $p \in [1, \frac{n}{n-2})$

$$R^{-\frac{n}{p}} \|u\|_{L^p(B_R(y))} \leq C \inf_{B_{R/2}(y)} u$$

Proof of Theorem 5.3

WLOG translate & rescale so that $y = 0$ & $R = 2$. So $u \in H^1(B_2)$, $u \geq 0$ & a supersolution. Like in the proof of Thm 5.2, assume $u \geq \varepsilon$, if not, we can apply the theorem to $u + \varepsilon$ & take $\varepsilon \to 0$.

Let $v = \frac{1}{u}$. Apply the definition of u being a supersolution with test fn $v^2 \varphi$, $\varphi \geq 0$ & $\varphi \in C_c^1(B_2)$:

$$\int_{B_2} a^{ij} D_i u \, D_j (v^2 \varphi) \geq 0 \quad \Longrightarrow$$

$$\int_{B_2} a^{ij} \frac{D_i v}{v^2} \left(2v D_j v \varphi + v^2 D_j \varphi \right) \leq 0$$

i.e.

$$\int_{B_2} a^{ij} D_i v D_j \varphi \leq -2 \int_{B_2} \frac{a^{ij} D_i v D_j v}{v} \varphi$$

$$\leq -2\lambda \int_{B_2} \frac{|Dv|^2}{v} \varphi \leq 0,$$

So v is a <u>subsolution</u>. So by Theorem 5.2 we have

$$\sup_{B_{3/4}} v \leq C \left(\int_{B_{3/2}} v^p \right)^{1/p},$$

$C = C(n, p, \lambda, \Lambda)$; reinstating $v = 1/u$ then gives

$$C \inf_{B_{3/4}} u \geqslant \left(\int_{B_{3/2}} u^{-p} \right)^{-\frac{1}{p}}$$

$$= \left(\int_{B_{3/2}} u^p \right)^{\frac{1}{p}} \times \underbrace{\left[\left(\int_{B_{3/2}} u^p \right) \left(\int_{B_{3/2}} u^{-p} \right) \right]^{-\frac{1}{p}}}_{(*)}.$$

wish to control (

If we can show $(*) \leq C(n, p, \Lambda, \lambda)$, then result will follow.

Claim $\exists\, p_0 > 0$ s.t. $\forall\, p \in (0, p_0]$

$$\left(\int_{B_{3/2}} u^{-p}\right)\left(\int_{B_{3/2}} u^{p}\right) \leq C(n, \lambda, \Lambda).$$

Proof of Claim To see this, let

$$w = \log(u) - \frac{1}{|B_{3/2}|}\int_{B_{3/2}} \log u$$

(recall we are assuming $u \geq \varepsilon$). Applying the definition of u being a supersolution with the test function $\frac{\varphi}{u}$, we have

$$\int_{B_2} a^{ij} D_i u D_j\left(\frac{\varphi}{u}\right) \geq 0 \implies$$

$D_i w = \frac{1}{u} D_i u$

$$\int_{B_2} a^{ij} u D_i w\left(\frac{1}{u} D_j \varphi - \frac{\varphi}{u^2} D_j u\right) \geq 0,$$

which implies

$$\int_{B_2} a^{ij} D_i w D_j \varphi \geq \int_{B_2} (a^{ij} D_i w D_j w) \varphi$$

$$\geq \lambda \int_{B_2} |Dw|^2 \varphi .$$

Can replace φ by φ^2 to get

$$2 \int_{B_2} (a^{ij} D_i w D_j \varphi) \varphi \geq \lambda \int_{B_2} |Dw|^2 \varphi^2 .$$

Hence

$$\int_{B_2} |Dw|^2 \varphi^2 \leq \frac{2\Lambda}{\lambda} \int_{B_2} |Dw| |D\varphi| |\varphi|$$

$$\overset{\text{Young}}{\leq} \frac{1}{2} \int_{B_2} |Dw|^2 \varphi^2 + C \frac{\Lambda}{\lambda} \int_{B_2} |D\varphi|^2$$

and so

$$\int_{B_2} |Dw|^2 \varphi^2 \leq C \int_{B_2} |D\varphi|^2 .$$

Since φ was an arbitrary positive test function, given any $B_\rho(y) \subset B_{3/2}(0)$ we can choose $\varphi|_{B_\rho(y)} \equiv 1$, $\varphi|_{B_2 \setminus B_{\frac{5}{4}\rho}(y)} \equiv 0$, with $|D\varphi| \le \frac{C}{\rho}$. Then the above gives

$$\int_{B_\rho(y)} |Dw|^2 \le C\rho^{n-2},$$

$C = C(n, \lambda, \Lambda)$. Using this, Hölder's ineq therefore gives

$$\int_{B_\rho(y)} |Dw| \le |B_\rho(y)|^{1/2} \left(\int_{B_\rho(y)} |Dw|^2 \right)^{1/2}$$

$$\le C\rho^{n/2} \cdot C(\rho^{n-2})^{1/2}$$

$$\le M\rho^{n-1}.$$

Thus w satisfies the conditions of Lemma 5.6, which gives (note $w_{B_{3/2}} = 0$)

$$\int_{B_{3/2}} e^{p_0|w|} \leq C \qquad (*)$$

(where we absorbed M into p_0). Thus

$b = u_{B_{3/2}}$

$$\left(\int_{B_{3/2}} u^{-p_0}\right)\left(\int_{B_{3/2}} u^{p_0}\right) = \left(\int_{B_{3/2}} e^{-p_0(w+b)}\right)$$

$$\times \left(\int_{B_{3/2}} e^{p_0(w+b)}\right)$$

$e^{\pm p_0 b}$ parts cancel as constant

$$\leq \left(\int_{B_{3/2}} e^{p_0|w|}\right)^2$$

$(*)$

$$\leq C$$

This gives the result for p_0. To obtain the result for $p < p_0$, apply Holder

$$\int_{B_{3/2}} u^{\pm p} = \|u^{\pm p}\|_{L^1(B_{3/2})}$$

$$\leq \| u^{\pm p} \|_{L^{p_0/p}(B_{3/2})} \cdot \| 1 \|_{L^{(p_0/p)^*}(B_{3/2})}$$

$$\leq C \| u^{\pm p_0} \|^{p/p_0}_{L^1(B_{3/2})} . \qquad \square$$

Combined with the previous lower bound on $\inf u$, this proves the theorem for $p \leq p_0$. To extend it for any $p \in (0, \frac{n}{n-2})$, apply Moser iterations to increase the power. That is, use the supersolution condition

$$\int_{B_2} a^{ij} D_i u D_j \varphi \geq 0$$

with $\varphi = u^{-\beta} \eta^2$, $\beta \in (0,1)$, and proceed as in the proof of Thm 5.2 to obtain

$$\left(\int_{B_{r'}} u^{\sigma \gamma} \right)^{\frac{1}{\sigma \gamma}} \leq \frac{1}{(r - r')^{2/\gamma}} C^{1/\gamma} \left(\int_{B_r} u^{\gamma} \right)^{\frac{1}{\gamma}},$$

where as before $\sigma = \frac{n}{n-2}$ if $n \geq 3$,

& $\sigma > 1$ arbitrary if $n = 2$, and

$$\gamma = 1 - \beta \in (0,1).$$

Then choosing $\gamma = \sigma^{J-1} \min\{1, p_0\}$,

$r' = 1 + \frac{1}{2^{j+1}}$, $r = 1 + \frac{1}{2^j}$, iterate

this a finite number of times to

ensure $\sigma\gamma = \sigma^J \min\{1, p_0\} \geq p_1$

for any given $p_1 \in \left(0, \frac{n}{n-2}\right)$. Then apply

Hölder's inequality to obtain the result for

$p \leq p_1$.

$\square$

END OF COURSE.