Supplementary Background Material

This course relies on various concepts from Topology, Algebraic Topology and Group Theory with which you may be unfamiliar or perhaps simply a bit rusty. This document will serve as a reference for some of these things. It is entirely possible that I have missed out some things, or assumed incorrectly that they are well-known to you; if so, contact me and I may expand this document to include them.

Bases and sub-bases

Let X be a set and let $\mathcal{T}$ be a topology on X—that is, $\mathcal{T}$ is the set of open subsets of X.

Definition 1. A *basis* for the topology $\mathcal{T}$ is a set $\mathcal{B} \subseteq \mathcal{T}$ of open subsets of X, such that every element of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. We may say that $\mathcal{T}$ is *generated by* $\mathcal{B}$.

A *neighbourhood basis*[1] *at a point* $x \in X$ is a collection $\mathcal{B}_x$ of open subsets of X, all containing x, such that any open subset of X which contains x contains some element of $\mathcal{B}_x$.

If X is a metric space, a standard example of a basis is given by the collection of open balls

$$\mathcal{B} = \{B(x,r) \mid x \in X, r > 0\}$$

in X. A neighbourhood basis at x in a metric space would be

$$\mathcal{B}_x = \{B(x,r) \mid r > 0\}$$

The property of 'being a basis' may be defined intrinsically.

Proposition 2. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. Then there is a (unique) topology on X for which $\mathcal{B}$ is a basis if and only if:*

- *for every $x \in X$ there exists some $B \in \mathcal{B}$ such that $x \in B$; and*
- *for all $B_1, B_2 \in \mathcal{B}$ and for all $x \in B_1 \cap B_2$ there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Proof. One direction is true by definition. For the other direction, if $\mathcal{B}$ satisfies the two properties above, then define $\mathcal{T}$ to be the collection of all subsets of X which are unions of elements of $\mathcal{B}$. It is not difficult to check that $\mathcal{T}$ is a topology. □

There is also a somewhat weaker notion of a sub-basis.

Definition 3. A *sub-basis* for the topology $\mathcal{T}$ is a set $\mathcal{S} \subseteq \mathcal{T}$ of open subsets of X, such that the set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for $\mathcal{T}$.

[1] Actually, the correct word is probably 'base'. Many mathematicians confuse 'base' and 'basis' in this context, possibly because of the influence of Linear Algebra. I personally use the mistaken form 'basis' so often in speech and text that I've just given up and decided to use 'basis' throughout.

The main use of this notion is to make it easier to check that maps are continuous.

Proposition 4. *Let Y and X be topological spaces and let $\mathcal{S}$ be a sub-basis for X. A function $f\colon Y \to X$ is continuous if and only if $f^{-1}(S)$ is open in Y for all $S \in \mathcal{S}$.*

Proof. One direction is clear. Suppose that the preimage of every sub-basic open set is open in Y.

Let $\mathcal{B}$ be the set of finite intersections of elements of $\mathcal{S}$. If $B \in \mathcal{B}$, write

$$B = \bigcap_{i=1}^{n} S_i$$

where $S_i \in \mathcal{S}$. Then

$$f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(S_i)$$

is an open set.

Let U be an open subset of X. Since $\mathcal{B}$ is a basis of X, we may write U as a union

$$U = \bigcup_{i \in I} B_i$$

of elements of $\mathcal{B}$, where I is a potentially infinite set. Then

$$f^{-1}(U) = \bigcup_{i \in I} f^{-1}(B_i)$$

is a union of open sets, hence open. Thus f is continuous. □

Compactness

Compactness should be familiar from IB Matric and Topological Spaces, so I will just recall a few useful properties here without troubling to prove them.

Definition 5. A topological space is *compact* if, for every family $\{U_i\}_{i \in I}$ of open sets of X such that $X = \bigcup_{i \in I} U_i$, there exists a finite subset $F \subseteq I$ such that $X = \bigcup_{f \in F} U_f$.

Proposition 6. *A closed subset of a compact space is compact. A compact subset of a Hausdorff topological space is closed.*

Proposition 7. *The image of a compact space under a continuous map is compact.*

Proposition 8. *Let X be a compact space and Y be a Hausdorff space. Let $f\colon X \to Y$ be a continuous bijection. Then f is a homeomorphism.*

Proposition 9 (Finite Intersection Property[2])**.** *Let X be a topological space. The X is compact if and only if X has the* finite intersection property:

> *if $\{C_i\}_{i \in I}$ is a collection of closed subsets of X such that $\bigcap_{f \in F} C_f$ is non-empty for every finite subset $F \subseteq I$, then $\bigcap_{i \in I} C_i \neq \emptyset$.*

[2] Technically I'm wrong to call this result 'the Finite Intersection Property'. According to correct usage it is the family of sets C_i that has the Finite Intersection Property, not the compact space X. But I think this is silly so I choose to ignore it.

Products

You will have seen in your first topology course the definition of the product of a finite number of topological spaces. In this course we will need to deal with infinite products. This is similar, but there is a subtlety in the definition.

Definition 10. Let $\{X_i\}_{i\in I}$ be a collection of topological spaces indexed by a set I. The *product topology* on $\prod X_i$ is the topology whose basis consists of sets of the form

$$\prod_{f\in F} U_f \times \prod_{i\in I\smallsetminus F} X_i$$

where $F \subseteq I$ is a finite set and U_f is an open subset of X_f.

An alternative way of describing this basis is via projection maps. Let $p_i\colon \prod X_i \to X_i$ be the projection map. Then a basic open set of $\prod X_i$ is a finite intersection

$$\bigcap_{f\in F} p_f^{-1}(U_f)$$

where $F \subseteq I$ is a finite set and U_f is an open subset of X_f.

The sets $p_i^{-1}(U_i)$ themselves form a sub-basis for the product topology.

The obvious question that comes to mind when one sees this definition is 'Why are the basic open sets not those of the form

$$\prod_{i\in I} U_i$$

for U_i open in X_i?'. Well, this is a perfectly valid topology, and it is occasionally studied under the name 'box topology'. However mathematical concepts are defined because they are useful, and the power and usefulness of the true product topology are shown by the next two propositions.

Proposition 11. *Let Z be a topological space and let $\{X_i\}_{i\in I}$ be a collection of topological spaces. The product topology is the unique topology on $\prod X_i$ with the following property:*

> *a function $f\colon Z \to \prod X_i$ is continuous if and only if all the compositions $p_i \circ f$ are continuous.*

Proof. First let us show that the product topology has the given property. The projection maps p_i are continuous directly from the definition, so if $f\colon Z \to \prod X_i$ is continuous then the compositions $p_i f$ are continuous. On the other hand, if all the compositions $p_i f$ are continuous, then for any sub-basic set $S = p_i^{-1}(U_i)$ in the product topology we have

$$f^{-1}(S) = (p_i f)^{-1}(U_i)$$

which is open. Hence f is continuous.

Now we note the uniqueness part of the proposition. Let $\mathcal{T}$ and $\mathcal{U}$ be two topologies on $\prod X_i$ with the given property, and let Y and Z denote the topological spaces $(\prod X_i, \mathcal{T})$ and $(\prod X_i, \mathcal{U})$ respectively.

The identity maps $Y \to Y$ and $Z \to Z$ are automatically continuous, so by the given property the projection maps $p_i^Y : Y \to X_i$ and $p_i^Z : Z \to X_i$ are continuous.

Let $\iota\colon Y \to Z$ be the identity function from $\prod X_i$ to itself. Then $\iota \circ p_i^Z = p_i^Y$ is continuous for all i, hence ι is continuous. By symmetry ι^{-1} is continuous also, so ι is a homeomorphism and there is a unique topology on $\prod X_i$ with the given property. □

The property given in this proposition should remind you of the categorical definition of a product.

The most important property of the product topology for our needs is the following theorem. The proof is beyond the scope of this pamphlet, but I will just mention that this is a fundamental theorem which is in fact equivalent to the Axiom of Choice. It has been described as 'the most important theorem of topology'.

Theorem 12 (Tychonoff's Theorem)**.** *Let X_i be a collection of compact topological spaces. Then $\prod X_i$ is compact.*

A much easier result is the analoguous statement for the Hasudorff property.

Proposition 13. *If all X_i are Hausdorff, then $\prod X_i$ is Hausdorff.*

Proof. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod X_i$ be distinct points. There exists i such that $x_i \neq y_i$. Since X_i is Hausdorff there exist disjoint open sets $U_i, V_i \subseteq X_i$ such that $x_i \in U_i$ and $y_i \in V_i$. Then $p_i^{-1}(U_i)$ and $p_i^{-1}(V_i)$ are disjoint open sets separating (x_i) and (y_i), showing that $\prod X_i$ is Hausdorff. □

Covering spaces

The following theorem is referred to in the Lecture Notes, but the proof is really too much about covering spaces to be included in lectures, so we deal with it here.

Theorem 14 (Nielsen-Schrier Theorem)**.** *Let F be a free group of rank r and let H be a subgroup of F of index n. Then H is a free group of rank $n(r-1)+1$.*

Proof. Let X be a graph with one vertex and r edges, so that $F \cong \pi_1 X$. By the theory of covering spaces, H is isomorphic to the fundamental group of some covering space Y of X of degree n. Then Y is a graph with n vertices and nr edges.

Let T be a maximal subtree of Y. Then T contains all n vertices of Y and has $n-1$ edges. Let Z be the graph obtained from Y by contracting T to a point. This is a homotopy equivalence so $\pi_1 Z \cong \pi_1 Y \cong H$. However Z is a graph with a single vertex and with $nr - (n-1)$ edges. Hence $\pi_1 Z$ is a free group of rank $nr - n + 1$. □

Free subgroups of linear groups

In this course I will make occasional reference to the fact that $\mathrm{SL}_2(\mathbb{Z})$ contains a free subgroup of rank 2, or that a free group injectively maps into $\mathrm{SL}_2(\mathbb{Z})$. This is not really essential knowledge for the course, so there is no particular need for you to see a proof. But it may be of interest to some of you, so it is included here.

Lemma 15 (Ping-Pong Lemma)**.** *Suppose a group G acts on a set X. Let $a, b \in G$. Suppose that X has non-empty disjoint subsets U and V such that:*

- $a^k \cdot u \in V$ *for all* $k \neq 0$ *and all* $u \in U$*; and*
- $b^k \cdot v \in U$ *for all* $k \neq 0$ *and all* $v \in V$.

Then $\langle a, b\rangle$ *is free.*

Proof. Let F be a free group on two generators x and y. We have a homomorphism $f\colon F \to G$ sending x to a and y to b. We wish to show that f is injective. Take an element $w \in F$, thought of as a word in x and y. To show $f(w) \neq 1$ it is sufficient to show $f(gwg^{-1}) \neq 1$ for some $g \in F$, so we may conjugate w by a suitable power of x to ensure that the new word w' starts and ends with a power of x. Now $f(w')$ is an element of G of the form

$$a^{k_1} b^{k_2} \cdots b^{k_{n-1}} a^{k_n}$$

where the k_i are non-zero integers.

To show that $f(w') \neq 1$, consider its action on a point $u \in U$. The a^{k_n} factor sends u to some element of V. The $b^{k_{n-1}}$ part then sends it back to U. This 'ping-pong' repeats until at last the final a^{k_1} sends our element back to V. So $f(w') \cdot u \in V$. Since V and U are disjoint, this means $f(w') \cdot u \neq u$, hence $f(w') \neq 1$ as required. □

Proposition 16. *There is a free subgroup of* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. We will apply the Ping-Pong Lemma to the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2$. Let

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$

and define subsets

$$U = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ such that } |y| > |x| \right\}, \quad V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ such that } |x| > |y| \right\}$$

of $\mathbb{R}^2$. Certainly U and V are non-empty and disjoint.

Let $k \neq 0$ and let $u = (x, y) \in U$, so that $|y| > |x|$. Then

$$a^k \cdot u = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2ky \\ y \end{pmatrix}$$

Since we have

$$|x + 2ky| \geq |2ky| - |x| > (2|k| - 1)|y| \geq |y|$$

we find that $a^k \cdot u \in V$. Similarly $b^k \cdot v \in U$ for all $v \in V$. Hence the Ping-Pong Lemma applies and $\langle a, b\rangle$ is a free subgroup of $\mathrm{SL}_2(\mathbb{Z})$. □

There is nothing particularly special about the choice of a and b here. There are many free subgroups inside $\mathrm{SL}_2(\mathbb{Z})$, but this is perhaps the most common one.