Profinite Properties of 3-Manifold Groups

Gareth Wilkes

St John's College

University of Oxford

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Dedicated to my family.

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Abstract

In this thesis we study the finite quotients of 3-manifold groups, concerning both residual properties of the groups and the properties of the 3-manifolds that can be detected using finite quotients of the fundamental group.

A key theme is the analysis of when two 3-manifold groups can have the same families of finite quotients. We make a detailed study of this 'profinite rigidity' problem for Seifert fibre spaces and prove complete classification results for these manifolds.

From Seifert fibre spaces we continue on this trajectory and extend our classification results to all graph manifolds. We illustrate this classification with examples and several consequences, including for graph knots and for mapping class groups.

The third part of the thesis concerns the behaviour of the finite p-group quotients of 3-manifold groups. In general these quotients may be scarce and poorly behaved. We give results showing that some of these issues may be resolved by passing to finite-sheeted covers of the manifold involved. We also prove theorems concerning the p-conjugacy separability of certain graph manifold groups.

The concluding chapter of the thesis collects other results linking low-dimensional topology and finite quotients of groups. In particular we prove that finite quotients of a right-angled Artin group distinguish it from other right-angled Artin groups, and we give an argument detecting the prime decomposition of certain 3-manifold groups from the finite p-group quotients.

Statement of Originality

Chapters 2 and 3 constitute an exposition of background material, written by the author but drawing on the sources cited within. Chapters 1, 4, 5 and 6 and Section 7.2 are entirely the author's own work. Section 7.1 was written by the author in collaboration with Robert Kropholler.

To the best of my knowledge all citations are complete and accurate.

Contents

Chapter 1

Introduction

It has long been known that one can often use the finite quotients of a given group to obtain information about that group. Mal'cev observed that a finitely-presented group G such that every element is non-vanishing in some finite quotient (i.e. such that G is residually finite) has solvable word problem. The conjugacy and membership problems have similar answers given conditions ensuring a plentiful supply of finite quotients. Such conditions may be referred to by the catch-all name of 'residual finiteness properties' or sometimes simply 'residual properties'.

For convenience one often assembles all the finite quotients of a group G into a single algebraic object $\widehat{G}$ called the *profinite completion* of G and attempts to study 'profinite invariants'—that is, those properties of G which can be recovered from the profinite completion. Of course in the broadest scope there is no hope of obtaining any significant invariants for the class of all groups, since taking a direct product or extension by an infinite simple group changes G without changing the finite quotients. One therefore looks to find classes of groups within which these problems are tractable.

Those groups arising in low-dimensional topology often have good residual properties. Work of Hempel [Hem87] together with Geometrization shows that all 3-manifold groups are residually finite, and the dramatic advances resulting from the theory of special cube complexes yield many powerful results. In dimension two, even the isomorphism type of a Fuchsian group (among lattices in connected Lie groups) is a profinite invariant [BCR16].

Profinite invariants and residual properties of 3-manifold groups have received much recent attention. Lackenby [Lac14] showed that the property of containing a pair of disjoint non-separating surfaces is detected by the pro-2 completion of the fundamental group of a 3-manifold. Wilton and Zalesskii [WZ17a] showed that the geometry of a 3-manifold is a profinite invariant. Bridson and Reid [BR15] and Boileau and Friedl [BF15b] showed that the fundamental group of the figure-eight knot complement is uniquely determined among 3-manifold groups by its finite quotients. This was later extended to all once-punctured torus bundles by Bridson, Reid and Wilton [BRW17]. These examples are 'profinitely rigid' in the following sense.

Definition 2.1.9. An (orientable) 3-manifold is *profinitely rigid* if the profinite completion distinguishes its fundamental group from all other fundamental groups of (orientable) 3-manifolds.

In the other direction, Funar [Fun13] building on Stebe [Ste72] found Sol manifolds which are not profinitely rigid; Hempel [Hem14] has found Seifert-fibred examples.

We now give an overview of the thesis, and state some of our key results. In Chapter 2 we recall the basic notions of the study of profinite groups, and in particular the cohomology theory of profinite groups. None of this chapter is new, except for the results of Section 2.2.3 which are adapted from similar results in [Nak94]. Chapter 3 gives an overview of the theory of profinite groups acting on profinite trees developed mainly by Ribes and Zalesskii. The primary source for this chapter is the book [Rib17]. The results of Section 3.2 are probably known to experts but are not well-attested in the literature.

In Chapter 4 we study profinite completions of Seifert fibre space groups. We prove the following strong theorem, in which the examples of Seifert fibre spaces which are not profinitely rigid are those found by Hempel [Hem14]. We also give a new proof that these examples do indeed fail to be rigid, and prove a version of Theorem A for Seifert fibre spaces with boundary. These results first appeared in the author's paper [Wil17b].

Theorem A. *Let M be a closed orientable Seifert fibre space. Then either:*

- *M is profinitely rigid; or*

- *M has the geometry $\mathbb{H}^2 \times \mathbb{R}$, and is a surface bundle with periodic monodromy ϕ;*

and in the latter case the 3-manifolds whose fundamental groups have the same finite quotients as $\pi_1 M$ are precisely the surface bundles with monodromy ϕ^k, for k coprime to the order of ϕ.

In Chapter 5, which is adapted from the author's papers [Wil18a] and [Wil18b], we investigate 3-manifolds with a non-trivial JSJ decomposition. We pay particular attention to the case of graph manifolds. We obtain the following classification theorem for graph manifold groups with isomorphic profinite completions:

Theorem B. *Let M and N be closed orientable graph manifolds with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively. Suppose $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$. Then the graphs X and Y are isomorphic, such that corresponding vertex groups have isomorphic profinite completions. Furthermore:*

- *If X is not a bipartite graph, then M is profinitely rigid and so $\pi_1 M \cong \pi_1 N$.*
- *If X is a bipartite graph then there is an explicit finite list of numerical equations defined in terms of M and N which admit a solution if and only if $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$.*

In particular, for a given M there are only finitely many N (up to homeomorphism) such that $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$.

We further establish control over graph manifold groups with isomorphic pro-p completions when the graph manifold groups have well-behaved pro-p topologies.

We also use our analysis to distinguish graph manifolds from mixed manifolds. This implies that the profinite completion sees which geometries are involved in the geometric decomposition of an aspherical 3-manifold, which may be seen as an extension of [WZ17a]. We use the following terminology: a manifold whose JSJ decomposition is non-trivial and has no Seifert-fibred pieces at all will be called *totally hyperbolic*; when the JSJ decomposition has at least one Seifert-fibred piece and at least one hyperbolic piece the manifold is called *mixed*.

Theorem C. *Let M be a mixed or totally hyperbolic 3-manifold and let N be a graph* *manifold. Then $\pi_1 M$ and $\pi_1 N$ do not have isomorphic profinite completions.*

This theorem implies that the profinite completion detects whether or not the simpicial volume of a 3-manifold vanishes or not [BF15a, Question 3.19]. When there are no Seifert-fibred pieces at all in the JSJ decomposition, we can extract some information about the configuration of the hyperbolic pieces.

Theorem D. *Let M and N be aspherical manifolds with $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ and with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ all of whose pieces are hyperbolic. Then the graphs X and Y have equal numbers of vertices and edges and equal first Betti numbers.*

These last two theorems, as well as Theorem 5.2.2, are subsumed in later, independent results in [WZ17b].

After the classification result (Theorem B), the chapter continues with consequences and adaptations of this result. In Section 5.8 we prove the following corollary (Proposition 5.8.1).

Theorem E. *Every closed orientable graph manifold has a finite-sheeted cover with profinitely rigid fundamental group. Hence if two graph manifold groups have isomorphic profinite completions, then they are commensurable.*

Following this we will use the techniques from Theorem B to investigate two other entities of great interest to low dimensional topologists—knots and mapping classes. In Section 5.9 we will study those knot exteriors in $\mathbb{S}^3$ which are graph manifolds and prove that they are all determined by the profinite completions of their fundamental groups. Strikingly this result does not assume any condition on the behaviour with respect to the peripheral structures of the groups of the isomorphisms of profinite completions. The following appears as Theorem 5.9.1.

Theorem F. *Let M_K be the exterior of a graph knot K. Let N be another compact orientable 3-manifold and assume that $\widehat{\pi_1 M_K} \cong \widehat{\pi_1 N}$. Then $\pi_1 M_K \cong \pi_1 N$. In particular if K is prime and N is also a knot exterior then N is homeomorphic to M_K.*

Finally in Section 5.10 we use the behaviour of profinite completions of fibred graph manifolds to deduce the following result (Theorem 5.10.4) concerning mapping classes.

Theorem G. *If ϕ_1 and ϕ_2 are piecewise periodic, but not periodic, automorphisms of a closed surface group $\pi_1 S$ which are not conjugate in $\mathrm{Out}(\pi_1 S)$, then ϕ_1 is not conjugate to ϕ_2^κ in $\mathrm{Out}(\widehat{\pi_1 S})$ for any $\kappa \in \widehat{\mathbb{Z}}^\times$.*

Here a 'piecewise periodic' mapping class is one for which the corresponding mapping torus is a graph manifold or Seifert fibre space; excluding periodic maps excludes Seifert fibre spaces and restricts to *bona fide* graph manifolds.

Chapter 6 discusses what might be termed 'virtual pro-p' properties of certain groups. While it is not true, for example, that all 3-manifold groups are residually p, they do have a finite index subgroup which is residually p as was proved in [AF13]. In this spirit we investigate some properties which, while known in the profinite world, were not yet known in the virtual pro-p world and were contained in the author's paper [Wil17c]. The principal difference is that the pro-p context fails to have such sledgehammer concepts as LERF or double coset separability, and we must be more circumspect when checking that a specific subgroup or double coset is separable. The first property of study is p-efficiency. This is a property saying that the JSJ decomposition of a 3-manifold is 'well-behaved' with respect to the pro-p topology. See Definition 3.3.10 for the precise definition. Virtual p-efficiency was proved for graph manifolds in [AF13]. We exploit the fact that all other aspherical 3-manifolds are virtually fibred [Ago13, PW18] to deal with the other cases.

Theorem H. *Let M be a compact fibred 3-manifold with fibre Σ and monodromy ϕ, where Σ is a surface of negative Euler characteristic. Let p be a prime. Then M has a finite-sheeted cover with p-efficient JSJ decomposition.*

Compact 3-manifold groups are known to be conjugacy separable [WZ10, HWZ12]. We prove that graph manifolds are virtually conjugacy p-separable—that is, their fundamental groups have a finite index subgroup in which conjugacy is detectable in finite p-group quotients (see Section 2.1.4). Lack of information about the hyperbolic pieces prevents us from extending this to all 3-manifolds.

Theorem J. *Let M be a compact orientable graph manifold with toroidal boundary.*

Then $\pi_1 M$ has a finite-index subgroup which is conjugacy p-separable.

Conjugacy p-separability was previously unproven for Fuchsian groups (except surface groups [Par09]). In the course of proving Theorem J we prove conjugacy p-separability for Fuchsian groups and (most) Seifert fibre space groups.

Theorem K. *Let G be the fundamental group of a 2-orbifold or of a Seifert fibre space that is not of geometry* Nil*. Then G is conjugacy p-separable precisely when G is residually p.*

Finally, in Chapter 7 we collect some other results connecting low-dimensional topology and profinite groups. In Section 7.1 we prove a profinite rigidity result for right-angled Artin and Coxeter groups:

Theorem L. *Let X and Y be finite graphs and let p be a prime. Let $A(X)$ and $C(X)$ denote the right-angled Artin and Coxeter groups associated to X, and similarly for Y. Then $\widehat{A(X)}_{(p)} \cong \widehat{A(Y)}_{(p)}$ if and only if $X \cong Y$, and $\widehat{C(X)}_{(2)} \cong \widehat{C(Y)}_{(2)}$ if and only if $X \cong Y$.*

This theorem originally appeared in the paper [KW16], co-authored with Robert Kropholler. In Section 7.2 we prove a 'prime decomposition theorem' for the pro-p completions of certain 3-manifold groups:

Theorem M. *Let M and N be closed connected 3-manifolds with prime decompositions $M = \#_{i=1}^{m} M_i$ and $N = \#_{i=1}^{n} N_i$ respectively. Suppose that $\pi_1 M$ and $\pi_1 N$ are both residually p and have isomorphic pro-p completions. Then after possibly reordering the factors, $n = m$ and $\widehat{\pi_1(M_i)}_{(p)} \cong \widehat{\pi_1(N_i)}_{(p)}$ for all i.*

We end this introduction with a remark about conventions and notation. Through this thesis many categories of mathematical objects are used: discrete, pro-p, and profinite groups, abstract and profinite graphs, and two notions of graphs of groups. This unfortunately makes allocating a consistent set of symbols to each type of object rather awkward, and the Roman and Greek alphabets barely provide enough room.

Conventions. The following conventions and notation will be in force for the rest of this thesis.

- Abstract groups will be assumed finitely presented and will be denoted with Roman letters G, H, K and so on.
- Profinite groups will generally be given capital Greek letters Γ, Δ, Θ and Λ. Due to the shortage of *bona fide* Greek capital letters, we may sometimes co-opt A as a 'capital alpha', for example.
- Profinite graphs will be given capital Greek letters such as Ξ, Υ, or T, and abstract graphs will usually be X or Y.
- A finite graph of profinite groups will be denoted $\mathcal{G} = (X, \Gamma_\bullet)$ where X is a finite graph and $\Gamma_\bullet$ will be an edge or vertex group (with similar notation $\mathcal{G} = (X, G_\bullet)$ for graphs of discrete groups).
- The symbols $\lhd_{\rm f}$, $\lhd_{\rm o}$ and $\lhd_p$ will denote 'normal subgroup of finite index', 'open normal subgroup', 'normal subgroup of index a power of p' respectively. Similar symbols will be used for not necessarily normal subgroups.
- There is a divergence in notation between profinite group theorists, who use $\mathbb{Z}_p$ to denote the p-adic integers, and manifold theorists for whom $\mathbb{Z}_p$ is usually the cyclic group of order p. To avoid any doubt, the cyclic group of order p will be denoted $\mathbb{Z}/p$, $\mathbb{Z}/p\mathbb{Z}$ or sometimes as $\mathbb{F}_p$ when thought of as a field.
- For two elements g and h of a group, g^h will denote $h^{-1}gh$—that is, conjugation will be a right action.
- All manifolds and orbifolds will be assumed compact and connected. All 3-manifolds will be orientable.
- For us, a graph manifold will be required to be non-geometric: i.e. it is not a single Seifert fibre space or a Sol-manifold and hence not a torus bundle.

Chapter 2

Preliminaries

2.1 Inverse limits and profinite completions

2.1.1 Inverse limits

We recall some definitions and facts about inverse limits for use later.

Definition 2.1.1. Let $(I, \leq)$ be a partially ordered set such that for any $i, j \in I$ there is some $k \geq i, j$. An *inverse system of groups* is a collection of groups $(A_i)_{i\in I}$ together with maps $\phi_{ji}\colon A_j \to A_i$ whenever $j \geq i$, such that $\phi_{ii} = \mathrm{id}_{A_i}$ and $\phi_{ji} \circ \phi_{kj} = \phi_{ki}$ whenever $k \geq j \geq i$. The *inverse limit* of this system is the group

$$\varprojlim_{i\in I} A_i = \left\{ (g_i) \in \prod_{i\in I} A_i \text{ such that } \phi_{ji}(g_j) = g_i \text{ for all } j \geq i \right\}$$

In category-theoretic terms we have a functor from the poset category I (where there is an arrow from i to j precisely when $i \geq j$) to the category of groups, and we take the limit of this functor—that is, an object $\varprojlim A_i$ equipped with maps ψ_i to each A_i such that $\phi_{ij} \circ \psi_i = \psi_j$ and which is universal with this property.

We can similarly define *direct limits* to be colimits of contravariant functors from I to the category of groups.

The category to which we map is of course unimportant for this definition, and the explicit definition above suffices for most concrete categories such as Sets, Groups, Rings etc. The following results will be of use later.

Lemma 2.1.2. *Let A_i be an inverse system of non-empty finite sets, such that each transition map ϕ_{ji} is surjective. Then $\varprojlim A_i$ is non-empty.*

This is in fact a special case of the same result for compact topological spaces, and is essentially equivalent to the finite intersection property. A similar argument proves the following.

Proposition 2.1.3. *Let A_i be an inverse system of finite abelian groups. Then $\varprojlim A_i = 0$ if and only if for all $i \in I$ there is some $j \geq i$ such that ϕ_{ji} is the zero map.*

Proof. The 'if' direction is trivial. For the other direction, topologise $\prod A_i$ as the product of discrete sets, and suppose there exists $i \in I$ such that no ϕ_{ji} is trivial. It follows that there is some non-zero $x \in A_i$ which lies in the image of ϕ_{ji} for all $j \geq i$. For otherwise we may choose a j_x for each $x \in A_i \setminus \{0\}$ such that x does not lie in the image of $\phi_{j_x i}$—then for some k larger than all j_x, we have $\phi_{ki} = 0$ giving a contradiction.

Now if $x \in A_i \setminus \{0\}$ has the above property, then for any finite set $J \subseteq I$ the set

$$B_J = \left\{ (g_k) \in \prod_{k \in I} A_k \text{ such that } g_i = x, \phi_{jk}(g_j) = g_k \text{ for all } j \in J, j \geq k \right\}$$

is non-empty. For we may choose some l_J greater than every element of $J \cup \{x\}$, and take $g_{l_J} \in A_{l_J}$ mapping to x. Then taking g_j to be the image of g_{l_J} when $l_J \geq j$ and defining g_j arbitrarily elsewhere we obtain an element of B_J.

Finally note that for a finite collection of finite subsets $J_1, \ldots, J_n$ we have

$$B_{J_1} \cap \cdots \cap B_{J_n} = B_{J_1 \cup \cdots \cup J_n} \neq \emptyset$$

so that the B_J are a collection of closed subsets of the compact set $\prod A_i$ with the finite intersection property; hence their intersection is non-empty. This intersection is precisely the set of elements of $\varprojlim A_i$ with A_i-coordinate equal to x, so $\varprojlim A_i \neq 0$. $\square$

There is a dual result for direct limits, that a direct limit of finite abelian groups is zero if and only if some map out of each group is trivial.

Since the limit is intuitively determined by the 'long term behaviour' of the system, we expect some process analogous to taking a subsequence. The correct notion is as follows.

Definition 2.1.4. Given a poset I such that for any $i, j \in I$ there is some k larger than both, a subset J of I is *cofinal* if for every $i \in I$ there is $j \in J$ such that $j \geq i$.

Proposition 2.1.5. *Let A_i be an inverse system of groups indexed over I, and let J be a cofinal subset of I. Then*

$$\varprojlim_{i \in I} A_i = \varprojlim_{j \in J} A_j$$

Definition 2.1.6. A *profinite group* is any inverse limit of finite groups. If all these finite groups are drawn from some class $\mathcal{C}$ of finite groups, the inverse limit is called a pro-$\mathcal{C}$ group. In particular, a pro-p group for p prime (respectively pro-π group, for π a set of primes) will be an inverse limit of finite p-groups (respectively finite groups whose orders are divisible only by primes in π).

Giving each finite group the discrete topology, a profinite group Γ is a compact Hausdorff totally disconnected topological group. A profinite group Γ is *(topologically) finitely generated* if some finite subset of Γ generates a dense subgroup.

Profinite groups form a category in which the natural maps are continuous homomorphisms. Therefore two profinite groups being 'isomorphic' will always mean 'isomorphic as *topological* groups'. By a deep theorem of Nikolov and Segal [NS07] all group homomorphisms between finitely generated profinite groups are in fact continuous, so it is not generally necessary to worry about this distinction. However all maps defined in the thesis are naturally continuous anyway, so this work does not rely on [NS07].

2.1.2 Profinite completions

The central motivating question of this subject was 'how much information is contained in the finite quotients of a group?'. In this section we discuss how to package that information into a profinite group. Proofs for most of the statements can be found in [RZ00b] or [DdSMS03].

Definition 2.1.7. Let $\mathcal{C}$ be a variety of finite groups—that is, a family of finite groups closed under taking finite direct products, subgroups, and quotients. For a (discrete) group G, the *pro-$\mathcal{C}$ completion* of G is the inverse limit of the system of groups

$$\{G/N \mid N \triangleleft_{\mathrm{f}} G,\, G/N \in \mathcal{C}\}$$

This completion is denoted $\widehat{G}_{\mathcal{C}}$.

If $\mathcal{C}$ is the collection of all finite groups, $\widehat{G}_{\mathcal{C}}$ is called the *profinite completion* and is written simply as $\widehat{G}$. If $\mathcal{C}$ is the collection of finite p-groups (respectively π-group, for π a set of primes), we write $\widehat{G}_{(p)}$ and call it the *pro-p completion* (respectively the pro-π completion $\widehat{G}_{\pi}$).

If $G = \pi_1 M$ is the fundamental group of a manifold, we may refer to $\Gamma = \widehat{G}$ as the 'profinite fundamental group' as a contraction of the phrase 'profinite completion of the fundamental group'. We may contrast this with the term 'discrete fundamental group' when referring to G itself.

Note that, by the categorical definition of inverse limits, there is a unique natural map $G \to \widehat{G}_{\mathcal{C}}$ induced by the quotient maps $G \to G/N$.

Definition 2.1.8. A property $\mathcal{P}$ of some class $\mathcal{F}$ of finitely generated groups is said to be a *profinite invariant* if, given finitely generated groups G_1 and G_2 in $\mathcal{F}$ with isomorphic profinite completions, G_1 has $\mathcal{P}$ if and only if G_2 does.

In this thesis $\mathcal{F}$ will generally be the class of closed 3-manifold groups and we will use the term 'profinite invariant' without qualification.

Definition 2.1.9. An (orientable) 3-manifold is *profinitely rigid* if the profinite completion of its fundamental group distinguishes its fundamental group from all other fundamental groups of (orientable) 3-manifolds.

Definition 2.1.10. A group G is *residually finite* if for all $g \in G$ there is a homomorphism $G \to F$ with F finite and the image of g non-trivial.

Some authors make use of the following notion.

Definition 2.1.11. For a group G in some family of groups $\mathcal{F}$, the *genus* of G in $\mathcal{F}$ is the set of all (isomorphism types of) groups in $\mathcal{F}$ whose profinite completion is isomorphic to $\widehat{G}$.

The present author's personal opinion is that the word 'genus' is already rather overcommitted, and we shall not make use of this terminology. This notion is sometimes referred to as the 'Pickel genus' [CL95, Pic71].

Examples of residually finite groups are surface groups (see for example [Hem72]) and all 3-manifold groups (proved by Hempel [Hem87] in the Haken case, which can be extended to all cases using geometrization).

Proposition 2.1.12. *Let G be a finitely generated discrete group and let $\iota\colon G \to \widehat{G}$ be the natural map. Then:*

- *the image of ι is dense;*
- *ι is injective if and only if G is residually finite; and*
- *ι is an isomorphism if and only if G is finite.*

The following proposition gives explicitly the strong links connecting the subgroup structures of groups with isomorphic profinite completions.

Proposition 2.1.13. *Let G_1 and G_2 be finitely generated residually finite groups, and suppose $\phi\colon \widehat{G}_1 \to \widehat{G}_2$ is an isomorphism of their profinite completions. Then there is an induced bijection ψ between the set of finite index subgroups of G_1 and the set of finite index subgroups of G_2, such that if $K \leq_{\mathrm{f}} H \leq_{\mathrm{f}} G_1$, then:*

- *$[H : K] = [\psi(H) : \psi(K)]$;*
- *$K \lhd H$ if and only if $\psi(K) \lhd \psi(H)$;*
- *if $K \lhd H$, then $H/K \cong \psi(H)/\psi(K)$; and*
- *$\widehat{H} \cong \widehat{\psi(H)}$.*

This follows immediately from the following proposition, which relates the subgroup structure of a group to that of its profinite completion:

Proposition 2.1.14. *Let G be a finitely generated residually finite group, and $\widehat{G}$ its profinite completion. Identify G with its image under the canonical inclusion $G \hookrightarrow \widehat{G}$. Let ψ be the mapping sending a finite index subgroup $H \leq_{\mathrm{f}} G$ to its closure $\overline{H}$. If $K \leq_{\mathrm{f}} H \leq_{\mathrm{f}} G$ then:*

1. *$\psi\colon \{H \leq_{\mathrm{f}} G\} \to \{U \leq_{\mathrm{o}} \widehat{G}\}$ is a bijection;*

2. $[H : K] = [\overline{H} : \overline{K}]$*;*

3. $K \lhd H$ *if and only if* $\overline{K} \lhd \overline{H}$*;*

4. if $K \lhd H$*, then* $H/K \cong \overline{H}/\overline{K}$*; and*

5. $\widehat{H} \cong \overline{H}$*.*

In fact, every finite-index subgroup of $\widehat{G}$ is open when $\widehat{G}$ is topologically finitely generated, by the Nikolov-Segal theorem [NS07].

Questions concerning profinite completions are often naïvely stated in terms of the 'set of isomorphism classes of finite quotients' $\mathcal{C}(G)$. These formulations are in fact equivalent.

Theorem (Dixon, Formanek, Poland, Ribes [DFPR82])**.** *Let* G_1 *and* G_2 *be finitely generated groups. If* $\mathcal{C}(G_1) = \mathcal{C}(G_2)$*, then* $\widehat{G}_1 \cong \widehat{G}_2$*.*

Corollary 2.1.15. *If* G_1 *and* G_2 *are finitely generated groups and* $\widehat{G}_1 \cong \widehat{G}_2$*, then* $H_1(G_1;\mathbb{Z}) \cong H_1(G_2;\mathbb{Z})$*.*

The profinite completion of the integers plays, as one might expect, a central role in much of the theory. It is the inverse limit

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n$$

of all finite cyclic groups. For π a set of primes the pro-π completion, denoted $\mathbb{Z}_\pi$, is the inverse limit of those finite cyclic groups whose orders only involve primes from π. Because of the Chinese Remainder Theorem, these finite cyclic groups split as products of cyclic groups of prime power order, and these splittings are natural with respect to the quotient maps $\mathbb{Z}/mn \to \mathbb{Z}/n$. It follows that the profinite completion of $\mathbb{Z}$ splits as the direct product, over all primes p, of the inverse limits

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i$$

Similarly

$$\mathbb{Z}_\pi = \prod_{p\in\pi} \mathbb{Z}_p$$

This splitting as a direct product, which of course is not a feature of $\mathbb{Z}$ itself, means that ring-theoretically $\widehat{\mathbb{Z}}$ behaves less well than $\mathbb{Z}$. In particular, there exist zero-divisors, which are precisely those elements with vanish under some projection to $\mathbb{Z}_p$. The rings $\mathbb{Z}_p$ themselves do not have any zero-divisors, and $\mathbb{Z}$ injects into each $\mathbb{Z}_p$ (for every natural number n and prime p, some power of p does not divide n), so no element of $\mathbb{Z}$ is a zero-divisor in $\widehat{\mathbb{Z}}$. Furthermore, the group of units of $\widehat{\mathbb{Z}}$ is rather large, being the inverse limit of the multiplicative groups of the rings $\mathbb{Z}/n$. In particular $\mathrm{Aut}(\widehat{\mathbb{Z}})$ is an infinite profinite group much larger than $\mathrm{Aut}(\mathbb{Z})$.

Elements of profinite groups may be raised to powers with exponents in $\widehat{\mathbb{Z}}$. For if Γ is a profinite group and $x \in \Gamma$, the map $1 \mapsto x$ extends to a map $\mathbb{Z} \to \Gamma$ by the universal property of $\mathbb{Z}$, which by continuity extends to a map $\widehat{\mathbb{Z}} \to \Gamma$ (that is, $\widehat{\mathbb{Z}}$ is a free profinite group on the element 1). The image of λ under this map is then denoted x^λ. This operation has all the expected properties. See Section 4.1 of [RZ00b] for more details.

As we are discussing $\widehat{\mathbb{Z}}$, this seems a fitting place to include the following easy but useful lemma.

Lemma 2.1.16. *If $\lambda \in \widehat{\mathbb{Z}}$, $n \in \mathbb{Z} \smallsetminus \{0\}$, and $\lambda n \in \mathbb{Z}$, then $\lambda \in \mathbb{Z}$. In particular if $\lambda \in \widehat{\mathbb{Z}}^\times$ then $\lambda = \pm 1$.*

Proof. Since λn is an integer congruent to 0 modulo n, there exists $l \in \mathbb{Z}$ such that $\lambda n = ln$. Since n is not a zero-divisor in $\widehat{\mathbb{Z}}$, it follows that $\lambda = l \in \mathbb{Z}$. $\square$

2.1.3 The profinite topology

Whenever profinite properties of groups are discussed, it is usually necessary to have some control over subgroup separability. Here we recall prior results that will be used heavily in the sequel.

Definition 2.1.17. The *(full) profinite topology* on a discrete group G is the topology whose neighbourhood basis at the identity consists of the finite-index normal subgroups of G. A subset of G is *separable* if and only if it is closed in the profinite topology.

If H is a subgroup of G we say that G *induces the full profinite topology on H* if the profinite topology on H agrees with the subspace topology—or more explicitly, if for every finite index normal subgroup U of H there is a finite index normal subgroup V of G such that $H \cap V \subseteq U$.

Every finite-index subgroup of a finitely generated group G contains a finite-index normal subgroup, so a different neighbourhood basis for the identity in this topology consists of the set of all finite-index subgroups of G. Note also that the profinite topology is precisely the topology induced on G by the map $G \to \widehat{G}$.

If G induces the full profinite topology on a subgroup H then the natural map $\widehat{H} \to \overline{H}$ to the closure of H in G is in fact an isomorphism.

Many concepts may be expressed in terms of the profinite topology. For instance, a group is residually finite if and only if the subset $\{1\}$ is separable. Separability of subgroups of G is of particular importance. There are various equivalent re-phrasings of the definition of separability; for instance a set $S \subseteq G$ is separable if and only if for every $g \in G \smallsetminus S$ there exists a finite quotient $\phi\colon G \to Q$ such that $\phi(g) \notin \phi(S)$.

Seifert fibred and hyperbolic 3-manifold groups have very good separability properties:

Definition 2.1.18. A group G is *LERF (locally extended residually finite)* if every finitely-generated subgroup H of G is separable in G.

Many 3-manifold groups are LERF.

Theorem ([Sco78])**.** *The fundamental group of a compact Seifert fibre space is LERF.*

Theorem (Agol, Wise, Kahn, Markovic, and others)**.** *The fundamental group of a compact hyperbolic 3-manifold is LERF.*

Unfortunately graph manifold groups are not in general LERF; in fact a generic graph manifold group is not LERF. See [BKS87, NW01]. However, those subgroups of primary concern are well behaved in the profinite topology. In particular:

Theorem (Hamilton [Ham01])**.** *Let M be a Haken 3-manifold. Then the abelian subgroups of $\pi_1 M$ are separable in $\pi_1 M$ (and thus $\pi_1 M$ induces the full profinite topology on them).*

Theorem 2.1.19 (Wilton and Zalesskii [WZ10]). *Let M be a compact, orientable, irreducible 3-manifold with boundary a (possibly empty) collection of incompressible tori, and let $(X, M_\bullet)$ be the graph of spaces corresponding to the JSJ decomposition of M. Then the vertex and edge groups $\pi_1 M_\bullet$ are closed in the profinite topology on $\pi_1 M$, and $\pi_1 M$ induces the full profinite topology on them.*

Remark. This theorem is stated as [WZ10, Theorem A] for closed manifolds, but the proof also works for toroidal boundary with no alteration.

2.1.4 The pro-p topology

Let p be a prime. One has a topology similar to the profinite topology derived from the pro-p completion. The *pro-p topology* on G is the topology whose neighbourhood basis at the identity consists of normal subgroups N of G with $[G : N]$ a power of p. Since an intersection of normal subgroups of index a power of p again has index a power of p, each normal subgroup of index a power of p contains a characteristic subgroup of index a power of p (that is, a subgroup invariant under all automorphisms of G). Thus the characteristic subgroups with index a power of p also form a neighbourhood basis at the identity. We will freely move between these two definitions of the pro-p topology. In contrast to the case of the full profinite topology, one cannot omit the word 'normal' from the definition above (a subgroup with index a power of p need not contain a normal subgroup with index a power of p).

A subset S of G is *p-separable* in G if S is closed in the pro-p topology; equivalently, if for every $g \in G \smallsetminus S$ there is $N \triangleleft_p G$ such that under the quotient map $\phi\colon G \to G/N$, the image of S does not contain the image of g.

For a subset S of G, suppose that for every $g \in G$ not conjugate to any element of S, there exists a finite p-group P and a surjection $\phi\colon G \to P$ such that $\phi(g)$ is not conjugate to any element of $\phi(S)$; equivalently suppose that the union of the conjugacy classes of elements in S is p-separable. Then we say S is *conjugacy p-distinguished* in G. If $g \in G$, we say g is conjugacy p-distinguished in G if $\{g\}$ is conjugacy p-distinguished. If all elements of G are conjugacy p-distinguished, then G is called *conjugacy p-separable.*

For H a subgroup of G, we say that G *induces the full pro-p topology on H*, or that H *is topologically p-embedded in G*, if the induced topology on H agrees with its pro-p topology. That is, we require that for any $N \triangleleft_p H$ there is $N' \triangleleft_p G$ such that $N' \cap H \leq N$. Note that if H is a normal subgroup of G with index a power of p, then G induces the full pro-p topology on H, because any characteristic normal subgroup of H is a normal subgroup of G.

The above conjugacy p-separability notions of course have analogues for the profinite topology.

We remark that the pro-p version of the LERF property is not a useful concept—it is almost never the case that every finitely-generated subgroup of a given group is p-separable.

2.2 Cohomology of profinite groups

Profinite groups have a homology and cohomology theory sharing many features with that for discrete groups. See [RZ00b] or [Ser13] for a full treatment. We provide here only what results we need. One definition of cohomology we shall use is the following. Take a profinite group Γ and a discrete abelian group A on which Γ acts continuously (i.e. a Γ-module). Then define cochain groups C^n and coboundary maps $d\colon C^n \to C^{n+1}$ by:

$$C^n(\Gamma, A) = \{\text{continuous functions } f\colon \Gamma^n \to A\}$$

$$\begin{aligned}(df)(g_1, \ldots, g_{n+1}) = {} & g_1 \cdot f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \ldots, g_n)\end{aligned}$$

This chain complex gives well-defined cohomology groups $H^n(\Gamma; A)$. The usual functoriality properties still hold, and a notion of cup product is still defined. There are other equivalent definitions. In particular, one can use projective resolutions of $\widehat{\mathbb{Z}}$ and apply functors to compute the (co)homology. The above chain complex may be obtained in this way, and this viewpoint will be exploited later.

2.2.1 Goodness and cohomological dimension

Serre [Ser13] made the following definition.

Definition 2.2.1. A finitely generated group G is *good* if for all finite G-modules A, the natural homomorphism

$$H^n(\widehat{G}; A) \to H^n(G; A)$$

induced by $G \to \widehat{G}$ is an isomorphism for all n.

One may similarly define '*p-good*' (sometimes called '*cohomologically p-complete*') by replacing the profinite completion by the pro-p completion and only considering finite p-primary modules over the pro-p completion.

The definition is stated in terms of cohomology, however under a finiteness condition the dual notion in homology holds too. Suppose G is of type FP_∞ and take a resolution of $\mathbb{Z}$ by finitely generated projective (left) $\mathbb{Z}[G]$-modules $P_\bullet$. We denote by $P_n^\perp$ the canonical right G-module associated to P_n; this has the same underlying abelian group as P_n, with the right G-action given by

$$p \cdot g = g^{-1} p$$

for $p \in P_n, g \in G$.

Then for any finite G-module M, we have

$$\begin{aligned} H_n(G, M)^* = H_n(P_\bullet^\perp \otimes_G M)^* = H^n((P_\bullet^\perp \otimes_G M)^*) \\ = H^n(\mathrm{Hom}_G(P_\bullet, M^*)) = H^n(G, M^*) \end{aligned}$$

noting that $(-)^* = \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact when applied to the sequences of finite modules $P_\bullet \otimes_G M$. So we have Pontrjagin duality for the (co)homology of the discrete group. Since it holds for the profinite group as well by [RZ00b, Proposition 6.3.6] we have the following proposition.

Proposition 2.2.2. *If G is a good group of type FP_∞ then for every finite G-module M the map $\iota\colon G \to \widehat{G}$ induces isomorphisms*

$$H_\bullet(G, M) \cong H_\bullet(\widehat{G}, M)$$

Many groups connected with low-dimensional topology are known to be good.

Theorem (Grunewald, Jaikin-Zapirain, Zalesskii [GJZZ08])**.** *All finitely generated Fuchsian groups are good.*

Theorem (Grunewald, Jaikin-Zapirain, Zalesskii [GJZZ08])**.** *Fully residually free groups are good.*

Under certain finiteness assumptions which hold in our cases of interest, an extension of a good group by a good group is itself good (see [Ser13, Exercise 2.6.2]). Furthermore, finite index subgroups of good groups are good.

Corollary 2.2.3. *The fundamental groups of Seifert fibre spaces are good.*

The next theorem is of somewhat disputed attribution. It was proved by Wilton and Zalesskii [WZ10] that a 3-manifold has good fundamental group if all pieces of its JSJ decomposition do. Seifert-fibred pieces are covered by the above corollary. That hyperbolic 3-manifold groups are good follows from the Virtually Compact Special Theorem and its various consequences. There are numerous ways one may deduce this; one may use the Virtual Fibring Theorem of Agol [Ago13] or another route as outlined in [Cav12]. In stating the theorem as due to Agol, Wilton and Zalesskii, and Wise we aim to recognise those who contributed most. A full account and list of references may be found in Section 5.2 of [AFW15].

Theorem (Agol, Wilton-Zalesskii, Wise)**.** *Fundamental groups of compact 3-manifolds are good.*

Note that in the case where the action of G on A is trivial, and A is finite, we have the identifications

$$H^1(\widehat{G}; A) \cong \mathrm{Hom}(\widehat{G}, A) \cong \mathrm{Hom}(G, A) \cong H^1(G; A)$$

so goodness is only important when working with higher cohomology groups. Moreover, the second cohomology group is always well-behaved to a more limited extent. The following proposition is stated in the category of pro-p groups where it will be used later (in Section 7.1), but is easily seen to hold in the category of profinite groups or with more general coefficient groups than $\mathbb{Z}/p$.

Proposition 2.2.4. *Let G be a discrete group and let p be a prime.*

- $H^1(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^1(G;\mathbb{Z}/p)$ *is an isomorphism;*
- $H^2(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^2(G;\mathbb{Z}/p)$ *is injective; and*
- *if H^{1+1} denotes that part of second cohomology generated by cup products of elements of H^1, then $H^{1+1}(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^{1+1}(G;\mathbb{Z}/p)$ is an isomorphism.*

Here all the maps are the natural ones induced by $G \to \widehat{G}_{(p)}$.

Proof. The first point is a trivial consequence of the fact that $H^1(-;\mathbb{Z}/p)$ is naturally isomorphic to $\mathrm{Hom}(-;\mathbb{Z}/p)$ in either the category of discrete groups or the category of pro-p groups. The third point follows from the first two and naturality of the cup product. The second is a special case of Exercise 2.6.1 of [Ser13]. We give here an explicit proof in dimension two in terms of extensions.

Recall that $H^2(-;\mathbb{Z}/p)$ classifies central extensions of G by $\mathbb{Z}/p$ both for discrete and pro-p groups (see Section 6.8 of [RZ00b] for the profinite theory). Take a central extension Γ of $\widehat{G}_{(p)}$ by $\mathbb{Z}/p$ representing $\xi \in H^2(\widehat{G}_{(p)};\mathbb{Z}/p)$. Then the pull-back

$$P = \{(g,h) \in G \times \Gamma \text{ such that } \pi(h) = \iota(g)\}$$

(where $\pi\colon \Gamma \to \widehat{G}_{(p)}$ and $\iota\colon G \to \widehat{G}_{(p)}$ are the obvious maps) gives a central extension of G by $\mathbb{Z}/p$ representing $\iota^*(\xi)$. If $\iota^*(\xi) = 0$ in H^2 then the extension splits; so there is a group-theoretic section $s\colon G \to P$. Now s induces a map $\hat{s}\colon \widehat{G}_{(p)} \to \widehat{P}_{(p)}$. Furthermore Γ is a pro-p group so that the projection $\mathrm{pr}_2\colon P \to \Gamma$ induces a map $\widehat{\mathrm{pr}_2}\colon \widehat{P}_{(p)} \to \Gamma$; then $\widehat{\mathrm{pr}_2}\hat{s}$ is a section of $\Gamma \to \widehat{G}_{(p)}$ and so ξ represented a trivial extension also.

$$\begin{tikzcd}
\mathbb{Z}/p \arrow[rr, hook] \arrow[dd, equal] \arrow[dr, equal] & & P \arrow[rr, two heads] \arrow[dd, "\mathrm{pr}_2"'] \arrow[dr] & & G \arrow[dd, "\iota"] \arrow[dr] \arrow[ll, dotted, bend left, "s"] & \\
& \mathbb{Z}/p \arrow[rr] \arrow[dl, equal] & & \widehat{P}_{(p)} \arrow[rr, two heads] \arrow[dl, "\widehat{\mathrm{pr}_2}"] & & \widehat{G}_{(p)} \arrow[dl, equal] \arrow[ll, dotted, bend left, "\hat{s}"] \\
\mathbb{Z}/p \arrow[rr, hook] & & \Gamma \arrow[rr, two heads, "\pi"] & & \widehat{G}_{(p)} &
\end{tikzcd}$$

∎

Remark. Note that a diagram chase applied to the lower parallelogram in the above diagram shows that in fact $\Gamma \cong \widehat{P}_{(p)}$. Thus the above analysis also illustrates why the map on H^2 may fail to be surjective—for any central extension of $\widehat{G}_{(p)}$ yielding a given extension P of G must be $\widehat{P}_{(p)}$. However there is no *a priori* reason that the map from $\mathbb{Z}/p$ to $\widehat{P}_{(p)}$ need be injective. See the example at the end of [Lor10] for a case where the map on H^2 fails to be surjective.

The notion of cohomological dimension of a group also extends well to profinite groups.

Definition 2.2.5. If H is an abelian group, its *p-primary component* $H(p)$ for a prime p is the subgroup of H consisting of all elements whose order is a power of p.

Definition 2.2.6. Let Γ be a profinite group, and p a prime. The *p-cohomological dimension* $\mathrm{cd}_p(\Gamma)$ of Γ is the smallest integer n such that, for all *finite* Γ-modules A,

$$H^i(\Gamma; A)(p) = 0 \quad \text{for all } i > n$$

The *cohomological dimension* $\mathrm{cd}(\Gamma)$ is the supremum of $\mathrm{cd}_p(\Gamma)$ over all primes p.

Because the cohomological dimension of a profinite group only relies on finite modules in its definition, we also have the following result.

Proposition 2.2.7. *If G is a good group then* $\mathrm{cd}(G) \geq \mathrm{cd}(\widehat{G})$

Proposition 2.2.8. *If M is a closed aspherical 3-manifold then* $\mathrm{cd}(\widehat{\pi_1 M}) = 3$.

Proof. Let $\Gamma = \widehat{\pi_1 M}$. We already have $\mathrm{cd}(\Gamma) \leq 3$ by the previous proposition. But $\pi_1 M$ is good and $H^3(M; \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$, so $H^3(\Gamma; \mathbb{Z}/2)(2) \neq 0$, noting that M is aspherical so that the cohomology of M and its fundamental group are the same. Hence $\mathrm{cd}(\Gamma) \geq \mathrm{cd}_2(\Gamma) \geq 3$ also. □

The same result holds in dimensions one and two.

Proposition 2.2.9. *If F is a free group, and Σ is a closed surface with $\chi(\Sigma) < 0$, then* $\mathrm{cd}(\widehat{F}) = 1$ *and* $\mathrm{cd}(\widehat{\pi_1 \Sigma}) = 2$.

Corollary 2.2.10. *Let M be a closed 3-manifold and let N be a compact 3-manifold with non-empty boundary. Then $\pi_1 M$ and $\pi_1 N$ do not have isomorphic profinite completions.*

Proof. We have $\mathrm{cd}(\widehat{\pi_1 M}) = 3$ by Proposition 2.2.8, while $\mathrm{cd}(\widehat{\pi_1 N}) \leq 2$ by Proposition 2.2.7. $\square$

We will also need the following result from [Ser13] and its corollary.

Proposition 2.2.11 (Proposition 14 of [Ser13])**.** *Let p be prime let Γ be a profinite group, and let Δ be a closed subgroup of Γ. Then $\mathrm{cd}_p(\Delta) \leq \mathrm{cd}_p(\Gamma)$.*

Corollary 2.2.12 (see also Corollary VIII.2.5 of [Bro12])**.** *Let G be a residually finite good group of finite cohomological dimension (over $\mathbb{Z}$). Then $\widehat{G}$ is torsion-free.*

2.2.2 Spectral sequence

Another useful property of the cohomology of profinite groups is that the Serre spectral sequence of group cohomology holds even in the profinite world. Given an exact sequence of profinite groups

$$1 \to \Delta \to \Gamma \to \Gamma/\Delta \to 1$$

and a continuous Γ-module A, then the natural continuous action of Γ on Δ by conjugation descends to an action of Γ/Δ on $H^q(\Delta; A)$. The cohomology of Γ is then given by a spectral sequence

$$E_2^{p,q} = H^p(\Gamma/\Delta; H^q(\Delta; A)) \Rightarrow H^n(\Gamma; A)$$

Any spectral sequence induces an exact sequence in the low-dimensional homology (or cohomology) groups. Here it is the *five term exact sequence*

$$\begin{aligned} 0 \to H^1(\Gamma/\Delta; H^0(\Delta; A)) \to H^1(\Gamma; A) \to H^0(\Gamma/\Delta; H^1(\Delta; A)) \\ \to H^2(\Gamma/\Delta; H^0(\Delta; A)) \to H^2(\Gamma; A) \end{aligned}$$

When Γ is a free profinite group and A a trivial module, in particular when a presentation

$$1 \to R \to F \to G \to 1$$

of an abstract group yields a short exact sequence of profinite groups

$$1 \to \overline{R} \to \widehat{F} \to \widehat{G} \to 1$$

we get the exact sequence

$$0 \to H^1(\widehat{G}; A) \to H^1(\widehat{F}; A) \to H^1\left(\frac{\overline{R}}{[\overline{R}, \widehat{F}]}; A\right) \to H^2(\widehat{G}; A) \to 0$$

2.2.3 Chain complexes

It will be necessary later to work with certain exact sequences of modules over a 'group ring' of a profinite group. In this section we will recall and prove some of the necessary tools.

Definition 2.2.13. Given a profinite ring A (usually $\widehat{\mathbb{Z}}$, $\mathbb{Z}_p$ or a finite ring) and a profinite group Γ, the *completed group ring* $A[\![\Gamma]\!]$ is defined as the inverse limit

$$\varprojlim_{A', N} A/A'[\Gamma/N]$$

of group rings indexed over the finite index open normal subgroups A' and N of A and Γ respectively. It is a compact Hausdorff totally disconnected topological ring.

A finite abelian group M with a continuous Γ-action now becomes a (left- or right-) $\widehat{\mathbb{Z}}[\![\Gamma]\!]$-module in the usual way. If $A = \widehat{\mathbb{Z}}$ and $\Gamma = \widehat{G}$ for G residually finite then $\widehat{\mathbb{Z}}[\![\widehat{G}]\!]$ naturally contains a copy of $\mathbb{Z}[G]$ as a dense subring.

These modules over $A[\![\Gamma]\!]$, together with continuous module maps, constitute an abelian category with the same formal properties as the category of R-modules for a ring R. Thus the machinery of homological algebra works and we can define profinite group cohomology with coefficients in a finite $\widehat{\mathbb{Z}}[\![\Gamma]\!]$-module M by starting from an arbitrary resolution of $\widehat{\mathbb{Z}}$ by projective (left) $\widehat{\mathbb{Z}}[\![\Gamma]\!]$-modules and applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}[\![\Gamma]\!]}(-, M)$ giving the continuous homomorphisms from a module to M. If M is a module with trivial Γ-action, we can factor this through the functor $\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\Gamma]\!]} -$ which 'forgets the Γ-action' on the chain complex.

We will need to show that, under certain conditions, a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules yields a free resolution of $\widehat{\mathbb{Z}}$ by $\widehat{\mathbb{Z}}[\![\widehat{G}]\!]$-modules. To this end we use the following propositions, which are adapted from results in [Nak94].

Definition 2.2.14. A discrete group G is of type $\mathrm{FP}(n)$ if there is a resolution of the trivial module $\mathbb{Z}$ by projective $\mathbb{Z}[G]$-modules $P_\bullet$ such that P_i is finitely generated for $0 \leq i \leq n$.

Proposition 2.2.15. *Let G be a discrete group which is good. Then:*

- $\varinjlim_{K \leq_{\mathrm{f}} G} H^q(K; M) = 0$ *for every finite G-module M and all $q \geq 1$*
- *If G is of type FP(n), then* $\varprojlim_{K \leq_{\mathrm{f}} G} H_q(K; M) = 0$ *for every finite G-module M and all $1 \leq q \leq n$.*

Proof. First note we may restrict to the case of trivial modules, as any finite G-module M becomes trivial over K for a cofinal subset of $\{K \leq_{\mathrm{f}} G\}$. Thus we may view M interchangeably as a left or right module. The maps $\mathrm{res}^K_{K'} \colon H^q(K; M) \to H^q(K'; M)$ are given by restriction of cochains. By the dual of Lemma 2.1.3 the direct limit in question (categorically a colimit) is zero if all elements of $H^q(K; M)$ are 'eventually zero'—that is, for all $x \in H^q(K; M)$ there is some $K' \leq K$ such that x is mapped to zero under the restriction map $H^q(K; M) \to H^q(K'; M)$.

So let $x \in H^q(K; M)$ for some $q \geq 1$, some $K \leq_{\mathrm{f}} G$ and some trivial finite G-module M. By goodness of K, there is a natural identification $H^q(K; M) \cong H^q(\widehat{K}; M)$ so we may represent x as a continuous cochain $\xi \colon \widehat{K}^q \to M$. The preimage of 0 under ξ is some open subset of $\widehat{K}^q$. Products of open subgroups of $\widehat{K}$ form a neighbourhood basis in $\widehat{K}^q$, so we may choose $\Delta \leq_o \widehat{K}$ such that $\xi|_{\Delta^q} = 0$. Then setting $K' = K \cap \Delta$ (so that $\Delta = \widehat{K'}$) the commuting diagram

$$\begin{tikzcd} H^q(K; M) \arrow[r, "\mathrm{res}^K_{K'}"] & H^q(K'; M) \\ H^q(\widehat{K}; M) \arrow[u, "\cong"] \arrow[r, "\mathrm{res}^{\widehat{K}}_{\widehat{K}'}"] & H^q(\widehat{K}'; M) \arrow[u, "\cong"] \end{tikzcd}$$

shows that $\mathrm{res}^K_{K'}(x) = 0$. Hence $\varinjlim_{K \leq_{\mathrm{f}} G} H^q(K; M) = 0$.

For the second conclusion, assume G is of type $\mathrm{FP}(n)$. Then $H_q(K; M)$ is finite for all $0 \leq q \leq n$, $K \leq_{\mathrm{f}} G$ and all finite G-modules M, and similarly for the cohomology. Recall that by Proposition 2.1.3 the condition that an inverse limit of finite abelian groups A_i is trivial is equivalent to the existence, for each i in the indexing set, of

$j \geq i$ such that $A_j \to A_i$ is the zero map (and similarly for a direct limit of finite groups).

So let K be a finite index subgroup of G, and take K' such that the restriction map $\mathrm{res}^K_{K'}$ is zero on each $H^q(-, M)$ for some finite trivial G-module M and $1 \leq q \leq n$. We show that we can dualise this to find that the corestriction map is also zero. Note that a finite-index subgroup of a group of type $\mathrm{FP}(n)$ is also of type $\mathrm{FP}(n)$. Let $P_\bullet$ be a projective resolution of $\mathbb{Z}$ by left $\mathbb{Z}K$-modules, which is finitely generated in dimensions at most n. There is a natural isomorphism (see [CE99], Proposition II.5.2)

$$\mathrm{Hom}_{\mathbb{Z}K}(P_\bullet, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes P_\bullet, \mathbb{Q}/\mathbb{Z})$$

Now take homology. Because $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor and commutes with homology. Hence we get a natural isomorphism

$$H^q(K; M^*) \cong (H_q(K; M))^*$$

where N^* denotes the dual $\mathrm{Hom}(N, \mathbb{Q}/\mathbb{Z})$ of an abelian group. Finite abelian groups are isomorphic to their dual and canonically isomorphic to their double-dual. So we get a natural isomorphism

$$H^q(K; M^*)^* \cong H_q(K; M)$$

since in dimensions $1 \leq q \leq n$ the right hand side is finite. The inclusion $K' \to K$ induces the zero map on the left hand side by assumption, noting that M is isomorphic to M^* so the restriction map with M^* coefficients also vanishes. Hence the map on the right hand side, the corestriction map, is zero. $\square$

To prove the next proposition, we will need some exactness properties of the functor $\varprojlim$. In general this functor will not be exact and so will not commute with homology. A well-known condition for exactness is the *Mittag-Leffler condition*—roughly, this is an 'eventual stability' condition. See [Wei95, Section 3.5] for a full treatment. Here we merely state the definition and consequence.

Definition 2.2.16. An inverse system $(A_i)_{i\in I}$ where $(I, \leq)$ is a *totally* ordered set (not merely partially ordered) satisfies the *Mittag-Leffler condition* if for all i there exists $j \geq i$ such that

$$\mathrm{im}(A_k \to A_i) = \mathrm{im}(A_j \to A_i)$$

for all $k \geq j$. That is, the images of the transition maps into A_i are eventually stable.

Take an inverse system of chain complexes $C_{\bullet,i}$. If for each N the systems $\{C_{n,i}\}_{i\in I}$ satisfies the Mittag-Leffler condition then we will have

$$\varprojlim_i H_n(C_{\bullet,i}) = H_n(\varprojlim_i C_{\bullet,i})$$

for all n. In our case, all the groups $C_{n,i}$ will be finite, so that the Mittag-Leffler condition holds (a decreasing sequence of subsets of a finite set is eventually constant). Our indexing set $I = \{(m, K) \,|\, m \in \mathbb{N}, K \leq_{\mathrm{f}} G\}$ need not be totally ordered; however if G is finitely generated we may pass to the cofinal subset $J = \{(m!, K_n)\}$, where K_n is the intersection of the (finitely many) subgroups of G of index at most n, to obtain a totally ordered indexing set without affecting the limits.

Proposition 2.2.17. *Let G be a finitely generated good group and let $(C_i)_{0\leq i\leq n} \to \mathbb{Z}$ be a partial resolution of $\mathbb{Z}$ by free finitely generated $\mathbb{Z}[G]$-modules $C_i = \mathbb{Z}[G]^{\oplus r_i}$. Then $(\widehat{C}_i)_{0\leq i\leq n} \to \widehat{\mathbb{Z}}$ is a partial resolution of $\widehat{\mathbb{Z}}$ by free $\widehat{\mathbb{Z}}[\![\widehat{G}]\!]$-modules*

$$\widehat{C}_i = \widehat{\mathbb{Z}}[\![\widehat{G}]\!]^{\oplus r_i}$$

Proof. For each $m \in \mathbb{N}$ and $K \leq_{\mathrm{f}} G$, set

$$A_{i,m,K} = (\mathbb{Z}/m)[G/K] \otimes_{\mathbb{Z}[G]} C_i = (\mathbb{Z}/m)[G/K]^{\oplus r_i}$$

so that the new chain groups are

$$\widehat{C}_i = \varprojlim_{m,K} A_{i,m,K}$$

The groups $A_{i,m,K}$ are finite, so the homology of each chain complex $(A_{\bullet,m,K})$ is finite. As described above we may now use the Mittag-Leffler condition to conclude

$$H_i(\widehat{C}_\bullet) = H_i(\varprojlim_{m,K} A_{i,m,K}) = \varprojlim_{m,K} H_i(A_{i,m,K})$$

Regarding $(C_\bullet)$ as an exact complex of free finitely generated $\mathbb{Z}[K]$-modules and noting that

$$A_{i,m,K} = (\mathbb{Z}/m) \otimes_{\mathbb{Z}[K]} C_i$$

these homology groups $H_i(A_{i,m,K})$ are precisely $H_i(K;\mathbb{Z}/m)$. By the goodness of G we can now use Proposition 2.2.15 to conclude

$$H_i(\widehat{C}_\bullet) = \varprojlim_{m,K} H_i(K;\mathbb{Z}/m) = 0$$

for $n-1 \geq i \geq 1$. For $i = 0$ we have

$$H_0(\widehat{C}_\bullet) = \varprojlim_{m,K} H_0(K;\mathbb{Z}/m) = \varprojlim_{m,K} \mathbb{Z}/m = \widehat{\mathbb{Z}}$$

so $(\widehat{C}_\bullet)$ is indeed a free partial resolution of $\widehat{\mathbb{Z}}$. $\square$

Chapter 3

Profinite Bass-Serre Theory

3.1 Groups acting on profinite graphs

3.1.1 Profinite graphs

We state here the definitions and basic properties of profinite graphs for convenience. The primary source for this section is [Rib17, Chapter 2]. See this reference for proofs and more detail.

Definition 3.1.1. An *abstract graph* X is a set with a distinguished subset $V(X)$ and two retractions $d_0, d_1 \colon X \to V(X)$. Elements of $V(X)$ are called vertices, and elements of $E(X) = X \smallsetminus V(X)$ are called edges. Note that under this definition a graph comes with an orientation on each edge.

If an abstract graph is in addition a profinite space Ξ (that is, an inverse limit of finite discrete topological spaces), $V(\Xi)$ is closed and d_0 and d_1 are in addition continuous, then Ξ is called a *profinite graph*. Note that $E(\Xi)$ need not be closed.

A morphism between graphs X and Y is a function $f \colon X \to Y$ such that $d_i f = f d_i$ for each i. In particular f sends vertices to vertices. However f need not send edges only to edges. A morphism of profinite graphs Ξ and Υ is a continuous map $f \colon \Xi \to \Upsilon$ which is a morphism of abstract graphs.

A profinite graph Ξ may equivalently be described as an inverse limit of finite abstract graphs X_i, and $V(\Xi) = \varprojlim V(X_i)$. If $E(\Xi)$ happens to be closed then we can choose the inverse system X_i so that the transition maps send edges to edges. In this case we have $E(\Xi) = \varprojlim E(X_i)$. If Y is another finite graph then any morphism of Ξ onto Y factors through some X_i.

A *path of length* n in a graph X is a morphism into X of a finite graph consisting of n edges $e_1, \ldots, e_n$ and $n+1$ vertices $v_0, \ldots, v_n$ such that the endpoints (with some choice of orientations on the edges) of each edge e_i are v_{i-1} and v_i. Note that such a morphism into a profinite graph is automatically continuous.

An abstract graph X is *path-connected* if any two vertices lie in the image of some path in X. A profinite graph Ξ is *connected* if any finite quotient graph is path-connected. Equivalently a connected profinite graph is the inverse limit of path-connected finite graphs. A path-connected profinite graph is connected.

We will generally only work in practice with profinite graphs Ξ with $E(\Xi)$ closed as this reduces the number of pathologies. For example, if the set of edges of a connected profinite graph is closed, then each vertex must have an edge incident to it unless the graph is a single point [Rib17, Proposition 2.1.6].

Example 3.1.2 (A connected profinite graph with a vertex with no edge incident to it). Let $I = \mathbb{N} \cup \tilde{\mathbb{N}} \cup \{\infty\}$ be the one-point compactification of two copies $\mathbb{N}$ and $\tilde{\mathbb{N}}$ of the natural numbers. Let $V(I) = \mathbb{N} \cup \{\infty\}$ and define d_0 and d_1 by setting $d_i(n) = n$ for $n \in \mathbb{N}$, $d_i(\infty) = \infty$, $d_0(\tilde{n}) = n$ and $d_1(\tilde{n}) = n+1$ for $\tilde{n} \in \tilde{\mathbb{N}}$. This is the inverse limit of the system of finite line segments I_n of length n, where the maps $I_{n+1} \to I_n$ collapse the final edge to the vertex n. Hence I is connected, but no edge is incident to ∞.

Example 3.1.3 (A connected graph whose proper connected subgraphs are finite). Consider the Cayley graphs $C_n = \mathrm{Cay}(\mathbb{Z}/n, 1)$ together with the natural maps $C_{mn} \to C_n$ coming from the natural group homomorphisms. Then $\Xi = \varprojlim C_n$ is the Cayley graph of $\widehat{\mathbb{Z}}$ with respect to the generating set 1 (see below). If Υ is a connected proper subgraph of Ξ, then the image Υ_n of Υ in some C_n is not all of C_n—hence it is a line segment in C_n. In each C_{mn}, the preimage of this line segment is m disjoint copies of Υ_n, and by connectedness Υ_{mn} is precisely one of these copies. That is, $\Upsilon_{mn} \to \Upsilon_n$ is an isomorphism for all m. Thus $\Upsilon = \varprojlim \Upsilon_{mn}$ is simply a line segment (of finite length).

Example 3.1.4 (Cayley graph). Let Γ be a profinite group and X be a closed subset of Γ, possibly containing 1. The *Cayley graph* of Γ with respect to X is the profinite graph $\Xi = \Gamma \times (X \cup \{1\})$, where $V(\Xi) = \Gamma \times \{1\}$, $d_0(g,x) = (g,1)$ and $d_1(g,x) = (gx,1)$.

The Cayley graph Ξ is the inverse limit of the Cayley graphs of finite quotients of Ξ, with respect to the images of the set X in these quotients. The set X generates Γ topologically if and only if its image in each finite quotient is a generating set. Hence Ξ is connected if and only if every finite Cayley graph is connected. In other words Ξ is connected if and only if X generates Γ.

3.1.2 Profinite trees

As one might expect from the reduced importance of paths in the theory of profinite graphs, a 'no cycles' condition does not give a good definition of 'tree'. Instead a homological definition is used. Let $\mathbb{F}_p$ denote the finite field with p elements for p a prime. The following definitions and statements can be found in Section 2 of [RZ00a] or Chapter 2 of [Rib17]. For more on profinite modules and chain complexes, see Chapters 5 and 6 of [RZ00b].

Definition 3.1.5. Given a profinite space $X = \varprojlim X_i$ where the X_i are finite spaces, define the *free profinite* $\mathbb{F}_p$*-module on* X to be

$$[\![\mathbb{F}_p X]\!] = \varprojlim [\mathbb{F}_p X_i]$$

the inverse limit of the free $\mathbb{F}_p$-modules with basis X_i. Similarly for a pointed profinite space $(X,*) = \varprojlim (X_i,*)$ define

$$[\![\mathbb{F}_p (X,*)]\!] = \varprojlim [\mathbb{F}_p (X_i,*)]$$

These modules satisfy the expected universal property, that a map from X to a profinite $\mathbb{F}_p$-module M (respectively, a map from $(X,*)$ to M sending $*$ to 0) extends uniquely to a continuous morphism of modules from the free module to M.

Definition 3.1.6. Let Ξ be a profinite graph. Let $(E^*(\Xi),*)$ be the pointed profinite space $\Xi/V(\Xi)$ with distinguished point the image of $V(\Xi)$. Consider the chain complex

$$0 \to [\![\mathbb{F}_p (E^*(\Xi),*)]\!] \xrightarrow{\delta} [\![\mathbb{F}_p V(\Xi)]\!] \xrightarrow{\epsilon} \mathbb{F}_p \to 0$$

where the map ϵ is the evaluation map and δ sends the image of an edge e in $(E^*(\Xi), *)$ to $d_1(e) - d_0(e)$. Then define $H_1(\Xi, \mathbb{F}_p) = \ker(\delta)$ and $H_0(\Xi, \mathbb{F}_p) = \ker(\epsilon)/\mathrm{im}(\delta)$.

Proposition 3.1.7 (see Section 2.3 of [Rib17])**.** *Let $\Xi = \varprojlim \Xi_i$ be a profinite graph and let p be a prime.*

- *Ξ is connected if and only if $H_0(\Xi, \mathbb{F}_p) = 0$.*
- *The homology groups H_0 and H_1 are functorial and*

$$H_j(\Xi, \mathbb{F}_p) = \varprojlim H_j(\Xi_i, \mathbb{F}_p)$$

Definition 3.1.8. A profinite graph Ξ is a *pro-p tree*, or simply p-tree, if Ξ is connected and $H_1(\Xi, \mathbb{F}_p) = 0$. If π is a non-empty set of primes then Ξ is a *π-tree* if it is a p-tree for every $p \in \pi$. We say simply 'profinite tree' if π consists of all primes.

Note that a finite graph is a p-tree if and only if it is an abstract tree. It follows immediately from Proposition 3.1.7 that an inverse limit of finite trees is a p-tree. However this is not the only source of p-trees, and the notion of p-tree is not independent of the prime p. We also remark that the relative strengths of the adjectives 'pro-p' and 'profinite' differ between the contexts of groups and trees: any pro-p group is a profinite group, and any profinite tree is a pro-p tree, but not vice versa.

Example 3.1.9. Let Ξ be the Cayley graph of $\widehat{\mathbb{Z}}$ with respect to the generator 1, written as the inverse limit of cycles C_n of length n. For a finite set X,

$$[\![\mathbb{F}_p(X \cup \{*\}, *)]\!] = [\![\mathbb{F}_p X]\!] = [\mathbb{F}_p X]$$

so for each n, $H_1(C_n, \mathbb{F}_p) = \mathbb{F}_p$ as this is now the simplicial homology of the realisation of C_n as a topological space. Thus the map $H_1(C_{mn}, \mathbb{F}_p) \to H_1(C_n, \mathbb{F}_p)$ is just multiplication by m. In particular, $C_{pn} \to C_n$ induces the zero map on homology. It follows that the inverse limit of these groups is trivial, i.e.

$$H_1(\Xi, \mathbb{F}_p) = \varprojlim H_1(C_n, \mathbb{F}_p) = 0$$

so that Ξ is a p-tree.

Now let Ξ_q be the Cayley graph of the q-adic integers $\mathbb{Z}_q$ with respect to 1, where q is a prime. Again Ξ_q is the inverse limit of the cycles C_{q^i} of length q^i, and the maps induced on homology by $C_{q^j} \to C_{q^i}$ for $j > i$ are multiplication by q^{j-i}. If $q = p$ these maps are trivial so again

$$H_1(\Xi_q, \mathbb{F}_p) = \varprojlim H_1(C_{q^i}, \mathbb{F}_p) = 0$$

so that Ξ_q is a q-tree. On the other hand if $q \neq p$ then multiplication by q^{j-i} is an isomorphism from $\mathbb{F}_p$ to itself, so that

$$H_1(\Xi_q, \mathbb{F}_p) = \varprojlim H_1(C_{q^i}, \mathbb{F}_p) \cong \mathbb{F}_p$$

and Ξ_q is *not* a p-tree for $p \neq q$.

It transpires [GR78, Theorem 1.2] that the Cayley graph of any free pro-π group with respect to a free basis (that is, a pro-π group satisfying the appropriate universal property in the category of pro-π groups) is a π-tree. Note that the examples above show that a profinite group acting freely on a profinite tree need not be free profinite. For the subgroup $\mathbb{Z}_p \leq \widehat{\mathbb{Z}}$ acts freely on the above Cayley graph Ξ, but is not a free object in the category of all profinite groups (for instance, if $q \neq p$ is another prime, the map $1 \mapsto 1 \in \mathbb{Z}/q$ does not extend to a map $\mathbb{Z}_p \to \mathbb{Z}/q$). Instead (by [Rib17, Theorem 4.1.2]) such a group will be *projective* in the sense of category theory. A pro-p group acting freely on a pro-p tree is a free pro-p group however [RZ00a, Theorem 3.4] because in the category of pro-p groups any projective group is free pro-p. This is one of the ways in which the pro-p theory is more amenable than the general profinite theory.

Some of the topological properties of abstract trees do carry over well to the world of profinite trees.

Proposition 3.1.10 (Lemma 1.16 and Proposition 1.18 of [ZM89b])**.** *Let T be a π-tree.*

- *Every connected profinite subgraph of T is a π-tree*
- *Any intersection of π-subtrees of T is a (possibly empty) π-subtree*

It follows that for every subset W of a π-tree T there is a unique smallest subtree of T containing W. If W consists of two vertices v and w then this smallest subtree is called the *geodesic* from v to w and is denoted $[v, w]$. Note that if a profinite tree T is path-connected and hence is also an abstract tree, then $[v, w]$ will coincide with the usual notion of geodesic, a shortest path from v to w. However there is no requirement for a geodesic to be a path. For instance, our above analysis (Example 3.1.3) of the connected subgraphs of the profinite tree $\mathrm{Cay}(\widehat{\mathbb{Z}}, 1)$ shows that if $\lambda, \mu \in \widehat{\mathbb{Z}}$ then either $\lambda - \mu \in \mathbb{Z}$ or the geodesic $[\lambda, \mu]$ is the entire tree $\mathrm{Cay}(\widehat{\mathbb{Z}}, 1)$.

3.1.3 Group actions on profinite trees

The theory of profinite groups acting on profinite trees is less tractable than the classical theory, but still parallels it in many respects. In this section we will recall results from the book [Rib17], and prove others which will be of use. The theory was originally developed in [GR78], [Zal89], [ZM89b], and [ZM89a] among others, and the pro-p version of the theory may also be found in [RZ00a].

Definition 3.1.11. A profinite group Γ is said to *act* on a profinite graph Ξ if Γ acts continuously on the profinite space Ξ in such a way that

$$\gamma \cdot d_i(x) = d_i(\gamma \cdot x)$$

for all $\gamma \in \Gamma, x \in \Xi$ and $i = 0, 1$. For each $x \in \Xi$ the stabiliser $\{\gamma \in \Gamma \mid \gamma \cdot x = x\}$ will be denoted Γ_x. For subsets $X \subseteq \Xi$ and $\Delta \subseteq \Gamma$ the set of points in X fixed by every element of Δ will be denoted X^{Δ}.

Note that given an edge e, we cannot have $\gamma \cdot e = e$ without fixing both endpoints of e, as $\gamma \cdot d_i(e) = d_i(e)$ for each i. In particular, the qualification 'without inversion' applied to group actions in the classical theory of [Ser03] is here subsumed in the definition.

If Γ acts on Ξ, the quotient space $\Gamma \backslash \Xi$ is a well-defined profinite graph. As one might expect, such an action may be represented as an 'inverse limit of finite group actions on finite graphs'.

Definition 3.1.12. Let a profinite group Γ act on a profinite graph Ξ. A decomposition of Ξ as an inverse limit of finite graphs $\Xi = \varprojlim X_i$ is said to be a Γ-*decomposition* if Γ acts on each X_i in such a way that all quotient maps $\Xi \to X_i$ and transition maps $X_j \to X_i$ are Γ-equivariant.

All such actions admit Γ-decompositions (Proposition 2.2.2 of [Rib17]). Furthermore, the following is true and provides a Γ-decomposition whenever the quotient $\Gamma \backslash \Xi$ is finite.

Lemma 3.1.13 (Lemma 2.2.1 of [Rib17])**.** *Let a profinite group* Γ *act on a profinite graph* Ξ*. Then the graphs* $\{\Lambda \backslash \Xi \mid \Lambda \lhd_o \Gamma\}$ *form an inverse system and*

$$\Xi = \varprojlim_{\Lambda \lhd_o \Gamma} \Lambda \backslash \Xi$$

Proof. The first statement is clear. By the universal property of inverse limits, we have a natural continuous surjection

$$\Xi \twoheadrightarrow \varprojlim_{\Lambda \lhd_o \Gamma} \Lambda \backslash \Xi$$

and it remains to show that this is injective. If $v, w \in \Xi$ are identified in every $\Lambda \backslash \Xi$, then for all Λ there is some $\lambda \in \Lambda$ such that $\lambda \cdot v = w$. Thus the closed subsets

$$\Lambda_V v, q = \{\lambda \in \Lambda \mid \lambda \cdot v = w\}$$

of Γ are all non-empty, and the collection of all open normal subgroups Λ of Γ is closed under finite intersections, so the subsets $\Lambda_{v,w}$ have the finite intersection property. Hence by compactness of Γ their intersection is non-empty, so there is some $\gamma \in \bigcap \Lambda$ such that $\gamma \cdot v = w$. However the intersection of all open normal subgroups of Γ is trivial, so $v = w$. ☐

Definition 3.1.14. If Γ is a profinite group acting on a profinite graph T, the action is said to be:

- *faithful*, if the only element of Γ fixing every vertex of T is the identity;
- *irreducible*, if no proper subgraph of T is invariant under the action of Γ;

- *free* if $\Gamma_x = 1$ for all $x \in \Xi$.

Note that a group Γ acts freely on its Cayley graph with respect to any closed subset X, with quotient the 'bouquet of circles' on the pointed profinite space $(X \cup \{1\}, 1)$—i.e. the profinite graph with vertex space $\{1\}$ and edge space X.

Faithful and irreducible actions are the most important actions. Indeed, given a group Γ acting on a profinite tree Ξ, we can quotient Γ by the kernel $\bigcap \Gamma_x$ of the action (i.e. those group elements fixing all of Ξ) and then pass to a minimal Γ-invariant subtree of Ξ to get a faithful and irreducible action on a profinite tree. Such a subtree exists by the following proposition.

Proposition 3.1.15 (Lemma 1.5 of [Zal91]; Proposition 2.4.12 of [Rib17])**.** *Let a profinite group Γ act on a pro-π tree Ξ. Then there exists a minimal Γ-invariant subtree Υ of Ξ, and if $|\Upsilon| > 1$, it is unique.*

We now collect additional useful results concerning actions on profinite trees.

Theorem 3.1.16 (Theorem 2.8 of [ZM89b]; Theorem 4.1.5 of [Rib17])**.** *Suppose a pro-π group Γ acts on a π-tree T. Then the set T^Γ of fixed points under the action of Γ is either empty or a π-subtree of Γ.*

Theorem 3.1.17 (Theorem 2.10 of [ZM89b]; Theorem 4.1.8 of [Rib17])**.** *Any finite group acting on a profinite tree fixes some vertex.*

Proposition 3.1.18. *Let a profinite group Γ act on a profinite tree T. Let $\Delta \leq_o \Gamma$ be an open subgroup of Γ with $[\Gamma : \Delta] < \infty$, and suppose Δ fixes a vertex v of T. Then Γ fixes some vertex of T.*

Proof. Since Δ is open there exists an open normal subgroup Λ of Γ contained in Δ. Then the finite group Γ/Λ acts on the set T^Λ of fixed points of Λ. This is non-empty by assumption, so T^Λ is a subtree by Proposition 3.1.16. Then Γ/Λ (and hence Γ) fixes a point of T^Λ by Theorem 3.1.17 and we are done. □

Proposition 3.1.19. *Let Γ act on a profinite tree T and let $x \in \Gamma$. If x^λ fixes a vertex of T for some $\lambda \in \widehat{\mathbb{Z}} \smallsetminus \{0\}$ which is not a zero-divisor then x also fixes some vertex of T.*

Proof. Let S be the subtree of T fixed by x^λ, which is non-empty by assumption. Consider the action of $C = \overline{\langle x \rangle}$ on S. The closed (normal) subgroup of C generated by x^λ acts trivially on S, so there is a quotient action of $C/\overline{\langle x^\lambda \rangle}$ on S. Now this quotient group is $\widehat{\mathbb{Z}}/\lambda\widehat{\mathbb{Z}}$. Using the splitting $\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p$, we find

$$\widehat{\mathbb{Z}}/\lambda\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p/\pi_p(\lambda)\mathbb{Z}_p$$

where π_p is the projection onto each factor. For every p the number $\pi_p(\lambda) \in \mathbb{Z}_p$ is non-zero because λ is not a zero-divisor in $\widehat{\mathbb{Z}}$. Closed subgroups of $\mathbb{Z}_p$ are finite index or trivial (see Proposition 2.7.1 of [RZ00b]), so $\widehat{\mathbb{Z}}/\lambda\widehat{\mathbb{Z}}$ is a direct product of finite cyclic groups.

The subtree fixed by a direct product of groups is the intersection of the subtrees fixed by each group. Any finite product of finite cyclic groups fixes some vertex by Theorem 3.1.17, so the subtrees fixed by each finite cyclic group are a collection of closed subsets of S with the finite intersection property. By compactness, the intersection of all of them is non-empty—but this is the subtree fixed by $C/\overline{\langle x^\lambda \rangle}$, which is the same as the subtree fixed by x. So x fixes some vertex of S (hence of T). □

Remark. The condition that λ is not a zero-divisor is necessary. For instance, $\widehat{\mathbb{Z}}$ (written multiplicatively with generator x) acts freely on its Cayley graph, but if $\lambda\mu = 0$ for λ and μ non-zero then x^μ fixes no vertex but $(x^\mu)^\lambda$ is the identity. Recall however that no element of $\mathbb{Z}$ is a zero divisor in $\widehat{\mathbb{Z}}$, so that the proposition applies in particular when $\lambda \in \mathbb{Z}$.

Proposition 3.1.20 (Lemma 2.4 of [Zal91]; Proposition 4.2.3(b) of [Rib17])**.** *Let* Γ *be an abelian profinite group acting faithfully and irreducibly on a profinite tree. Then* Γ *acts freely and* $\Gamma \cong \mathbb{Z}_\pi$ *for some set of primes* π*.*

3.2 Acylindrical actions

Actions on profinite trees are particularly malleable when the action is acylindrical.

Definition 3.2.1. Let a profinite group Γ act on a profinite tree T. The action is *k-acylindrical* if the stabiliser of any injective path of length greater than k is trivial.

For instance an action with trivial edge stabilisers is 0-acylindrical. In [WZ17a] Wilton and Zalesskii exploited the fact that if edge groups are malnormal in the adjacent vertex groups then the action on the standard graph is 1-acylindrical.

We now prove some results about acylindrical actions on profinite trees. The following lemma is taken from the Appendix to [HZ12] and will be used to remove some of the pathologies associated with profinite graphs; we reproduce it here for completeness.

Lemma 3.2.2. *Let Ξ be a profinite graph in which there are no paths longer than m edges for some integer m. Then the connected components of Γ (as a profinite graph) are precisely the path components (that is, the connected components as an abstract graph). In particular if Ξ is connected then it is path-connected.*

Proof. First define the composition of two binary relations R and S on a set X to be the relation that $xRSy$ if and only if there exists z such that xRz and zSy and inductively set $R^{n+1} = R^nR$. Further define R^{op} to be the relation that $xR^{\mathrm{op}}y$ if and only if yRx. Let $\Delta = \{(x,x) \in X \times X\}$ be the identity relation.

For an abstract graph Ξ define $R_0 = \{(x,y) \in \Xi \times \Xi \,|\, d_1(x) = d_0(y)\}$ (where x, y could be vertices), and set $R = R_0 \cup \Delta \cup R_0^{\mathrm{op}}$. If xR^ny then there is some path of length at most n containing x and y. Conversely if there is a path of n edges containing x and y then $xR^{2n+1}y$. This discrepancy ('n' versus '$2n+1$') is due to our convention that graphs are oriented, so that we may need to include vertices in addition to the edges in a path of length n to obtain a chain of R-related elements of Ξ. The path-components of Ξ are then the equivalence classes of the equivalence relation $S = \bigcup_n R^n$.

Now, in our profinite graph Ξ we have $S = R^n$ for some n as there is a uniform bound on the length of paths in Ξ. One can show that the continuity of the maps d_0 and d_1 and compactness of Ξ imply that R, and all R^n, are closed compact subsets of $\Xi \times \Xi$. In particular, the equivalence classes of $S = R^n$ are closed subsets of Ξ; that is, the path-components of Ξ are closed. The quotient profinite graph Ξ/S has no edges, hence its maximal connected subgraphs are points. Thus connected components of Ξ (as a profinite graph) are contained in, hence equal to, a path-component of Ξ. $\square$

Proposition 3.2.3. *Let a profinite group* Γ *act irreducibly on a profinite tree* T*. Then either* T *is a single vertex or it contains paths of arbitrary length.*

Proof. Assume that there is a bound on the lengths of paths in T. Then by the previous result T is path-connected, hence is a tree when considered as an abstract graph. Elementary graph theory shows that any group acting on a finite-diameter abstract tree fixes a vertex or edge: by removing all vertices of valence 1 one obtains a subtree of smaller diameter on which Γ acts, and inductively Γ acts on a subtree of diameter 0 or 1. By our convention that graphs are oriented, if Γ fixes an edge then it fixes each endpoint. So Γ fixes some vertex of T and irreducibility forces T to be a single point. □

Corollary 3.2.4. *Let a profinite group* Γ *act* k*-acylindrically on a profinite tree* T *for some* k*. Suppose* $g \in \Gamma \smallsetminus \{1\}$ *fixes two vertices* v *and* w*. Then* v *and* w *are in the same path-component of* T*.*

Proof. By Theorem 3.1.16 the set T^g of points fixed by g is a subtree of T as it is non-empty; therefore it contains the geodesic $[v, w]$. By acylindricity of the action, $[v, w]$ contains no paths of length longer than k and hence it is path-connected by Lemma 3.2.2. □

Proposition 3.2.5. *Let a profinite abelian group* A *act acylindrically on a profinite tree* T*. Then either* $A \cong \mathbb{Z}_\pi$ *for some set of primes* π *or* A *fixes some vertex of* T*.*

Proof. Let S be a minimal invariant subtree for the action of A on T. Then A acts irreducibly on S so, by Proposition 3.2.3, either S is a point (whence A fixes a point of T) or S contains paths of arbitrary length. By Proposition 3.1.20, if A is not a projective group $\mathbb{Z}_\pi$ then the action is not faithful, so some non-trivial element of A fixes S and hence fixes paths of arbitrary length. But this is impossible by acylindricity. □

The following concept will be useful later.

Definition 3.2.6. Given a profinite group Γ and a copy Θ of $\widehat{\mathbb{Z}}$ contained in it, the *restricted normaliser* of Θ in Γ is the closed subgroup

$$\mathcal{N}'_\Gamma(\Theta) = \{g \in \Gamma \,|\, h^g = h \text{ or } h^g = h^{-1}\}$$

where h is a generator of Θ.

Note that this is a closed subgroup of Γ containing the centraliser as an index 1 or 2 subgroup. We deal with the restricted normaliser to avoid certain technicalities in later proofs—in particular since $\mathrm{Aut}(\widehat{\mathbb{Z}})$ is infinite the centraliser of Θ may not be of finite index in its full normaliser. There is a continuous homomorphism from the full normaliser $\mathcal{N}_\Gamma(\Theta)$ to $\mathrm{Aut}(\widehat{\mathbb{Z}})$. The centraliser is the kernel of this map and the reduced normaliser is the preimage of the unique order 2 subgroup $\mathrm{Aut}(\mathbb{Z}) \leq \mathrm{Aut}(\widehat{\mathbb{Z}})$ which acts non-trivially on each $\mathbb{Z}_p$.

Proposition 3.2.7. *Let a profinite group Γ act k-acylindrically on a profinite tree T. Let $\Theta \cong \widehat{\mathbb{Z}}$ be a subgroup of Γ fixing some vertex v of T. Then the centraliser $\mathcal{Z}_\Gamma(\Theta)$ and the restricted normaliser $\mathcal{N}'_\Gamma(\Theta)$ both fix some vertex of T.*

Proof. The reduced normaliser has the centraliser as an index 2 subgroup, so by Proposition 3.2.7 it suffices to consider the centraliser of Θ. Now by acylindricity the tree of fixed points T^Θ has bounded diameter. Furthermore $\mathcal{Z}_\Gamma(\Theta)$ acts on this tree, so passing to a $\mathcal{Z}_\Gamma(\Theta)$-invariant subtree of T^Θ and applying Proposition 3.2.3 yields the result. Note that by Corollary 3.2.4 any fixed point of $\mathcal{Z}_\Gamma(\Theta)$ lies in the same path-component of T as v and is joined to it by a path of length less than k. □

3.3 Graphs of profinite groups

3.3.1 Free profinite products

Definition 3.3.1. A *sheaf of pro-$\mathcal{C}$ groups* is a triple $(\mathcal{G}, \pi, X)$ where $\mathcal{G}$ and X are profinite spaces, $\pi\colon \mathcal{G} \to X$ is a continuous function, and where each 'fibre' $\mathcal{G}(x) = \pi^{-1}(x)$ is equipped with a group operation which varies continuously with x in the following sense. Let $\mathcal{G}^{(2)}$ be the space $\{(g,h) \in \mathcal{G}^2 \,|\, \pi(g) = \pi(h)\}$. Then the map

$\mathcal{G}^{(2)} \to \mathcal{G}$ sending (g, h) to $g^{-1}h$ is required to be continuous, where the multiplication and inversion are performed in the group $\mathcal{G}(\pi(g)) = \mathcal{G}(\pi(h))$.

A *morphism of sheaves* $(\phi, f)\colon (\mathcal{G}, \pi, X) \to (\mathcal{H}, \rho, Y)$ is a pair of continuous functions $\phi\colon \mathcal{G} \to \mathcal{H}$ and $f\colon X \to Y$ such that $f \circ \pi = \rho \circ \phi$ and such that the restriction of ϕ to any fibre $\mathcal{G}(x) \to \mathcal{H}(f(x))$ is a group homomorphism. Regarding a profinite group as a sheaf over a singleton space gives a notion of a morphism from a sheaf to a profinite group.

We will often abuse notation and refer to 'the sheaf $\mathcal{G}$'. Note that a sheaf over a finite space X is simply a finite collection of pro-$\mathcal{C}$ groups indexed by X, with no additional constraints.

Definition 3.3.2. Let $(\mathcal{G}, \pi, X)$ be a sheaf of pro-$\mathcal{C}$ groups. A *free pro-$\mathcal{C}$ product of $\mathcal{G}$* is a pro-$\mathcal{C}$ group Γ together with a morphism $\omega\colon \mathcal{G} \to \Gamma$ (sometimes called a 'universal morphism') with the following universal property: for every sheaf morphism $\beta\colon \mathcal{G} \to \Delta$ to a pro-$\mathcal{C}$ group Δ there is a unique continuous homomorphism $\overline{\beta}\colon \Gamma \to \Delta$ such that $\overline{\beta}\omega = \beta$. A free pro-$\mathcal{C}$ product will be denoted $\coprod_X \mathcal{G}$ or $\coprod_{x\in X} \mathcal{G}(x)$.

Note that a standard inverse limit argument shows that one need only verify this universal property for finite groups $\Delta \in \mathcal{C}$. If X is a finite space this definition is simply a rephrasing of the usual universal property of a free product in terms of sheaves and sheaf morphisms. See Section 9.1 of [RZ00b] for properties of finite free pro-$\mathcal{C}$ products. The following proposition may be seen as some justification for the sheaf-theoretic definition.

Proposition 3.3.3 (Proposition 5.1.7 of [Rib17])**.** *Let I be a directed poset and let $(\mathcal{G}_i, \pi_i, X_i)_{i\in I}$ be an inverse system of sheaves over I. Then*

$$(\mathcal{G}, \pi, X) = (\varprojlim \mathcal{G}_i, \varprojlim \pi_i, \varprojlim X_i)$$

is a sheaf. Note that the morphisms in the inverse system of sheaves induce homomorphisms making $(\coprod \mathcal{G}_i)_{i\in I}$ into an inverse system. Then

$$\coprod_X \mathcal{G} \cong \varprojlim \coprod_{X_i} \mathcal{G}_i$$

In this way one may decompose any free pro-$\mathcal{C}$ product as a certain inverse limit of pro-$\mathcal{C}$ free products with finitely many factors.

We will also need a characterization of when a group may be expressed as a free product of a family of subgroups. One may make an internal definition of free product—this is done in Section 5.3 of [Rib17]. The internal and external notions of free product are of course equivalent, so for brevity we will use this equivalence as our definition.

Definition 3.3.4. Let Γ be a pro-$\mathcal{C}$ group and $\{\Gamma_x\}_{x\in X}$ a family of closed subgroups of Γ indexed by a profinite space X. This family is said to be *continuously indexed by* X if the subset

$$\mathcal{G} = \{(\gamma, x) \in \Gamma \times X \text{ such that } \gamma \in \Gamma_x\} \subseteq \Gamma \times X$$

is closed. Note that in this situation $\mathcal{G}$ is a sheaf of pro-$\mathcal{C}$ groups over X and the map $\omega\colon \mathcal{G} \to \Gamma$ sending $(\gamma, x) \mapsto \gamma \in \Gamma_x$ is a sheaf morphism. Then Γ is the *internal free pro-$\mathcal{C}$ product* of the family Γ_x if ω is a universal morphism exhibiting Γ as the free pro-$\mathcal{C}$ product of the sheaf $\mathcal{G}$. In this case we write $\Gamma = \coprod_{x\in X} \Gamma_x$ or $\Gamma = \coprod_X \Gamma_x$.

When $\{\Gamma_x\}_{x\in X}$ is a family of closed subgroups of Γ continuously indexed by X, note that there is a closed equivalence relation $\sim$ on $\Gamma \times X$ defined by

$$(\gamma_1, x_1) \sim (\gamma_2, x_2) \text{ if and only if } x_1 = x_2 = x \text{ and } \gamma_1^{-1}\gamma_2 \in \Gamma_x$$

such that as a set

$$(\Gamma \times X)/\!\sim\; = \bigcup_{x\in X} \Gamma/\Gamma_x$$

Hence the union of coset spaces on the right acquires a topology making it into a profinite space. Note that if Δ is another subgroup of Γ then the action of Δ on $\Gamma \times X$ by left multiplication on Γ preserves $\sim$ so that the union of double coset spaces

$$\Delta\backslash(\Gamma \times X)/\!\sim\; = \bigcup_{x\in X} \Delta\backslash\Gamma/\Gamma_x$$

also acquires a topology. We can now finally state a form of Kurosh Theorem.

Theorem (Kurosh Subgroup Theorem for pro-p groups; Theorem 9.6.2 of [Rib17]). *Suppose $\Gamma = \coprod_{x\in X} \Gamma_x$ is a free pro-p product of pro-p groups Γ_x indexed by a profinite space X. Let $\Delta \leq_c \Gamma$. Then for each $x \in X$ there is a family $\{g_{x,\tau}\}$ of representatives of the double cosets $\tau \in \Delta\backslash\Gamma/\Gamma_x$ such that the family $\{\Delta \cap g_{x,\tau}\Gamma_x g_{x,\tau}^{-1}\}$ of subgroups of Γ is continuously indexed by*

$$\Delta\backslash(\Gamma \times X)/\sim \; = \bigcup_{x\in X} \Delta\backslash\Gamma/\Gamma_x = \{H g_{x,\tau}\Gamma_x\}$$

and H is the free pro-p product

$$\Delta = \coprod_{\Delta\backslash(\Gamma\times X)/\sim} (\Delta \cap g_{x,\tau}\Gamma_x g_{x,\tau}^{-1}) \amalg \Phi$$

where Φ is a free pro-p subgroup of Γ.

In the category of profinite groups this theorem does not hold; for instance a p-Sylow subgroup of a free profinite group is not expressible as a free profinite product. However for open subgroups of free products of finitely many factors, a subgroup theorem does hold.

Theorem (Kurosh Open Subgroup Theorem; Theorem 9.1.9 of [RZ00b]). *Suppose $\Gamma = \coprod_{j=1}^{n} \Gamma_j$ is a free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups Γ_j. Let $\Lambda \leq_o \Gamma$. Then*

$$\Lambda = \coprod_{j=1}^{n} \coprod_{\tau\in\Lambda\backslash\Gamma/\Gamma_j} (\Lambda \cap g_{j,\tau}\Gamma_j g_{j,\tau}^{-1}) \amalg F$$

where F is a free pro-$\mathcal{C}$ group of rank

$$\mathrm{rk}(F) = 1 + (n-1)[\Gamma : \Lambda] - \sum_{j=1}^{n} |\Lambda\backslash\Gamma/\Gamma_j|$$

and $g_{j,\tau}$ is a representative for the double coset $\tau \in \Lambda\backslash\Gamma/\Gamma_j$.

3.3.2 Graphs of groups

The theory of profinite graphs of groups can be defined for general profinite graphs X. We shall only consider finite graphs X here as this considerably simplifies the theory and is sufficient for our needs.

Definition 3.3.5. A *finite graph of profinite groups* $\mathcal{G} = (X, \Gamma_\bullet)$ consists of a finite graph X, a profinite group Γ_x for each $x \in X$, and two (continuous) monomorphisms $\partial_i \colon \Gamma_x \to \Gamma_{d_i(x)}$ for $i \in \{0,1\}$ which are the identity when $x \in V(X)$. We will occasionally suppress the graph X and refer to 'the graph of groups $\Gamma_\bullet$'.

Definition 3.3.6. Given a finite graph of profinite groups $\mathcal{G} = (X, \Gamma_\bullet)$, choose a maximal subtree Y of X. A *profinite fundamental group of the graph of groups $\mathcal{G}$ with respect to* Y consists of a profinite group Δ, and a map

$$\phi\colon \coprod_{x\in X} \Gamma_x \amalg \coprod_{e\in E(X)} \overline{\langle t_e \rangle} \to \Delta$$

such that

$$\phi(t_e) = 1 \text{ for all } e \in E(Y)$$

and

$$\phi(t_e^{-1}\partial_0(\gamma)t_e) = \phi(\partial_1(\gamma)) \text{ for all } e \in E(X), \gamma \in \Gamma_e$$

and with (Δ, ϕ) universal with these properties. The profinite group Δ will be denoted $\Pi_1(\mathcal{G})$ or $\Pi_1(X, \Gamma_\bullet)$.

The group so defined exists and is independent of the maximal subtree Y (see Section 6.2 of [Rib17]). Note that in the category of discrete groups this is precisely the same as the classical definition as a certain presentation. Free products are a special case of this, namely they are fundamental groups of graphs of groups in which all edge groups are trivial.

In the classical Bass-Serre theory, a graph of discrete groups $(X, G_\bullet)$ gives rise to a fundamental group $\pi_1(X, G_\bullet)$ and an action on a certain tree T whose vertices are cosets of the images $\phi(G_v)$ of the vertex groups in $\pi_1(X, G_\bullet)$ and whose edge groups are cosets of the edge groups. Putting a suitable topology and graph structure on the corresponding objects in the profinite world and proving that the result is a profinite tree is rather more involved than the classical theory—however the conclusion is much the same. We collate the various results into the following theorem.

Theorem 3.3.7 (Proposition 3.8 of [ZM89b]; see also Section 6.3 of [Rib17])**.** *Let $\mathcal{G} = (X, \Gamma_\bullet)$ be a finite graph of profinite groups. Let $\Pi = \Pi_1(\mathcal{G})$ and $\Pi(x) = \operatorname{im}(\Gamma_x \to$*

Π). *Then there exists an (essentially unique) profinite tree* $S(\mathcal{G})$*, called the* standard graph *of* $\mathcal{G}$*, on which* Π *acts with the following properties.*

- *The quotient graph* $\Pi\backslash S(\mathcal{G})$ *is isomorphic to* X*.*
- *The stabiliser of a point* $s \in S(\mathcal{G})$ *is a conjugate of* $\Pi(\zeta(s))$ *in* Π*, where* $\zeta\colon S(\mathcal{G}) \to X$ *is the quotient map.*

Conversely (see [ZM89a], or Section 6.4 of [Rib17]) an action of a profinite group on a profinite tree with quotient a finite graph gives rise to a decomposition as a finite graph of profinite groups. However no analogous result holds when the quotient graph is infinite.

In the classical theory one tacitly identifies each G_x with its image in the fundamental group $\pi_1(X, G_\bullet)$ of a graph of groups. In general in the world of profinite groups the maps $\phi_x\colon \Gamma_x \to \Pi_1(\mathcal{G})$ may not be injective, even for simple cases such as amalgamated free products. We call a graph of groups *injective* if all the maps ϕ_x are in fact injections. An example where injectivity fails may be found in [RZ00b, Example 9.2.10].

Let $\mathcal{G} = (X, G_\bullet)$ be a finite graph of abstract groups. We can then form a finite graph of profinite groups $\widehat{\mathcal{G}} = (X, \widehat{G}_\bullet)$ by taking the profinite completion of each vertex and edge group. We have not yet addressed whether the 'functors'

$$\mathcal{G} \to \widehat{\mathcal{G}} \to \Pi_1(\widehat{\mathcal{G}})$$

and

$$\mathcal{G} \to \pi_1(\mathcal{G}) \to \widehat{\pi_1(\mathcal{G})}$$

on $\mathcal{G}$ yield the same result—that is, whether the order in which we take profinite completions and fundamental groups of graphs of groups matters. In general the two procedures do not give the same answer. We require some additional separability properties.

Definition 3.3.8. A graph of discrete groups $(X, G_\bullet)$ is *efficient* if $\pi_1(X, G_\bullet)$ is residually finite, each group G_x is closed in the profinite topology on $\pi_1(X, G_\bullet)$, and $\pi_1(X, G_\bullet)$ induces the full profinite topology on each G_x.

Theorem 3.3.9 (Exercise 9.2.7 of [RZ00b]). *Let $(X, G_\bullet)$ be an efficient finite graph of discrete groups. Then $(X, \widehat{G}_\bullet)$ is an injective graph of profinite groups and*

$$\widehat{\pi_1(X, G_\bullet)} \cong \Pi_1(X, \widehat{G}_\bullet)$$

Notice that the above-quoted Theorem 2.1.19 proved by Wilton and Zalesskii [WZ10] may be rephrased as 'the JSJ decomposition of a graph manifold group is efficient'. Therefore the profinite completion of a 3-manifold group acts in a well-controlled fashion on a profinite tree.

It should be noted that, the above problems notwithstanding, free profinite products are generally quite well-behaved. For instance for discrete groups G_1, G_2 we have

$$\widehat{G}_1 \amalg \widehat{G}_2 = \widehat{G_1 * G_2}$$

One may of course consider the pro-p version of the above theory, taking all universal properties and constructions in the category of pro-p groups. This theory closely parallels the profinite version of the theory. In particular one obtains an action on a standard tree. There is also a corresponding notion of p-efficiency, with similar consequences.

Definition 3.3.10. A graph of discrete groups $\mathcal{G} = (X, G_\bullet)$ is *p-efficient* if $G = \pi_1(\mathcal{G})$ is residually p, each group G_x is closed in the pro-p topology on G, and G induces the full pro-p topology on each G_x.

The appropriate reformulations of the statements of Theorems 3.3.7 and 3.3.9 now hold in the context of pro-p groups.

Chapter 4

Seifert Fibre Spaces

4.1 Background on Seifert fibre spaces

We first recall some information about the invariants of a Seifert fibre space before moving on to profinite matters. For a more comprehensive introduction to Seifert fibre spaces see [Bri07], [Sco83] and [Orl06]. Recall that, by convention, we only study orientable 3-manifolds in this thesis.

Recall that a *fibred solid torus* is formed as a quotient of $\mathbb{D}^2 \times [0,1]$ by identifying the two end discs by a rotation by $2\pi q/p$ where p and q are coprime integers, called the *fibre invariants* of the fibred solid torus. The foliation of $\mathbb{D}^2 \times [0,1]$ by lines $\{x\} \times [0,1]$ descends to a foliation of the torus by circles. Such pieces form a local model for a Seifert fibre space—that is, a *Seifert fibre space* is a 3-manifold admitting a foliation by circles (called 'fibres' or 'Seifert fibres') such that each fibre has a closed neighbourhood which is homeomorphic (preserving the foliations) to a fibred solid torus. A fibre is *exceptional* if it has such a neighbourhood where it is mapped to the core curve $\{0\} \times \mathbb{S}^1$ in a fibred solid torus with $p \geq 2$. The fibres which are not exceptional are called *regular*, and are all isotopic.

Note that the quotient of a fibred solid torus obtained by collapsing each fibre naturally has an orbifold structure where the image of the exceptional fibre is a cone point of order p. After fixing an orientation for the disc and fibre the number q becomes well-defined in the range $0 < q < p$; if no orientations are chosen then it is well-defined only in the range $0 < q \leq p/2$. To give the standard presentation for the fundamental group it is conventional to define the *Seifert invariants* of the exceptional fibre to be (α, β) where $\alpha = p$ and $\beta q \equiv 1 \bmod p$.

The orbifold quotients of neighbourhoods of each fibre piece together to form the quotient of the whole manifold M by the foliation—this is the *base orbifold* O of the Seifert fibre space. This quotient induces a short exact sequence

$$1 \to \langle h \rangle \to \pi_1 M \to \pi_1^{\mathrm{orb}} O \to 1$$

where $\pi_1^{\mathrm{orb}} O$ is the orbifold fundamental group and h is an element of π_1 represented by a regular fibre. This subgroup $\langle h \rangle$ may be finite or infinite cyclic, and is either central (if O is orientable) or $\pi_1 M$ has an index 2 subgroup which contains h as a central element (if O is non-orientable).

There is one final invariant needed to classify closed Seifert fibre spaces. It is a rational number e, defined up to sign, called the *Euler number* of the Seifert fibre space. It has various different formulations: see [Bri07, Sco83, NR78]. For the purposes of this thesis the formula given in Proposition 4.1.3 relating e to a presentation of the fundamental group will be considered to be a definition.

The Euler number is in some sense the 'obstruction to a section'. In the case when there are no exceptional fibres and the Seifert fibre space is therefore a *bona fide* fibre bundle over a surface S, then e coincides with the classical Euler number $e \in H^2(S; \mathbb{Z}) \cong \mathbb{Z}$ of the fibration, which again is the obstruction to a section. The connection between Seifert fibre spaces, the Euler number and cohomology classes will be discussed in more detail in Sections 4.3.2 and 4.3.3.

The key properties of the Euler number are the above behaviour when there are no exceptional fibres, and the following naturality property.

Proposition 4.1.1 (Theorem 3.6 of [Sco83])**.** *If $\widetilde{M} \to M$ is a degree d cover, where the base orbifold cover $\widetilde{O} \to O$ has degree m and a regular fibre of $\widetilde{M}$ covers a regular fibre of M with covering degree l, yielding the commutative diagram:*

$$\begin{tikzcd}
1 \arrow[r] & \langle \widetilde{h} \rangle \arrow[r] \arrow[d, "l"] & \pi_1 \widetilde{M} \arrow[r] \arrow[d, "d"] & \pi_1^{orb} \widetilde{O} \arrow[r] \arrow[d, "m"] & 1 \\
1 \arrow[r] & \langle h \rangle \arrow[r] & \pi_1 M \arrow[r] & \pi_1^{orb} O \arrow[r] & 1
\end{tikzcd}$$

then the Euler numbers of M and $\widetilde{M}$ are related by

$$e(\widetilde{M}) = \frac{m}{l} \cdot e(M)$$

The Euler number has no well-defined sign *a priori.* Given a choice of orientation on M the Euler number e acquires a sign; reversing the orientation (by flipping the direction along the fibres) changes this sign. This is consistent with the interpretation as the obstruction to a section; when there are no exceptional fibres, circle bundles with orientable total space are classified by elements of $H^2(\Sigma; \mathbb{Z})$, where the $\mathbb{Z}$ coefficients are twisted by the orientation homomorphism for Σ; this group is $\mathbb{Z}$ whether or not Σ is orientable.

The vanishing of the Euler number gives important topological information:

Proposition 4.1.2 (Lemma 3.7 of [Sco83])**.** *Let M be a closed Seifert fibre space. The Euler number $e(M)$ vanishes if and only if M is virtually a surface bundle over the circle with periodic monodromy.*

Finally, we can state the classification results of Seifert fibre spaces and characterisations of their fundamental groups from these invariants.

Proposition 4.1.3 (See [Orl06], Sections 1.10 and 5.3)**.** *A closed Seifert fibre space is uniquely determined by the symbol*

$$(b, \Sigma; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$$

where

- $b \in \mathbb{Z}$ *and* $e = -(b + \sum \beta_i/\alpha_i)$*;*
- Σ *is the underlying surface of the base orbifold; and*
- (α_i, β_i) *are the Seifert invariants of the exceptional fibres, and* $0 < \beta_i < \alpha_i$ *are coprime.*

If Σ is closed and orientable of genus g then $\pi_1 M$ has presentation

$$\big\langle a_1, \ldots, a_r, u_1, v_1, \ldots, u_g, v_g, h \,\big|\, h \in Z(\pi_1 M),\ a_i^{\alpha_i} h^{\beta_i} = 1,\ a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g] = h^b \big\rangle$$

If Σ is closed and non-orientable of genus g then $\pi_1 M$ has presentation

$$\left\langle a_1, \ldots, a_r, v_1, \ldots, v_g, h \,\middle|\, h^{a_i} = h,\, h^{v_i} = h^{-1},\, a_i^{\alpha_i} h^{\beta_i} = 1,\, a_1 \ldots a_r v_1^2 \ldots v_g^2 = h^b \right\rangle$$

In these presentations h represents the regular fibre. Killing h gives a presentation of the orbifold fundamental group of the base. When the Seifert fibre space has boundary we have similar presentations without the last relation. The base orbifold group is then just a free product of (finite or infinite) cyclic groups. Note also that by reversing the orientation of the fibre h and 'renormalising' to get the β_i back into the correct range, we find an ambiguity in the above symbol for a Seifert fibre space given by the transformation

$$(b, \Sigma; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)) \to (-b - r, \Sigma; (\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_r, \alpha_r - \beta_r))$$

which flips the sign of e. When the orbifold is orientable, this will be the only ambiguity provided there is a unique Seifert fibre space structure on the manifold.

Proposition 4.1.4 ([Sco83], Theorem 3.8)**.** *If a closed manifold M has two distinct Seifert fibre space structures, then it is covered by $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.*

Proposition 4.1.5 ([Sco83], Lemma 3.2)**.** *If h is a regular fibre then the subgroup $\langle h \rangle$ is infinite cyclic unless M is covered by $\mathbb{S}^3$.*

Proposition 4.1.6 ([Sco83], Table 4.1)**.** *A manifold M is Seifert fibred if and only if it has one of the six geometries in Figure 4.1. The geometry is determined by the Euler characteristic of the base orbifold and the Euler number of M.*

	$\chi^{\text{orb}} > 0$	$\chi^{\text{orb}} = 0$	$\chi^{\text{orb}} < 0$
$e = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	$\mathbb{S}^3$	Nil	$\widetilde{\mathrm{SL}_2(\mathbb{R})}$

Figure 4.1: The geometry of a Seifert fibre space is determined by the base orbifold and Euler number

4.2 Profinite completions of 2-orbifold groups

In this section we recall the results of Bridson, Conder and Reid [BCR16] concerning Fuchsian groups (i.e. orbifold fundamental groups of hyperbolic 2-orbifolds), and show that they extend to the case of Euclidean 2-orbifolds.

Theorem 4.2.1 (Theorem 1.1 of [BCR16])**.** *Let G_1 be a finitely-generated Fuchsian group and let G_2 be a lattice in a connected Lie group. If $\widehat{G}_1 \cong \widehat{G}_2$ then $G_1 \cong G_2$.*

Corollary 4.2.2. *Let O_1 and O_2 be closed 2-orbifolds. If $\widehat{\pi_1^{orb}(O_1)} \cong \widehat{\pi_1^{orb}(O_2)}$ then $\pi_1^{orb}(O_1) \cong \pi_1^{orb}(O_2)$. If $\chi^{orb}(O_1) \leq 0$, then O_1 and O_2 are homeomorphic as orbifolds.*

Proof. Since $\pi_1^{\mathrm{orb}}(O_1)$ is finite if and only if the orbifold Euler characteristic is positive, we can safely ignore these cases as the profinite completion is then simply the original group. Otherwise assume $\widehat{\pi_1^{\mathrm{orb}}(O_1)} \cong \widehat{\pi_1^{\mathrm{orb}}(O_2)}$.

Recall from Proposition 2.1.13 that there is an induced correspondence between finite-index subgroups of the orbifold fundamental groups, and hence between finite-sheeted covers of the orbifolds themselves. The orbifold O_1 has a finite cover which is a surface. Take such a cover of O_1 and the corresponding cover of O_2. If necessary pass to a further finite cover of O_2 (and hence O_1) so that both O_1 and O_2 are covered (with degree d, say) by surfaces with isomorphic profinite fundamental groups. A surface group is determined by its first homology, which is seen by the profinite completion, so the two surfaces are homeomorphic to the same surface Σ. Orbifold Euler characteristic is multiplicative under finite covers, so $\chi^{\mathrm{orb}}(O_1) = \chi(\Sigma)/d = \chi^{\mathrm{orb}}(O_2)$. Hence Euclidean and hyperbolic orbifolds are distinguished from each other.

In light of the above theorem of Bridson, Conder and Reid it only remains to distinguish the Euclidean 2-orbifolds from each other. The profinite completion detects first homology, and a direct computation shows that this suffices to distinguish all the Euclidean 2-orbifolds except $(\mathbb{S}^2; 2, 4, 4)$ and $(\mathbb{P}^2; 2, 2)$. Recall that an isomorphism of profinite completions would induce a correspondence between the index 2 subgroups, with corresponding subgroups having the same profinite completions. But $(\mathbb{P}^2; 2, 2)$ is covered by the Klein bottle with degree 2, and the Klein bottle is distinguished from the other 2-orbifolds by its profinite completion, but does not cover $(\mathbb{S}^2; 2, 4, 4)$. So the two remaining Euclidean orbifolds also have distinct profinite completions. □

Theorem 4.2.3 (Theorem 5.1 of [BCR16]). *Let G be a finitely generated Fuchsian group. Every finite subgroup of $\widehat{G}$ is conjugate to a subgroup of G, and if two maximal finite subgroups of G are conjugate in $\widehat{G}$ then they are already conjugate in G.*

Proposition 4.2.4. *Let G be the fundamental group of a closed Euclidean 2-orbifold X. Every torsion element of $\widehat{G}$ is conjugate to a torsion element of G, and if two torsion elements of G are conjugate in $\widehat{G}$ then they are already conjugate in G.*

Proof. The second statement is a special case of the fact that a virtually abelian group is conjugacy separable [Ste72, Lemma 3.8].

We proceed on a case-by-case basis. If X is a torus or Klein bottle, then G is good and has finite cohomological dimension. Hence $\widehat{G}$ is torsion free by Corollary 2.2.12. If $X = (\mathbb{S}^2; 2, 2, 2, 2)$ then G is the amalgamated free product of two copies of the infinite dihedral group. The result then holds by the same argument as in Theorem 5.1 of [BCR16]: a finite subgroup of the fundamental group of a graph of groups must be conjugate into one of the vertex groups, which here are the copies of $\mathbb{Z}/2$; the same result holds profinitely. Similarly if $X = (\mathbb{P}^2; 2, 2)$ then the fundamental group is an amalgamated free product.

In [BCR16] the triangle orbifolds were dealt with by passing to certain finite covers which decompose as amalgams, and whose fundamental group contains the torsion element of interest. However for Euclidean orbifolds, it may happen that no such covers exist—indeed no Euclidean orbifold whose fundamental group is an amalgam has any cone points of order greater than 2. We will instead exploit the fact that our triangle groups are virtually abelian. We give in detail the proof for the orbifold $X = (\mathbb{S}^2; 3, 3, 3)$. The other two triangle orbifolds are similar but involve checking more cases, so it would be uninformative to include the proofs.

Let $G = \left\langle a, b \,\middle|\, a^3, b^3, (ab)^3 \right\rangle$. We have a short exact sequence

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "\mathrm{id}"] & H_1(G) \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & \mathbb{Z}^2 \arrow[r] & G \arrow[r] & (\mathbb{Z}/3)^2 \arrow[r] & 1
\end{tikzcd}$$

The kernel N is contained in the translation subgroup of G. The translation subgroup is generated by the translations $x = a^{-1}b$ and $y = ba^{-1}$. The action of conjugation is

$$x^a = x^b = y^{-1}x^{-1}, \quad x^{a^{-1}} = x^{b^{-1}} = y \quad \text{etc}$$

The subgroup $N = \langle\!\langle aba^{-1}b^{-1} \rangle\!\rangle$ is then generated by the elements $u = y^{-1}x$, $v = x^3$. To guide our calculations, note that an element au^rv^s of G acts on the plane by rotation about the centroid of a certain triangle, whose location turns out to be that of the rotation a translated by $u^{r+s}v^{r/3}$. So in G, we have

$$au^rv^s = a^{u^{r+s}v^{r/3}} = a^{y^{-r-s}x^s}$$

and we expect similar equations to hold in $\widehat{G}$.

We have a short exact sequence for $\widehat{G}$ induced from the one above:

$$1 \to \widehat{\mathbb{Z}}^2 \to \widehat{G} \to (\mathbb{Z}/3)^2 \to 1$$

and see that any torsion element of $\widehat{G}$ is of the form $a^ib^ju^\rho v^\sigma$ where $i, j = 0, 1, 2$ are not both zero and $\rho, \sigma \in \widehat{\mathbb{Z}}$. For example, take $i = 1, j = 0$; the other cases are very similar. We now calculate

$$\begin{aligned} a^{y^{-\rho-\sigma}x^\sigma} &= x^{-\sigma}y^{\rho+\sigma} \cdot a \cdot y^{-\rho-\sigma}x^\sigma \\ &= a \cdot (x^{-\sigma}y^{\rho+\sigma})^a y^{-\rho-\sigma}x^\sigma \\ &= a \cdot (y^{-1}x^{-1})^{-\sigma}x^{\rho+\sigma}y^{-\rho-\sigma}x^\sigma \\ &= a \cdot y^{-\rho}x^\rho x^{3\sigma} = au^\rho v^\sigma \end{aligned}$$

So that torsion elements of this form are indeed conjugates of elements in G. The rest of the proof consists of similar calculations for other cases and can be safely omitted. □

4.3 Profinite rigidity of Seifert fibre spaces

In this section we analyse the profinite completions of Seifert fibre space groups, with the ultimate goal of proving the following result.

Theorem 4.3.1. *Let M_1 and M_2 be closed, orientable Seifert fibre spaces. Then $\widehat{\pi_1 M_1} \cong \widehat{\pi_1 M_2}$ if and only if one of the following holds:*

- *$\pi_1 M_1 \cong \pi_1 M_2$, so that M_1 and M_2 are homeomorphic except possibly if they have $\mathbb{S}^3$-geometry;*
- *M_1 and M_2 have the geometry $\mathbb{H}^2 \times \mathbb{R}$ and, for some hyperbolic surface S and some periodic automorphism ϕ of S, the 3-manifolds M_1 and M_2 are S-bundles over the circle with monodromies ϕ and ϕ^k respectively, where k is coprime to the order of ϕ.*

Wilton and Zalesskii [WZ17a, Theorem B] have shown that the profinite completion distinguishes Seifert fibre spaces from other 3-manifold groups, hence this theorem implies Theorem A from the Introduction. The non-trivial part of the 'if' direction of this theorem was proved by Hempel [Hem14, Theorem 1.1]. Alternatively one can apply the argument of Theorem 4.3.9 below to get a new proof.

The solution of the problem will proceed in several stages. Firstly, we will show that, except in the 'trivial' geometries, an isomorphism of profinite completions of Seifert fibre spaces will induce an automorphism of the profinite completion of the base orbifold group $\widehat{B}$, which the two Seifert fibre spaces will share; and furthermore that both Seifert fibre spaces will have the same Euler number (up to sign). We will then constrain the automorphism of $\widehat{B}$ and compute the action of such an automorphism on $H^2(\widehat{B})$. Intuitively we will be considering what can happen to the 'fundamental class' of the orbifold. We will then be able to conclude the result by considering the cohomology classes giving the Seifert fibre spaces as central extensions of $\widehat{B}$.

The 'trivial' geometries mentioned above are $\mathbb{S}^3$, $\mathbb{E}^3$, and $\mathbb{S}^2 \times \mathbb{R}$. They are trivial for the profinite rigidity problem in the sense that spherical manifolds have finite fundamental group, and there are only six and two orientable manifolds of the latter two geometries respectively, all distinguished by their first homology. For the rest of the section, a *generic* Seifert fibre space will mean any Seifert fibre space not of the above geometries.

We will be using heavily the fact that the subgroup generated by a regular fibre is central; this is only true for orientable base orbifolds so first note that we can reduce to this case as follows. Suppose that we have a closed Seifert fibre space M. The base orbifold group B has a canonical index 2 subgroup corresponding to the orientation cover of the underlying surface of the orbifold. This induces an index 2 cover of the Seifert fibre space. Note that this cover contains all the information needed to recover the original Seifert fibre space M: in particular, for each exceptional fibre of M with Seifert invariants (p, q) where $1 \leq q < p/2$ the cover has 2 exceptional fibres with the same invariant (p, q), and has no other exceptional fibres. Because the index 2 subgroup is uniquely determined, it will follow that any isomorphism of the profinite completions of two Seifert fibre space groups will induce an isomorphism for these characteristic covers. We may then apply the theorem in the case of orientable base orbifold and recover the original manifolds as described above.

When the Seifert fibre space has boundary, the base orbifold group itself does not distinguish orientable base orbifold from non-orientable, and hence has no obvious characteristic subgroup. However if we assume that the peripheral subgroups of the base orbifold groups are preserved under the isomorphism of profinite completions, we can collapse each of them to obtain a closed orbifold and take the canonical index 2 cover of this, and hence of the original orbifold, to recover the above situation.

4.3.1 Preservation of the fibre

We first prove that the subgroup given by the fibre is still essentially unique for most Seifert fibre spaces. In the statement of the theorem, a 'virtually central' subgroup Z of a group G will mean that either Z is central in G or that the ambient group G has an index 2 subgroup containing Z in which Z is central. The fibre subgroup of a Seifert fibre space group G is such a subgroup: it is central when the base orbifold is orientable, or is central in the index 2 subgroup of G corresponding to the orientation cover of a non-orientable base orbifold.

Theorem 4.3.2. *Let M and N be closed Seifert fibre spaces and suppose that $\widehat{\pi_1(M)}$ is isomorphic to $\widehat{\pi_1(N)}$. Call this common completion Γ. Then:*

1. *M and N have the same geometry;*

2. *Γ has a unique maximal virtually central normal procyclic subgroup unless the geometry of M is $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, or $\mathbb{E}^3$; and*

3. *If the geometry is* Nil, $\mathbb{H}^2 \times \mathbb{R}$, *or* $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, *then M and N have the same base orbifold and Euler number.*

Remark. The first conclusion of this theorem was already known for all closed 3-manifolds by the above-cited theorem [WZ17a, Theorem B]. The proof here, specific to Seifert fibre spaces, is different in some respects, so we include it for completeness.

Proof. As usual, spherical manifolds are distinguished by having finite fundamental groups, hence finite profinite completions. The four model geometries $\mathbb{E}^3$, Nil, $\mathbb{H}^2 \times \mathbb{R}$, and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ are contractible, so the fundamental groups of all such manifolds have cohomological dimension exactly 3. All compact $\mathbb{S}^2 \times \mathbb{R}$-manifolds are finitely covered by $\mathbb{S}^2 \times \mathbb{S}^1$ and hence have a finite index subgroup of cohomological dimension 1. All 3-manifold groups are good, so these facts are detected by the profinite completion, hence $\mathbb{S}^2 \times \mathbb{R}$ is distinguished from the other geometries. Henceforth assume that M has one of the four relevant geometries with contractible universal cover.

Now suppose that Γ has two virtually central normal procyclic subgroups, $\overline{\langle h \rangle}$ and $\overline{\langle \eta \rangle}$, where h is represented by a regular fibre of M and $\overline{\langle \eta \rangle}$ is not contained in $\overline{\langle h \rangle}$. We will show first that the base orbifold O is Euclidean. Passing to the quotient by $\overline{\langle h \rangle}$, the image of $\overline{\langle \eta \rangle}$ is a normal procyclic subgroup of $\widehat{\pi_1^{\mathrm{orb}}(O)}$. By Corollary 5.2 of [BCR16] and Proposition 4.2.4 above, profinite completions of non-positively curved orbifold groups have no finite normal subgroups, so $\overline{\langle \eta \rangle}$ persists as an infinite procyclic subgroup of $\widehat{\pi_1^{\mathrm{orb}}(O)}$. It also follows that the subgroup $\overline{\langle h \rangle}$ is still maximal even in the profinite completion—that is, it is not contained in some larger normal procyclic subgroup.

We can now pass to a finite index subgroup of Γ whose intersections with $\overline{\langle h \rangle}$ and $\overline{\langle \eta \rangle}$ are central and non-trivial, and then to a further finite index subgroup Δ so that the corresponding cover of M has base orbifold an orientable surface Σ covering O. After taking a quotient by $\Delta \cap \overline{\langle h \rangle}$, the image of $\overline{\langle \eta \rangle}$ now gives a non-trivial central subgroup of $\widehat{\pi_1 \Sigma}$. But the profinite completion of a surface group has no centre unless the surface is a torus (see [And74], [Nak94] or [Asa01]). Hence O is Euclidean.

The base orbifold Σ is now a torus. Then the finite index subgroup $\Delta \leq \Gamma$ corresponds to a cover $\widetilde{M} \to M$ where the base orbifold of $\widetilde{M}$ is a torus. Then we have

$$\pi_1 \widetilde{M} = \left\langle u_1, v_1, h \,\middle|\, [u_1, v_1] = h^{-e}, h \text{ central} \right\rangle$$

where e is the Euler number of $\widetilde{M}$. Every element of $\pi_1 \widetilde{M}$ has the form $u_1^r v_1^s h^t$ for some integers $r, s, t \in \mathbb{Z}$. It follows that every element of the completion Δ may be written in the form $u_1^\rho v_1^\sigma h^\tau$ for some $\rho, \sigma, \tau \in \widehat{\mathbb{Z}}$. One may directly compute that the centre of this group is exactly $\overline{\langle h \rangle}$ unless $e = 0$. Hence unless $e = 0$ we find $\eta \in \overline{\langle h \rangle}$ giving a contradiction. So $e = 0$, and by naturality of the Euler number the original space M also has trivial Euler number.

We now deal with the case where Γ has a unique maximal virtually central procyclic normal subgroup. Note that in this case, the isomorphism $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$ preserves $\overline{\langle h \rangle}$, and hence induces an isomorphism of the profinite completions of the base orbifold group. Then by Theorem 4.2.1 and Corollary 4.2.2 the manifolds M and N have the same base orbifold O.

If we now show that M and N have the same Euler number, then we are finished as the geometries are distinguished by base orbifolds and whether the Euler number is non-zero. Again pass to an index d subgroup Δ of Γ with the corresponding cover of M being $\widetilde{M} \to M$, where M has base orbifold a surface. Then we may compute that for both M and N, the Euler number is given, up to sign, by the size of the torsion part of $H_1(\widetilde{M})$ divided by d, because the Euler number has the naturality property in Proposition 4.1.1. First homology is a profinite invariant, hence M and N have the same Euler number and the proof is complete. $\square$

Recall that the Euler number of the Seifert fibre space was of the form

$$e = -(b + \sum \frac{\beta_i}{\alpha_i})$$

with b an integer. Thus given the base orbifold (hence the α_i) and the Euler number, the only further ambiguity is whether we can change the β_i by values δ_i (with δ_i not congruent to 0 modulo α_i) such that $\sum \delta_i/\alpha_i$ is an integer. By the Chinese Remainder Theorem, there is no such collection of δ_i when all the α_i are coprime. Hence we have the following corollary, in which we change notation to follow the usual conventions for cone points.

Corollary 4.3.3. *Let M be a Seifert fibre space whose base orbifold is an orbifold $(\Sigma; p_1, \ldots, p_k)$ where $p_1, \ldots, p_k$ are coprime. Then $\pi_1 M$ is distinguished by its profinite completion from all other 3-manifold groups.*

The above theorem was stated and proved for closed Seifert fibre spaces. A similar result holds for Seifert fibre spaces with boundary. Much of the above argument holds just as well when the Seifert fibre space has boundary, except that we must rule out some cases with more than one geometry, and the Euler number is no longer defined. Furthermore, surfaces are no longer determined by their profinite completion unless we have some information about the boundary.

Theorem 4.3.4. *Let M and N be Seifert fibre spaces with non-empty boundary. Suppose that $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. Call this common completion Γ. Then:*

1. *Γ has a unique maximal virtually central normal procyclic subgroup unless M (and hence N) is a solid torus, $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ or the orientable $\mathbb{I}$-bundle over the Klein bottle; and*

2. *except in these cases, the base orbifolds of M and N have the same fundamental group. If in addition M and N have the same number of boundary components then M and N have the same base orbifold.*

Proof. The only positive Euler characteristic orbifolds with boundary are the disc with possibly one cone point—the Seifert fibre space is then a fibred solid torus.

The only zero Euler characteristic orbifolds with boundary are the annulus (giving the Seifert fibre space $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$), the Möbius band and disc with two order 2 cone points (both giving the orientable $\mathbb{I}$-bundle over the Klein bottle).

These three spaces all have different profinite completions of fundamental groups; one is $\widehat{\mathbb{Z}}$, one is $\widehat{\mathbb{Z}}^2$ and the other is non-abelian. None of the Seifert fibre spaces with hyperbolic base orbifold have virtually abelian fundamental group, so we can safely proceed assuming M and N are not any of the three exceptional manifolds above.

Part 1 of the proposition now follows from the same argument as in Theorem 4.3.1, replacing "virtually a non-abelian surface group" with "virtually a non-abelian free group" to get the lack of central subgroups of the base orbifold group. Now the base orbifold groups have isomorphic profinite completions, so by Theorem 4.2.1 they are the same group.

For the final part, note that Γ detects whether the unique maximal virtually central normal subgroup $\langle h\rangle$ is genuinely central or merely virtually so and hence whether the base orbifold is orientable or not. This, together with the additional knowledge that M and N and hence their base orbifolds have the same number of boundary components resolves the ambiguity in the base orbifold. $\square$

4.3.2 Central extensions

A *central extension* of a group B by a (necessarily abelian) group A consists of a short exact sequence

$$1 \to A \to G \to B \to 1$$

where the image of A is contained in the centre of G. Two such extensions are regarded as *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 1 \\ & & \big\downarrow \mathrm{id} & & \big\downarrow \cong & & \big\downarrow \mathrm{id} & & \\ 1 & \longrightarrow & A & \longrightarrow & G' & \longrightarrow & B & \longrightarrow & 1 \end{array}$$

Equivalence classes of central extensions are classified by elements of $H^2(B; A)$. The proof of this fact proceeds directly via cochains, but for what follows it will also be convenient to have the following interpretation.

Let $B = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$ be a presentation for B, let F be the free group on the x_i, and R the normal subgroup generated by the r_j. From the Serre spectral sequence for the short exact sequence

$$1 \to R \to F \to B \to 1$$

we obtain the five-term exact sequence

$$0 \to H^1(B; A) \to H^1(F; A) \to (H^1(R; A))^F \to H^2(B; A) \to 0 = H^2(F; A)$$

where the third non-zero term denotes those elements of $H^1(R;A)$ invariant under the conjugation action of F—in fact this is the group $H^1(R/[R,F];A)$. Given an element $\zeta \in H^2(B;A)$, consider a lift to a map

$$\xi \in (H^1(R;A))^F = (\mathrm{Hom}(R,A))^F$$

Then a central extension of B by A is given by the 'presentation' (abusing notation slightly):

$$G = \big\langle Y, x_1, \ldots, x_n \,\big|\, S,\, Y \subseteq Z(G),\, r = \xi(r)\ \forall r \in R \big\rangle$$

where $A = \big\langle Y \big| S \big\rangle$. As we will now show, the condition that the natural map $A \to G$ is genuinely an inclusion is in fact equivalent to the invariance of ξ under the action of F.

If ξ is not F-invariant then there exists $r \in R$ and $f \in F$ such that $\xi(r)$ and $\xi(r^f)$ are distinct elements of A. However in G we have

$$1 = (r\xi(r)^{-1})^f = r^f \xi(r)^{-1} = \xi(r^f)\xi(r)^{-1}$$

because $\xi(r)$ is central in G. So in this case $A \to G$ is not injective.

On the other hand if ξ is F-invariant consider an element a of A which vanishes in G. Then in $A \times F$ we must have an expression of a as a product of conjugates of relations of the form $r^{-1}\xi(r)$. One may use the fact that R is normal in F and the invariance of the homomorphism ξ to rearrange this into an expression $a = r^{-1}\xi(r)$ in $A \times F$. The projection to F then shows $r = 1$ and $a = 1$ as required.

When ξ is F-invariant the above presentation is equivalent to the simpler presentation

$$G = \big\langle Y, x_1, \ldots, x_n \,\big|\, S,\, Y \subseteq Z(G),\, r_1 = \xi(r_1), \ldots, r_m = \xi(r_m) \big\rangle$$

The ambiguity under choice of lift of ζ to an element ξ is an element $\psi \in H^1(F;A)$. However this ambiguity corresponds precisely to changing the generating set of G by replacing each x_i by the element $x_i' = x_i \cdot \psi(x_i)$. Conversely if two such extensions G and G' (given by maps ξ and ξ') are isomorphic *by an isomorphism* Φ *fixing* B *and* A, then ξ and ξ' differ by $\psi \in H^1(F;A)$ given by $\psi(x_i) = x_i \cdot (\Phi(x_i))^{-1}$.

The question of when two central extensions G and G' of B by A given by $\zeta, \zeta' \in H^2(B;A)$ can be isomorphic while allowing arbitrary automorphisms for B and A is more subtle; one needs to analyse whether any automorphisms of B and A can carry ζ to ζ' by the induced maps on H^2. This will be the central issue in the proof of Theorem 4.3.1.

The above theory of central extensions also holds when we replace B by a profinite group $\widehat{B}$, provided that the abelian group A is finite so that the cohomology group $H^2(\widehat{B}, A)$ is reasonably well-behaved. See [RZ00b, Section 6.8]. In fact one can apply the same theory when A is an infinite profinite abelian group given conditions on the profinite group $\widehat{B}$—namely that this profinite group has type FP_2 (which is defined in essentially the same way as for discrete groups). In this case $H^2(\widehat{B}, A)$ is well-defined and has various naturality properties with respect to expressing A as an inverse limit $A = \varprojlim A_i$ of finite abelian groups [SW00, Section 3.7]. Since central extensions also behave well under inverse limits it follows that the correspondence of second cohomology classes with central extensions still holds.

The fundamental groups of generic Seifert fibre spaces (over orientable base) are central extensions

$$1 \to \mathbb{Z} \to G \to B \to 1$$

classified by an element $\eta_G \in H^2(B; \mathbb{Z})$, where $B = \pi_1^{\mathrm{orb}} O$ is the fundamental group of the base orbifold. The profinite completion of a generic Seifert fibre space group is a central extension of $\widehat{B}$ by the infinite group $\widehat{\mathbb{Z}}$. Profinite completions of Fuchsian groups do have property FP_2; this follows from goodness and Proposition 2.2.17. So we could in principle work directly with $H^2(\widehat{B}, \widehat{\mathbb{Z}})$. However to avoid the complications raised by the presence of $\widehat{\mathbb{Z}}$, we restrict to a finite coefficient group as follows. Note that since an isomorphism of profinite completions of two Seifert fibre space groups G and G' preserves this central subgroup $\widehat{\mathbb{Z}}$ by Theorem 4.3.2, and since $\widehat{\mathbb{Z}}$ has a unique index t subgroup, any isomorphism $\widehat{G} \cong \widehat{G}'$ induces an isomorphism

$$\Gamma = \widehat{G}/\overline{\langle h^t \rangle} \cong \widehat{G}'/\overline{\langle (h')^t \rangle} = \Gamma'$$

where Γ and Γ' are now central extensions of $\widehat{B}$ by $\mathbb{Z}/t$. Hence they are classified by elements ζ and ζ' of $H^2(\widehat{B}; \mathbb{Z}/t)$. But B is a good group, hence $H^2(\widehat{B}; \mathbb{Z}/t)$ is canonically isomorphic to $H^2(B; \mathbb{Z}/t)$ and ζ_G and $\zeta_{G'}$ are the images of η_G and $\eta_{G'}$

under the maps

$$H^2(B;\mathbb{Z}) \to H^2(B;\mathbb{Z}/t) \cong H^2(\widehat{B};\mathbb{Z}/t)$$

It remains to show that no automorphisms of $\widehat{B}$ and $\mathbb{Z}/t$ can carry ζ_G to $\zeta_{G'}$ under the induced maps on $H^2(\widehat{B};\mathbb{Z}/t)$ for all t unless the manifolds M_1 and M_2 are homeomorphic or are covered by the theorem of Hempel [Hem14].

Before moving on, let us calculate the cohomology classes η_G in terms of the five-term exact sequence for use later. For a Seifert fibre space over orientable base with symbol

$$(b, \Sigma; (p_1, q_1), \ldots, (p_r, q_r))$$

the fundamental group has presentation

$$\Big\langle a_1, \ldots, a_r, u_1, v_1, \ldots, u_g, v_g, h \,\Big|\, h \in Z(\pi_1 M),\, a_i^{p_i} h^{q_i},\, a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g] = h^b \Big\rangle$$

Let $1 \to R \to F \to B \to 1$ be the corresponding presentation of the base orbifold group. Now $R/[R,F]$ is in fact the free $\mathbb{Z}$-module on the relations $y_0 = a_1 \cdots v_g^{-1}$ and $y_i = a_i^{p_i}$ for $1 \le i \le r$. Comparing to above general theory we see that the cohomology class η_G is the image in $H^2(B;\mathbb{Z})$ of the map

$$y_0 \mapsto b, \quad y_i \mapsto -q_i$$

in $\mathrm{Hom}(R/[R,F],\mathbb{Z})$. The chain complexes in the following section make rigorous our treatment of $R/[R,F]$ as a free abelian group on these generators. The calculation is similar for the bounded case, except that the y_0 term does not appear.

4.3.3 Action on cohomology

We first constrain the possible automorphisms of base orbifold that we need to consider.

Proposition 4.3.5. *Let M_1 and M_2 be generic closed Seifert fibre spaces and let Φ be an isomorphism from $\widehat{\pi_1(M_1)}$ to $\widehat{\pi_1(M_2)}$. Let the base orbifold group be*

$$B = \big\langle\, a_1, \ldots, a_r, u_1, v_1 \ldots, u_g, v_g \,\big|\, a_1^{p_1}, \ldots, a_r^{p_r}, a_1 \cdots a_r \cdot [u_1, v_1] \cdots [u_g, v_g] \,\big\rangle$$

Then there is an automorphism of $\pi_1(M_2)$ such that after post-composing Φ with the corresponding automorphism of $\widehat{\pi_1(M_2)}$, the induced automorphism ϕ of $\widehat{B}$ maps each a_i to a conjugate of $a_i^{k_i}$, where k_i is coprime to p_i.

Proof. Given Theorem 4.3.2 this is a simple corollary of Theorem 4.2.3 and Proposition 4.2.4. The induced automorphism of $\widehat{B}$ from any given isomorphism of the $\widehat{\pi_1(M_i)}$ must induce a bijection on conjugacy classes of maximal torsion elements. Hence a_i is sent to a conjugate of $a_{\sigma(i)}^{k_i}$ for some permutation σ with $p_{\sigma(i)} = p_i$ and k_i coprime to p_i. Permuting the a_i under the map σ^{-1} is an automorphism of B, hence of $\widehat{B}$, so we can force σ to be the identity—on the level of the Seifert fibre spaces we are simply relabelling the exceptional fibres and exploiting the invariance of the fundamental group under such relabellings. $\square$

Note that this proposition works just as well when there is boundary.

Proposition 4.3.6. *If ϕ is an automorphism of $\widehat{B}$ as in Proposition 4.3.5, then there exists $\kappa \in \widehat{\mathbb{Z}}$ such that κ is congruent to k_i modulo p_i for all $1 \leq i \leq r$ and for any n the action of ϕ^* on $H^2(\widehat{B}; \mathbb{Z}/n)$ is multiplication by κ.*

Proof. We construct a partial resolution of $\mathbb{Z}$ by free $\mathbb{Z}B$-modules, transport this to a partial resolution of $\widehat{\mathbb{Z}}$ by free $\widehat{\mathbb{Z}}[\![\widehat{B}]\!]$-modules, and use this to compute the action on cohomology of the above automorphisms of B. Fix the presentation

$$B = \left\langle a_1, \ldots, a_r, u_1, v_1 \ldots, u_g, v_g \,\middle|\, a_1^{p_1}, \ldots, a_r^{p_r}, a_1 \cdots a_r \cdot [u_1, v_1] \cdots [u_g, v_g] \right\rangle$$

of B, let F be the free group on the generators a_i, u_i, v_i and let $R = \ker(F \to B)$.

Set $C_0 = \mathbb{Z}B$, interpreted as the free $\mathbb{Z}$-module on the vertices of the Cayley graph of B with B-action given by left translation on $\mathrm{Cay}(B)$. Let $\epsilon\colon \mathbb{Z}B \to \mathbb{Z}$ be the evaluation map.

Let $C_1 = \mathbb{Z}B\{x_i, \bar{u}_j, \bar{v}_j\}$, the free $\mathbb{Z}B$-module with generators x_i, $\bar{u}_j$ and $\bar{v}_j$ for $1 \leq i \leq r$ and $1 \leq j \leq g$. The generator x_i represents the edge in $\mathrm{Cay}(B)$ starting at 1 and labelled by a_i, and similarly $\bar{u}_j$ and $\bar{v}_j$ represent the edges labelled u_j and v_j. Thus C_1 is the space of linear combinations of paths in $\mathrm{Cay}(B)$, with B-action given by left-translation.

The boundary map $d_1\colon C_1 \to C_0$ sends each path to the sum of its endpoints, so that for example $x_i \mapsto a_i - 1 \in \mathbb{Z}B$. Certainly $\epsilon d_1 = 0$. Exactness at C_0 now follows by connectedness of the Cayley graph.

Let $C_2 = \mathbb{Z}B\{y_0, \ldots, y_r\}$. We can interpret C_2 as representing 'all the relations of B'—that is, all closed loops in the Cayley graph. The generator y_0 will represent the relation $a_1 \cdots v_g^{-1}$ in the above presentation, and y_i the relation $a_i^{p_i}$. Now define $d_2\colon C_2 \to C_1$ by mapping each generator to the loop in the Cayley graph representing it—for instance,

$$\begin{aligned} d_2(y_i) &= x_i + a_1 \cdot x_i + a_1^2 \cdot x_i + \cdots + a_i^{p_i - 1} \cdot x_i \\ d_2(y_0) &= x_1 + a_1 \cdot x_2 + \cdots + a_1 \cdots a_{r-1} \cdot x_r \\ &\qquad + a_1 \cdots a_r \cdot \bar{u}_1 + \cdots - a_1 \cdots a_r [u_1, v_1] \cdots [u_g, v_g] \bar{v}_g \end{aligned}$$

Any loop in the Cayley graph represents some element of R, which can be expressed as a product of conjugates of the relations in the above presentation. Left conjugation of a relation corresponds to left-translating the loop around the Cayley graph, so any such product of conjugates can be realised in the Cayley graph as a $\mathbb{Z}B$-linear combination of the $d_2(y_i)$. Hence $d_1 d_2 = 0$ and the image of d_2 is precisely the kernel of d_1.

Let us analyse the kernel of d_2. Let

$$s = \sum_i \sum_b n_b^i b \cdot y_i \in \ker(d_2)$$

where $\sum_b n_b^i b \in \mathbb{Z}B$ for each i. The coefficient of x_i in $d_2(s)$ is

$$0 = \sum_b n_b^0 b a_1 \cdots a_{i-1} + \sum_b n_b^i b (1 + a_i + \cdots a_i^{p_i - 1})$$

Multiplying on the right by $(a_i - 1)$ kills the second sum. Re-parametrising the first sum yields $n_{ba_i}^0 = n_b^0$ for all $b \in B$. If $r > 1$, the a_i generate an infinite subgroup of B. Since $\sum n_b^0 b$ is a finite linear combination this forces $n_b^0 = 0$ for all b. If $r = 1$, we can instead analyse the coefficient of u_i since $g > 0$ for a non-spherical orbifold—or we can simply note that profinite rigidity in the cases $r = 0$ or 1 was already covered by Corollary 4.3.3, so that we need not worry any further about them. We are left to conclude that $\sum_b n_b^i b(1 + a_i + \cdots a_i^{p_i - 1}) = 0$, hence $\sum_b n_b^i b$ is some multiple of $(a_i - 1)$ and the kernel of d_2 is spanned by $(a_i - 1)y_i$.

Now set $C_3 = \mathbb{Z}B\{z_1, \ldots, z_r\}$ and $d_3(z_i) = (a_i - 1) \cdot y_i$ to find an exact sequence

$$C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}$$

i.e. a partial resolution of $\mathbb{Z}$ by free $\mathbb{Z}B$-modules as desired.

By Proposition 2.2.17 we have a partial resolution

$$\widehat{C}_3 \to \widehat{C}_2 \to \widehat{C}_1 \to \widehat{C}_0 \to \widehat{\mathbb{Z}}$$

where each $\widehat{C}_i$ is the free $\widehat{\mathbb{Z}}[\![\widehat{B}]\!]$-module on the same generators as C_i and the boundary maps are defined by the same formulae on these generators. We can thus use this resolution to compute the first and second (co-)homology on $\widehat{B}$.

Let $\phi \colon \widehat{B} \to \widehat{B}$ be an automorphism of $\widehat{B}$ as in Proposition 4.3.5. Construct maps $\phi_\sharp \colon \widehat{C}_i \to \widehat{C}_i$ for $i = 1, 2$ as follows. Lift ϕ to $\widetilde{\phi} \colon \widehat{F} \to \widehat{F}$ such that

$$\widetilde{\phi}(a_i) = (a_i^{k_i})^{g_i^{-1}}$$

for some $g_i \in \widehat{F}$. Write the image of each generator of $\widehat{F}$ under $\widetilde{\phi}$ as a limit of words on these generators, then map the corresponding generator of $\widehat{C}_1$ to the associated limit of paths in the Cayley graph. To define $\phi_\sharp$ on $\widehat{C}_2$, note that each relation of $\widehat{B}$ is mapped to an element of $\overline{R}$ under $\widetilde{\phi}$, hence can be written as a (limit of) products of conjugates of relations; now map this to an element of $\widehat{C}_2$ just like before. We have made a choice of expression of an element of $\overline{R}$ in terms of conjugates of relations—the ambiguity is by construction an element of $\ker(\widehat{d}_2) = \mathrm{im}(\widehat{d}_3)$, which image will soon vanish. For definiteness, choose

$$\phi_\sharp(y_i) = k_i g_i \cdot y_i \quad (1 \leq i \leq r)$$

coming from the obvious expression of $\widetilde{\phi}(a_i^{p_i})$ from above. Because the map on $\overline{R}$ was induced by the map on $\widehat{F}$ used to define $\phi_\sharp \colon \widehat{C}_1 \to \widehat{C}_1$, we get a commuting diagram

$$\begin{tikzcd}
\widehat{C}_3 \arrow[r] & \widehat{C}_2 \arrow[r] \arrow[d, "\phi_\sharp"] & \widehat{C}_1 \arrow[r] \arrow[d, "\phi_\sharp"] & \widehat{C}_0 \\
\widehat{C}_3 \arrow[r] & \widehat{C}_2 \arrow[r] & \widehat{C}_1 \arrow[r] & \widehat{C}_0
\end{tikzcd}$$

Now apply the functor $\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} -$ to the above diagram (that is, factor out the action of $\widehat{B}$) to get a commuting diagram

$$\begin{array}{ccccccc}
\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} \widehat{C}_3 & \xrightarrow{0} & \widehat{\mathbb{Z}}\{y_0, \dots, y_r\} & \xrightarrow{d_2'} & \widehat{\mathbb{Z}}\{x_i, \bar{u}_j, \bar{v}_j\} & \xrightarrow{0} & \widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} \widehat{C}_0 \\
& & \downarrow{\scriptstyle \phi_\sharp} & & \downarrow{\scriptstyle \phi_\sharp} & & \\
\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} \widehat{C}_3 & \xrightarrow{0} & \widehat{\mathbb{Z}}\{y_0, \dots, y_r\} & \xrightarrow{d_2'} & \widehat{\mathbb{Z}}\{x_i, \bar{u}_j, \bar{v}_j\} & \xrightarrow{0} & \widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} \widehat{C}_0
\end{array}$$

with the rows no longer exact, but with the maps marked as zero becoming trivial because the image of each generator of the chain group had a factor $(a_i - 1)$. We have some good control over the maps in the above, viz.

$$\begin{aligned}
\phi_\sharp(x_i) &= k_i x_i \\
\phi_\sharp(y_i) &= k_i y_i \\
d_2'(y_0) &= x_1 + \cdots + x_r \\
d_2'(y_i) &= p_i x_i
\end{aligned}$$

If $\phi_\sharp(y_0) = \kappa y_0 + \sum \mu_i y_i$, then tracking this around the diagram we find

$$\kappa + p_i \mu_i = k_i$$

for all i.

For $n \in \mathbb{N}$, we now apply $\mathrm{Hom}_{\widehat{\mathbb{Z}}}(-, \mathbb{Z}/n)$ to the above diagram, to get a commuting diagram

$$\begin{array}{ccccc}
\xleftarrow[0]{} & (\mathbb{Z}/n)\{y_0^*, \dots, y_r^*\} & \xleftarrow[d^2]{} & (\mathbb{Z}/n)\{x_i^*, \bar{u}_j^*, \bar{v}_j^*\} & \xleftarrow[0]{} \\
& {\scriptstyle \phi^\sharp}\uparrow & & {\scriptstyle \phi^\sharp}\uparrow & \\
\xleftarrow[0]{} & (\mathbb{Z}/n)\{y_0^*, \dots, y_r^*\} & \xleftarrow[d^2]{} & (\mathbb{Z}/n)\{x_i^*, \bar{u}_j^*, \bar{v}_j^*\} & \xleftarrow[0]{}
\end{array}$$

in which the homology of each row gives $H^2(\widehat{B}; \mathbb{Z}/n)$ and $\phi^\sharp$ gives an action on this cohomology group.

First let us note that this action is genuinely the functorial map ϕ^* induced by ϕ. By construction $\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} \widehat{C}_2$ is the free $\widehat{\mathbb{Z}}$-module on our relations. In this construction for the discrete group, this would be $R/[R, F]$. In the profinite world, $\overline{R}/[\overline{R}, \widehat{F}]$ may not be free abelian, as not every closed subgroup of a free profinite group is free; however we do get a canonical surjection

$$\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} \widehat{C}_2 \twoheadrightarrow \overline{R}/[\overline{R}, \widehat{F}]$$

since our chosen set of relations is a generating set for this latter group. But now the map $\phi_\sharp$ on $\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{B}]\!]} \widehat{C}_2$ is easily seen to induce the natural map on $\overline{R}/[\overline{R}, \widehat{F}]$ given by $\widetilde{\phi}$; and naturality of the quotient map

$$H^1(\overline{R}/[\overline{R}, \widehat{F}]; \mathbb{Z}/n) \twoheadrightarrow H^2(\widehat{B}; \mathbb{Z}/n)$$

coming from the five-term exact sequence shows that $\phi^\sharp$ will indeed give the correct action on H^2.

Finally, we can compute this action on $H^2(\widehat{B}; \mathbb{Z}/n)$. We have from above

$$\begin{aligned}
\phi^\sharp(y_0^*) &= \kappa y_0^* \\
\phi^\sharp(y_i^*) &= \mu_i y_0^* + k_i y_i^* \\
d^2(x_i^*) &= y_0^* + p_i y_i^* \\
d^2(\bar{u}_i^*) &= 0 = d^2(\bar{v}_i^*)
\end{aligned}$$

so that, given a cochain $\zeta = by_0^* - \sum_i q_i y_i^*$, we have

$$\begin{aligned}
\phi^*([\zeta]) = [\phi^\sharp(\zeta)] &= [(\kappa b - \sum q_i \mu_i) y_0^* - \sum q_i k_i y_i^*] \\
&= [\kappa(by_0^* - \sum q_i y_i^*) - \sum q_i \mu_i (y_0^* + p_i y_i^*)] \\
&= \kappa[\zeta]
\end{aligned}$$

as required. □

Proof of Theorem 4.3.1. Recall that we have reduced to the case of orientable base orbifold. As discussed in Section 4.3.2, our manifolds M_1 and M_2 are determined by cohomology classes $\eta_1, \eta_2 \in H^2(B; \mathbb{Z})$. If M_1 is given by the Seifert symbol

$$M_1 = (b, \Sigma; (p_1, q_1), \ldots, (p_r, q_r))$$

then as a cochain in the basis $y_0^*, \ldots, y_r^*$ of $\mathrm{Hom}_{\mathbb{Z}B}(C_2, \mathbb{Z})$ where $C_\bullet$ is the partial resolution defined above, we have (see Section 4.3.2)

$$\eta_1 = [by_0^* - \sum_{1 \le i \le r} q_i y_i^*]$$

and similarly for η_2. From these we get cohomology classes $\zeta_{i,n} \in H^2(\widehat{B}; \mathbb{Z}/n)$ for $i = 1, 2$. Suppose that $\Phi\colon \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is an isomorphism. Then by Theorem 4.3.2 the isomorphism Φ decomposes as an isomorphism of short exact sequences and so $\zeta_{1,n}$ is carried to $\zeta_{2,n}$ by the actions of $\mathrm{Aut}(\widehat{B})$ and $\mathrm{Aut}(\widehat{\mathbb{Z}})$. By the previous proposition both these actions are now known to be multiplication by scalars so there exists $\kappa \in \widehat{\mathbb{Z}}$ such that $\kappa\zeta_{1,n} = \zeta_{2,n}$ for all n.

Assume the M_i have non-zero Euler number $e > 0$ (by reversing the orientation on the fibres we can always force $e > 0$ for both manifolds). Choose $n = me\prod p_i$ for some integer m and define a group homomorphism $E\colon H^2(\widehat{B}; \mathbb{Z}/n) \to \mathbb{Z}/n$ by

$$E\left(\sum t_i y_i^*\right) = -t_0 \prod p_j + \sum_{i\neq 0} t_i \prod_{j\neq i} p_j$$

so that $E(\kappa\xi) = \kappa E(\xi)$. Since $e = -(b + \sum q_i/p_i)$, we have $E(\zeta_{1,n}) = e\prod p_j$ modulo n. Then

$$E(\kappa\zeta_{1,n} - \zeta_{2,n}) = (\kappa - 1)e \prod p_j = 0 \text{ modulo } n$$

Thus $\kappa\zeta_{1,n} = \zeta_{2,n}$ for all $n = me\prod p_j$ implies κ is congruent to 1 modulo m for all m. Hence $\kappa = 1$ and $\zeta_{1,n} = \zeta_{2,n}$ for all n, so that $\eta_1 = \eta_2$ and M_1 and M_2 are homeomorphic.

Otherwise the M_i have Euler number zero so that they are $(\mathbb{H}^2 \times \mathbb{R})$-manifolds. Let $n = \prod p_i$ and choose $k \in \mathbb{Z}$ such that k is congruent to κ modulo n. Then M_2 is a Seifert fibre space with zero Euler characteristic and Seifert invariants (p_i, kq_i); there is only one such Seifert fibre space, and Hempel [Hem14, Section 5] showed that these pairs of $\mathbb{H}^2 \times \mathbb{R}$ manifolds are precisely those surface bundles in the statement of the theorem. $\square$

The bounded case is rather easier, given sensible conditions on the boundary.

Definition 4.3.7. Let M and N be compact orientable 3-manifolds with incompressible boundary components $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_s$ respectively. An isomorphism $\Phi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$ *preserves peripheral systems* if $r = s$ and, possibly after relabelling, $\Phi(\widehat{P_i})$ is conjugate to $\widehat{Q}_i$ in $\widehat{\pi_1 N}$ for all i.

We remark that the closure of P_i in $\widehat{\pi_1 M}$ is indeed equal to $\widehat{P}_i$. This is a theorem of Hamilton [Ham01, Lemma 9] when the boundary is toroidal, and the most general version may be found as [Wil18c, Corollary 6.20].

Theorem 4.3.8. *Let M_1 and M_2 be compact orientable Seifert fibre spaces with non-empty boundary, and assume that there exists an isomorphism preserving peripheral systems.*

Let M_1 have Seifert invariants (p_i, q_i). Then M_2 is the Seifert fibre space with the same base orbifold and Seifert invariants (p_i, kq_i) for some $k \in \mathbb{Z}$ coprime to all p_i.

Proof. We can safely focus on hyperbolic base orbifolds since the other three Seifert fibre spaces with boundary are easily distinguished from these and each other by their first homology and hence by the profinite completion. As before we have already reduced to the case of orientable base orbifold. By Theorem 4.3.4 both Seifert fibre spaces share the same base orbifold O, and there is an induced automorphism of $\widehat{B} = \widehat{\pi_1^{\mathrm{orb}} O}$. As before, we can now consider the Seifert fibre spaces as being represented by elements of $H^2(\widehat{B}; \mathbb{Z}/n)$ for arbitrary n.

Take a presentation

$$B = \left\langle a_1, \ldots, a_r, b_1, \ldots, b_s, u_1, v_1, \ldots, u_g, v_g \,\middle|\, a_1^{p_1}, \ldots, a_r^{p_r} \right\rangle$$

for the base orbifold, where the a_i are the cone points and (the conjugacy classes of) the b_i give all but one of the boundary components. The remaining boundary component is

$$b_0 = \big(a_1 \cdots a_r \cdot b_1 \cdots b_s \cdot [u_1, v_1] \cdots [u_g, v_g]\big)^{-1}$$

As before, we are at liberty to permute cone points with the same order and to permute boundary components. Thus given Proposition 4.3.5 and the conditions of the theorem we may assume that the automorphism ϕ of $\widehat{B}$ induced by Φ is of the form

$$a_i \mapsto (a_i^{k_i})^{g_i},\ b_j \mapsto (b_j^{l_j})^{h_j}$$

for elements $g_i, h_j \in \widehat{B}$, $l_j \in \widehat{\mathbb{Z}}^{\times}$, and integers k_i coprime to p_i.

Now the induced automorphism of

$$H_1(\widehat{B}) = \widehat{B}_{\mathrm{ab}} = \bigoplus_{i=1}^{r} \mathbb{Z}/p_i \oplus \bigoplus_{j=1}^{s} \widehat{\mathbb{Z}} \oplus \widehat{\mathbb{Z}}^{2g}$$

sends the class of b_0 to

$$\phi_*([b_0]) = -\phi_*(\sum [a_i] + \sum_{j\neq 0} [b_j]) = -\sum k_i[a_i] - \sum_{j\neq 0} l_j[b_j]$$

and on the other hand to

$$l_0[b_0] = -\sum l_0[a_i] - \sum_{j\neq 0} l_0[b_j]$$

showing that all the k_i are congruent to l_0 modulo p_i and that all the l_i are equal to some $\mu \in \widehat{\mathbb{Z}}^{\times}$.

Let $n = \prod p_i$. Using essentially the same chain complex as in the closed case we can now compute that the action of ϕ on

$$H^2(\widehat{B}; \mathbb{Z}/n) = \bigoplus_{i=1}^{r} \mathbb{Z}/p_i$$

is multiplication by μ. Suppose that the map on fibres was given by multiplication by λ—that is, suppose we have a short exact sequence of isomorphisms

$$\begin{tikzcd}
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] \arrow[d, "\cdot\lambda"] & \widehat{\pi_1 M_1} \arrow[r] \arrow[d, "\Phi"] & \widehat{B} \arrow[r] \arrow[d, "\phi"] & 1 \\
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & \widehat{\pi_1 M_2} \arrow[r] & \widehat{B} \arrow[r] & 1
\end{tikzcd}$$

Let $\zeta_1 = (q_1, \ldots, q_r)$ be the cohomology class representing M_1 and let ζ_2 be the class representing M_2. We claim that $\zeta_2 = \mu^{-1}\lambda\zeta_1$, hence proving the theorem—where we take $k \in \mathbb{Z}$ to be congruent to $\mu^{-1}\lambda$ modulo all p_i.

Let Γ be a central extension of $\widehat{B}$ by $\widehat{\mathbb{Z}}$ corresponding to the cohomology class $\mu\zeta_2$. Since the action of ϕ^{-1} on cohomology is multiplication by μ^{-1}, and noting that this action is *contravariant*, we have

$$(\phi^{-1})^*(\mu\zeta_2) = \zeta_2$$

and hence we have a short exact sequence of isomorphisms

$$\begin{tikzcd}
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] \arrow[d, "\mathrm{id}"] & \widehat{\pi_1 M_2} \arrow[r] \arrow[d, "\cong"] & \widehat{B} \arrow[r] \arrow[d, "\phi^{-1}"] & 1 \\
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & \Gamma \arrow[r] & \widehat{B} \arrow[r] & 1
\end{tikzcd}$$

Pre-composing this with the short exact sequence coming from Φ there is an isomorphism

$$\begin{tikzcd}
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] \arrow[d, "\cdot\lambda"] & \widehat{\pi_1 M_1} \arrow[r] \arrow[d, "\cong"] & \widehat{B} \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & \Gamma \arrow[r] & \widehat{B} \arrow[r] & 1
\end{tikzcd}$$

which, since the map on $\widehat{\mathbb{Z}}$ gives a *covariant* map on cohomology, says that the cohomology class representing Γ is in fact $\lambda\zeta_1$. Hence

$$\lambda\zeta_1 = \mu\zeta_2$$

as was claimed. $\square$

We finally prove the converse to the last theorem. A mild adjustment to this argument, with the appropriate modification of the cohomology group considered, provides another proof of Hempel's theorem on closed Seifert fibre spaces. In the circumstances of Theorem 4.3.9 itself, we note that the original argument of [Hem14] would suffice to give an isomorphism of profinite fundamental groups, but would not give any information about the peripheral structure.

Theorem 4.3.9. *Let M_1 and M_2 be Seifert fibre spaces with non-empty boundary and with the same base orbifold O. Suppose M_1 has Seifert invariants (p_i, q_i) and M_2 has Seifert invariants (p_i, kq_i) where k is some integer coprime to every p_i. Then $\widehat{\pi_1 M_1} \cong \widehat{\pi_1 M_2}$ by an isomorphism preserving peripheral systems.*

Proof. Again it suffices to deal with the case of orientable base orbifold. Let $\Gamma_i = \widehat{\pi_1 M_i}$, let h_i be a generator of the centre of $\pi_1 M_i$, and let $\Gamma_{i,n} = \Gamma_i / \overline{\langle h^t \rangle}$ where $t = n \prod p_i$.

Note that for each i the $\Gamma_{i,n}$ form a natural inverse system with maps $\Gamma_{i,nm} \to \Gamma_{i,n}$. Furthermore, any map from Γ_i to a finite group must kill some power of h, and hence factors through some $\Gamma_{i,n}$. It follows that

$$\Gamma_i = \varprojlim_n \Gamma_{i,n}$$

Now k maps to an invertible element of $\mathbb{Z}/\prod p_i$, so there is some invertible element κ of $\widehat{\mathbb{Z}}$ congruent to k modulo each p_i. One can prove this by noting that by the Chinese Remainder Theorem the natural map $(\mathbb{Z}/mn)^\times \to (\mathbb{Z}/n)^\times$ is always surjective, hence so is the map $\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/n)^\times$.

As discussed in Section 4.3.2, $\Gamma_{i,n}$ is classified by an element

$$\zeta_i \in H^2(\widehat{B}; \mathbb{Z}/t) = \bigoplus_{j=1}^{r} \mathbb{Z}/p_j$$

where B is the base orbifold group. By assumption $\zeta_2 = k\zeta_1 = \kappa\zeta_1$. Multiplication by κ in the coefficient group gives an automorphism of the cohomology group taking ζ_1 to ζ_2, and hence induces an isomorphism $\Gamma_{1,n} \to \Gamma_{2,n}$. Moreover this isomorphism is compatible with the quotient maps $\Gamma_{i,nm} \to \Gamma_{i,n}$. Hence we have an isomorphism

$$\Gamma_1 = \varprojlim_n \Gamma_{1,n} \cong \varprojlim_n \Gamma_{2,n} = \Gamma_2$$

as required. Since the induced automorphism of $\widehat{B}$ is the identity it follows that the isomorphism preserves peripheral systems. $\square$

4.4 Distinguishing Seifert fibre spaces among 3-manifolds

We have already seen that Wilton and Zalesskii [WZ17a] proved that the profinite completion of a closed 3-manifold detects the geometry of the manifold. This theorem is a rather involved application of the theory of special cube complexes. The purpose of this section is to include a somewhat more elementary proof of the more limited result that Seifert fibre spaces are distinguished among 3-manifold groups by their profinite completions. Specifically we prove the following theorem.

Theorem 4.4.1. *Let M be a 3-manifold which is either a graph manifold or is virtually fibred. Assume M is not a Seifert fibre space and let p be any prime. Then there is some finite-sheeted cover $\widetilde{M}$ of M such that $G = \pi_1 \widetilde{M}$ is residually p and $\widehat{G}_{(p)}$ has trivial centre. In particular M does not have the same profinite completion as any Seifert fibre space group.*

Remark. In proving this theorem we will call upon some results from later in the thesis; however the reasoning is not circular and the theorem makes more sense in the present chapter. Furthermore the graph manifold case, as studied in Chapter 5, does not call upon results from [WZ17a].

Remark. The hypotheses of the theorem do actually cover all non-Seifert-fibred 3-manifolds by the celebrated Virtual Fibring Theorem(s) [Ago13, PW18]. Of course these theorems still rely on special cube complexes, so this theorem when stated for all 3-manifolds still does not have an entirely elementary proof. However the 'virtual pro-p' criterion proved could be regarded as being stronger than using the whole profinite completion, and may conceivably be quicker to compute algorithmically.

The proof will consist, in the fibred case, of a study of the centre of the profinite completion of a semidirect product and will call upon some residual properties of the mapping class group.

For the rest of this section let π be a collection of primes and let $\Theta = \mathbb{Z}_\pi$ be generated by t.

Definition 4.4.2. Let Δ be a profinite group and let $\psi \in \Delta$. We call ψ a *π-full element* of Δ if the composite map $\Theta \hookrightarrow \widehat{\mathbb{Z}} \to \Delta$ sending t to ψ is injective.

Remark. This notion is a refinement of the notion of 'infinite order'. For instance, when $|\pi| > 1$, if only one direct factor of $\Theta = \prod_{p\in\pi} \mathbb{Z}_p$ survives in Δ then one could consider this as saying that ψ has some form of 'torsion' even if the image of Θ is torsion-free.

Another way to give this definition is in terms of 'supernatural numbers' [Ser13, Section 1.3], which give a more precise notion of order for an element of a profinite group. Our π-fullness condition is then that $\prod_{p\in\pi} p^\infty$ divides the order of ψ.

The following lemma allows us to move easily between different notions of fullness.

Lemma 4.4.3. *Let Δ be a profinite group and let $\psi \in \Delta$. Then ψ is π-full if and only if ψ is p-full for every $p \in \pi$.*

Proof. Since every finite cyclic π-group splits canonically as a direct product of p-groups over $p \in \pi$, the image of $\Theta = \prod_{p\in\pi} \mathbb{Z}_p$ in Δ is the direct product of the images of the $\mathbb{Z}_p$ factors. Thus the map $\Theta \to \Delta$ is injective if and only if the restriction to each $\mathbb{Z}_p$ is injective. $\square$

For the following lemma, recall that the automorphism group of a finitely generated profinite group is itself naturally a profinite group [RZ00b, Corollary 4.4.4].

Proposition 4.4.4. *Let Γ be a finitely generated profinite group with trivial centre and let ψ be an element of $\mathrm{Aut}(\Gamma)$. Then the semidirect product $\Gamma \rtimes_\psi \Theta$ has trivial centre provided the image of ψ is π-full in $\mathrm{Out}(\Gamma)$.*

Proof. Assume ψ is π-full and suppose we have an element $\zeta = \gamma t^\kappa$ in the centre of $\Gamma \rtimes \Theta$ where $\gamma \in \Gamma, \kappa \in \mathbb{Z}_\pi$. Then for any element $\delta t^\lambda \in \Gamma \rtimes \Theta$ we have

$$\delta t^\lambda = t^{-\kappa}\gamma^{-1}\delta t^\lambda \gamma t^\kappa = t^{-\kappa}\gamma^{-1}\delta\psi^{-\lambda}(\gamma)t^{\lambda+\kappa} = \psi^\kappa\left(\gamma^{-1}\delta\psi^{-\lambda}(\gamma)\right)t^\lambda$$

Hence we have, for every δ and λ,

$$\delta = \psi^\kappa\left(\gamma^{-1}\delta\psi^{-\lambda}(\gamma)\right)$$

Setting $\delta = 1$ we find that $\psi^{-\lambda}(\gamma) = \gamma$ for all λ. Hence γ is fixed by ψ. Furthermore the equation above says that ψ^κ agrees with (left-)conjugation by γ and hence ψ^κ vanishes in $\mathrm{Out}(\Gamma)$. By π-fullness this implies $\kappa = 0$. Therefore $\zeta = \gamma$ is central in Γ and hence vanishes as required. $\square$

Proposition 4.4.5. *Let $G = \pi_1\Sigma$ where Σ is a closed surface of genus at least 2 and let $\psi \in \mathrm{Aut}(G)$ be an element which has infinite order in $\mathrm{Out}(G)$. Let π be a family of primes and let Γ be the pro-π completion of G. Then the image of ψ under the natural map*

$$\mathrm{Aut}(G) \to \mathrm{Aut}(\Gamma) \to \mathrm{Out}(\Gamma)$$

is π-full.

Proof. By Lemma 4.4.3 we must prove p-fullness for every $p \in \pi$. Let $p \in \pi$. It suffices to prove the proposition for $\pi = \{p\}$: if the composite map

$$\mathbb{Z}_p = \Theta \xrightarrow{\psi} \mathrm{Aut}(\Gamma) \to \mathrm{Out}(\Gamma) \to \mathrm{Out}(\widehat{G}_{(p)})$$

is injective then *a fortiori* the map to $\mathrm{Out}(\Gamma)$ is injective.

So assume $\pi = \{p\}$. Consider the commutative square

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[d] & \Theta \arrow[d] \\ \mathrm{Out}(G) \arrow[r, hook] & \mathrm{Out}(\Gamma) \end{tikzcd}$$

The injectivity of the map $\mathrm{Out}(G) \to \mathrm{Out}(\Gamma)$ follows from the arguments in [Par09] but is not explicitly stated there. The statement in [Par09, Theorem 1.4] is that $\mathrm{Out}(G)$ is virtually residually p; the injectivity of the given map is *a priori* stronger, but [Par09] in fact proves this stronger statement. A different proof of this injectivity is given by the construction of a 'pro-p curve complex' in [Wil17a, Theorem 4.6].

Now ψ has infinite order in $\mathrm{Out}(G)$. The image of Θ is therefore an infinite quotient of $\mathbb{Z}_p$. The only infinite quotient of $\mathbb{Z}_p$ is $\mathbb{Z}_p$ itself so ψ is indeed p-full as required. □

Corollary 4.4.6. *If ψ is an automorphism of $G = \pi_1\Sigma$ with infinite order in* $\mathrm{Out}\, G$ *then the centre of $\widehat{G} \rtimes_\psi \widehat{\mathbb{Z}}$ is trivial.*

Proof of Theorem 4.4.1. If M is a graph manifold then it has a finite cover $\widetilde{M}$ with a p-efficient JSJ decomposition by [AF13, Proposition 5.2]. The pro-p completion of $\pi_1\widetilde{M}$ acts acylindrically and faithfully on its JSJ tree and therefore is centre-free by Proposition 3.2.7.

Next suppose M is virtually fibred but not of geometry Sol. We may pass to a finite cover $\widetilde{M}$ whose fundamental group is a semidirect product $\pi_1\Sigma \rtimes_\psi \mathbb{Z}$ where Σ has genus at least 2, the automorphism ψ acts trivially on $H_1(\pi_1\Sigma; \mathbb{F}_p)$ and ψ has infinite order in $\mathrm{Out}(\pi_1\Sigma)$. By, for example, Proposition 6.1.10 the pro-p completion of $\pi_1\widetilde{M}$ is the semidirect product $\widehat{\pi_1\Sigma}_{(p)} \rtimes \mathbb{Z}_p$. The pro-$p$ completion of a surface group has no centre (as may be readily proved using profinite Bass-Serre theory and the p-efficient splittings in Chapter 6). The result now follows from Propositions 4.4.4 and 4.4.5.

Finally suppose M has Sol geometry. We may pass to a finite cover whose fundamental group is a semidirect product $\mathbb{Z}^2 \rtimes_\psi \mathbb{Z}$ where ψ acts trivially on $H_1(\mathbb{Z}^2; \mathbb{F}_p)$, has infinite order and fixes no element of $\mathbb{Z}^2$. Again the pro-p completion is simply $\mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$. It is elementary to prove that such a group has centre if and only if ψ fixes an element of $\mathbb{Z}_p^2$. This does not happen since

$$\det_{\mathbb{Q}_p}(\psi - \mathrm{id}) = \det_{\mathbb{Q}}(\psi - \mathrm{id}) \neq 0$$

and we are done.

The 'in particular' part of the theorem now follows easily. Suppose $\pi_1 M$ has the same profinite completion as some Seifert fibre space group. Then there is some finite-sheeted cover $\widetilde{M}$ whose profinite fundamental group is the same as that of a Seifert fibre space N with base orbifold a surface. Every finite sheeted cover of N also has this form and thus has residually p fundamental group—so its pro-p fundamental group has non-trivial centre. This contradicts the earlier part of the theorem (as applied to $\widetilde{M}$). $\square$

Chapter 5

JSJ Decompositions and Graph Manifolds

In this chapter all 3-manifolds will be assumed to be compact, connected, orientable and aspherical and have boundary a (possibly empty) collection of incompressible tori.

5.1 Background on JSJ decompositions

Let M be a 3-manifold. The *JSJ decomposition* of M is a graph of spaces $(X, M_\bullet)$ with realisation M with the following properties:

(1) the edge spaces M_e for $e \in E(X)$ form a collection of disjoint incompressible tori which are not isotopic to each other or to a boundary component of M;

(2) each vertex space is either Seifert fibred or is atoroidal (i.e. every incompressible torus is isotopic to a boundary component of the vertex space); and

(3) the number of edges of the graph is minimal with respect to property (2). That is, removing an edge space by gluing vertex spaces along it gives a graph-of-spaces with a vertex space which is neither atoroidal or Seifert fibred.

Such a decomposition exists and is unique up to isotopy [JS78, Joh79]. This graph of spaces gives rise to a graph-of-groups decomposition $(X, \pi_1 M_\bullet)$ of the fundamental group $\pi_1 M$.

If the graph X is non-trivial then the interiors of the atoroidal vertex spaces which are not Seifert fibred are cusped hyperbolic manifolds by Thurston's hyperbolisation theorem for Haken manifolds [Thu86]. We shall henceforward refer to these as the 'hyperbolic pieces'. The only manifolds with boundary which are both Seifert fibred and atoroidal are $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ and the orientable $\mathbb{I}$-bundle over a Klein bottle.

In the literature there are two inequivalent definitions of what is meant by a 'graph manifold'. In all cases, a *graph manifold* is a compact irreducible manifold with (possibly empty) toroidal boundary whose JSJ decomposition consists only of Seifert-fibred pieces. The JSJ decomposition is by definition minimal, so the fibrings of adjacent pieces of the decomposition never extend across the union of those pieces. Some authors additionally require that a graph manifold is not geometric—that is, it is not itself Seifert-fibred and is not a Sol manifold. An equivalent requirement is that the JSJ decomposition is non-trivial and the graph manifold is not finitely covered by a torus bundle. In this thesis we do impose this constraint. Furthermore we shall deal only with orientable graph manifolds.

At the other end of the spectrum, a manifold whose JSJ decomposition is non-trivial and has no Seifert-fibred pieces at all will be called *totally hyperbolic*. When the JSJ decomposition has at least one Seifert-fibred piece and at least one hyperbolic piece the manifold is called *mixed*.

For the rest of this chapter all 3-manifolds will be assumed to have non-trivial JSJ decompositions.

Let us now consider the Seifert fibre spaces with boundary which may arise as pieces of the JSJ decomposition of a 3-manifold. No positive characteristic base orbifolds can occur when the boundary is incompressible and non-empty. The Seifert fibre spaces with boundary which have Euclidean base orbifold are precisely $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$ and the orientable $\mathbb{I}$-bundle over a Klein bottle. The first of these can never arise in a JSJ decomposition of anything other than a torus bundle and we will henceforth ignore it. The fundamental group of the second has two maximal normal cyclic subgroups ('fibre' subgroups), one central (with quotient the infinite dihedral group) and one not central (with quotient $\mathbb{Z}$, here being the fundamental group of a Möbius band). We call the first of these fibre subgroups the *canonical* fibre subgroup. We refer to such pieces of the JSJ decomposition of a 3-manifold as *minor*, and to those Seifert fibred pieces with base orbifold of negative Euler characteristic as *major*—the terms ‘small Seifert fibre space’ and ‘large Seifert fibre space’ being already entrenched in the literature with a rather different meaning.

By definition all pieces of the JSJ decomposition of a graph manifold are Seifert fibre spaces, and by minimality of the decomposition the fibres of two adjacent Seifert fibred pieces do not match, even up to isotopy. Thus the canonical fibre subgroups of adjacent Seifert fibre spaces intersect trivially in the fundamental group of the graph manifold. If one piece is minor then neither of its two fibre subgroups intersect the fibre subgroup of the adjacent piece non-trivially.

Note that two minor pieces can never be adjacent. For each of these pieces has only one boundary component so the whole graph manifold would then be just two minor pieces glued together. Each has an index 2 cover which is a copy of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{I}$, so our graph manifold would be finitely covered by a torus bundle, and would thus be either a Euclidean, Nil or Sol manifold—contradicting our requirement that graph manifolds be non-geometric.

Many of these properties still hold in the profinite completion. For instance, when the base orbifold is of negative Euler characteristic, Theorem 4.3.4 guarantees that we still have a unique maximal procyclic subgroup which is either central or is central in an index 2 subgroup. We may directly check that the profinite completion of the Klein bottle group also still has just two maximal normal procyclic subgroups, one of which is central. Hence our notion of (canonical) fibre subgroup as a maximal procyclic group with the above property carries over to the profinite world.

5.2 The JSJ decomposition

Definition 5.2.1. Let M be a 3-manifold with JSJ decomposition $(X, M_\bullet)$. The *Seifert graph* $X_{\rm SF}$ of M is the full subgraph of X spanned by those vertices whose associated 3-manifold is a Seifert fibre space.

In this section we analyse the JSJ decomposition of an aspherical manifold, and show that ‘the Seifert-fibred part’ is a profinite invariant.

Theorem 5.2.2. *Let M and N be compact orientable aspherical 3-manifolds with (possibly empty) incompressible toroidal boundary. Let the respective JSJ decompositions be $(X, M_\bullet)$ and $(Y, N_\bullet)$. Assume that there is an isomorphism $\Phi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$. Then there is an isomorphism $\phi\colon X_{\mathrm{SF}} \cong Y_{\mathrm{SF}}$ such that $\Phi(\widehat{\pi_1 M_x})$ is a conjugate of $\widehat{\pi_1 N_{\phi(x)}}$ for every $x \in X_{\mathrm{SF}}$.*

If in addition M and N are graph manifolds then, after possibly post-composing Φ with an automorphism of $\widehat{\pi_1 N}$, the isomorphism Φ induces an isomorphism of JSJ decompositions in the following sense:

- *there is a graph isomorphism $\phi\colon X \to Y$;*
- *Φ restricts to an isomorphism $\widehat{\pi_1 M_x} \to \widehat{\pi_1 N_{\phi(x)}}$ for every $x \in V(X) \cup E(X)$.*

The proof will consist of a detailed analysis of the abelian subgroups of $\Gamma = \widehat{\pi_1 M}$ and their centralisers, via their actions on profinite trees. We begin with some technical 'malnormality'-type lemmas concerning profinite completions of vertex groups. These culminate in the acylindricity of the action of $\widehat{\pi_1 M}$ on its profinite Bass-Serre tree, which will pin down the location of non-cyclic abelian subgroups. From there we will state and prove some intrinsic properties of the subgroups of $\widehat{\pi_1 M}$ which arise from JSJ tori or from regular Seifert fibres in JSJ pieces, and prove that these are the only subgroups of $\widehat{\pi_1 M}$ with these properties. As the fundamental groups of Seifert-fibred pieces are, roughly speaking, the centralisers in $\widehat{\pi_1 M}$ of regular fibres, this allows us to reconstruct the Seifert-fibred portions of M from $\widehat{\pi_1 M}$.

We maintain the above notations for the rest of the section. Additionally let $S(\mathcal{G})$ be the standard graph of a graph of profinite groups $\mathcal{G}$, and let $\zeta\colon S(\mathcal{G}) \to X$ be the projection. Let Z_v denote the canonical fibre subgroup of a vertex stabiliser Γ_v ($v \in V(S(\mathcal{G}))$) which is a conjugate of the profinite completion of some Seifert fibre space group. By abuse of terminology we will refer to vertices of $S(\mathcal{G})$ as major, minor or hyperbolic when the corresponding vertex stabiliser is the profinite completion of a major or minor Seifert fibre space group or a cusped hyperbolic 3-manifold.

We will first show that the action on $S(\mathcal{G})$ is 4-acylindrical.

Remark. Wilton and Zalesskii use this fact in their paper [WZ10] for graph manifolds, as well as in [WZ14] and [HWZ12] more generally. The reader is warned that their original proof does however contain a gap. Specifically, their version of Lemma 5.2.3 below only allows for conjugating elements γ_i in the original group $\pi_1^{\mathrm{orb}}O$, rather than its profinite completion. There is a similar problem in the hyperbolic pieces, which we deal with in Lemma 5.2.6 below. The more recent published paper [WZ17a] by Wilton and Zalesskii contains a complete proof.

Lemma 5.2.3. *Let O be a hyperbolic 2-orbifold, and let l_1 and l_2 be curves representing components of ∂O. Let $\Delta = \widehat{\pi_1^{orb}O}$ and let Θ_i be the closure in Δ of $\pi_1 l_i$. Then for $\gamma_i \in \Delta$, either $\Theta_1^{\gamma_1} \cap \Theta_2^{\gamma_2} = 1$ or $l_1 = l_2$ and $\gamma_2\gamma_1^{-1} \in \Theta_1$.*

Proof. By conjugating by γ_1^{-1} we may assume that $\gamma_1 = 1$. For brevity we drop the subscript on $\gamma_2 = \gamma$. Note that $\Theta_1 \cap \Theta_2^{\gamma}$ is torsion-free, so it is sufficient to pass to a finite index subgroup Λ and show that $\Theta_1 \cap \Theta_2^{\gamma} \cap \Lambda = 1$.

Because O is hyperbolic, it has some finite-sheeted regular cover O' with more than two boundary components. Then given any pair of boundary components, $\pi_1^{\mathrm{orb}}O'$ has a decomposition as a free product of cyclic groups, among which are the two boundary components. Let Λ be the corresponding finite index normal subgroup of Δ. Note that for some set $\{h_i\}$ of coset representatives of $\pi_1^{\mathrm{orb}}O'$ in $\pi_1^{\mathrm{orb}}O$ (which give coset representatives of Λ in Δ), each group $\Lambda \cap \Theta_2^{h_i}$ is the closure of the fundamental group of a component of $\partial O'$. Set $\Theta_2' = \Theta_2^{h_i}$ where $\gamma = h_i\lambda$ for some $\lambda \in \Lambda$. Furthermore, if two boundary components of O are covered by the same boundary component of O', then they must coincide—that is, if $\Theta_1 \cap \Lambda = \Theta_2' \cap \Lambda$ then $\Theta_1 = \Theta_2'$.

By construction the intersections of Θ_1, Θ_2' with Λ are free factors of Λ—that is,

$$\Lambda = (\Theta_1 \cap \Lambda) \amalg (\Theta_2' \cap \Lambda) \amalg \Phi$$

where Φ is a free product of cyclic groups (unless $\Theta_1 = \Theta_2'$, when $\Lambda = (\Theta_1 \cap \Lambda) \amalg \Phi$). Let T be the standard graph for this free product decomposition of Λ. Then $\Theta_1 \cap \Lambda = \Lambda_v$ and $\Theta_2' \cap \Lambda = \Lambda_w$ for vertices $v, w \in T$. The action on T is 0-acylindrical because all edge stabilisers are trivial, so for $\lambda \in \Lambda$ the intersection

$$\Theta_1 \cap {\Theta_2'}^{\lambda} \cap \Lambda = \Lambda_v \cap \Lambda_{\lambda^{-1}\cdot w}$$

can only be non-trivial if $v = \lambda^{-1} \cdot w$, so that $\Theta_1 \cap \Lambda = \Theta_2' \cap \Lambda$ (hence $\Theta_1 = \Theta_2'$) and $\lambda \in \Theta_1$.

We have reduced to the case where $\Theta_2^{h_i} = \Theta_1$. The intersection of two distinct peripheral subgroups of $\pi_1^{\mathrm{orb}} O$ is trivial, and peripheral subgroups coincide with their normalisers in $\pi_1^{\mathrm{orb}} O$. But $\pi_1^{\mathrm{orb}} O$ is virtually free, so by Lemma 3.6 of [RZ96], the intersection of their closures in Δ is also trivial, so if $\Theta_2^{h_i} \cap \Theta_2 \neq 1$, then $\Theta_2^{h_i} = \Theta_2$, and h_i normalises Θ_2. Hence h_i normalises the intersection with $\pi_1^{\mathrm{orb}} O$, and so $h_i \in \Theta_2 = \Theta_1$ as required. $\square$

Lemma 5.2.4 (Proposition 5.4 of [WZ10])**.** *Let L be a major Seifert fibre space with boundary and let O be its base orbifold. Let $\Lambda = \widehat{\pi_1 L}$ and let Z be the canonical fibre subgroup of Λ. Let Δ_1 and Δ_2 be peripheral subgroups of Λ; that is, conjugates in Λ of the closure of peripheral subgroups of $\pi_1 L$. Then $\Delta_1 \cap \Delta_2 = Z$ unless $\Delta_1 = \Delta_2$.*

Proof. Certainly Z is contained in the intersection $\Delta_1 \cap \Delta_2$. If $\Delta_1 \cap \Delta_2$ is strictly larger than Z, then the images Θ_1 and Θ_2 of Δ_1 and Δ_2 in Λ/Z intersect non-trivially. Hence by the previous lemma $\Theta_1 = \Theta_2$, hence $\Delta_1 = \Delta_2$. $\square$

Lemma 5.2.5. *Let $e = [v, w]$ be an edge of $S(\mathcal{G})$ where v and w are Seifert fibred vertices. Then $Z_v \times Z_w \leq_{\mathrm{o}} \Gamma_e$, and so $Z_v \cap Z_w = 1$.*

Proof. After a conjugation in Γ, we may assume that e is an edge in the standard graph of the abstract fundamental group $\pi_1 M$, i.e. Γ_e is the closure in Γ of a peripheral subgroup of some $\pi_1 M_v$. Because $\widehat{H_1 \times H_2} \cong \widehat{H}_1 \times \widehat{H}_2$ for discrete groups H_1 and H_2, the result now follows from the corresponding result in the fundamental group $\pi_1 M$—the canonical fibre subgroups are distinct direct factors of the edge group, which therefore contains their product as a finite-index subgroup. $\square$

Lemma 5.2.6. *Let L be a hyperbolic 3-manifold with toroidal boundary. Let $\Lambda = \widehat{\pi_1 L}$, and let Δ_1 and Δ_2 be peripheral subgroups of L—that is, conjugates in Λ of the closure of maximal peripheral subgroups of $\pi_1 L$. Then $\Delta_1 \cap \Delta_2 = 1$ unless $\Delta_1 = \Delta_2$ and, moreover, each Δ_i is malnormal in Λ.*

Proof. Choose a basepoint for L and let $P_1, \ldots, P_n$ be the fundamental groups of the boundary components of L with this basepoint. These form a malnormal family of subgroups of $\pi_1 L$ and we wish to show that their closures are a malnormal family of subgroups of H. Suppose $\lambda \in \Lambda$ is such that $\widehat{P}_i \cap \widehat{P}_j{}^\lambda \neq 1$ and suppose that $i \neq j$ or $\lambda \notin \widehat{P}_i$. In the latter case let $q\colon \pi_1 L \to Q$ be a map to a finite group such that under its extension $\bar{q}\colon \Lambda \to Q$ to Λ, the image of λ does not lie in the image of $\widehat{P}_i$. Otherwise take q to be the map to the trivial group.

By Thurston's hyperbolic Dehn surgery theorem (see [BP12, Section E.5]) we may choose slopes p_k on the P_k such that Dehn filling along each slope gives a closed hyperbolic 3-manifold N; moreover, we may choose such slopes with enough freedom to ensure that the image of $\widehat{P}_i \cap \widehat{P}_j{}^\lambda$ is infinite in $\Lambda/\overline{\langle\langle p_1, \ldots, p_n\rangle\rangle}$. The images of the P_k in this hyperbolic 3-manifold group $\pi_1 N$ are a malnormal family of subgroups.

Now consider $K_0 = \ker(q) \cap \langle\langle p_1, \ldots, p_n\rangle\rangle$ and its closure in H. Note that $\pi_1 L/K_0$ is a hyperbolic virtually special group—indeed on the finite-index subgroup $\ker(q)$ we have just Dehn filled to get a closed hyperbolic manifold whose fundamental group is virtually special by a theorem of Agol building on work of Kahn-Markovic, Wise, Haglund-Wise and others (see Chapter 4 of [AFW15] for full citations). Furthermore the images of the P_k in $\pi_1 L/K_0$ are an almost malnormal family of subgroups. Then by [WZ17a, Corollary 3.4] their closures (i.e. the images of the $\widehat{P}_k$ in $H/\overline{K}_0$) are an almost malnormal family of subgroups of $\Lambda/\overline{K}_0$. But since the maps to $\widehat{\pi_1 N}$ and the map $\bar{q}$ both factor through the map $\Lambda \to \Lambda/\overline{K}_0$, we have that the images of $\widehat{P}_i$ and $\widehat{P}_j^\lambda$ intersect in an infinite subgroup, but $i \neq j$ and the image of λ does not lie in the image of $\widehat{P}_i$. This contradiction completes the proof. □

Lemma 5.2.7. *Let L be a hyperbolic 3-manifold with toroidal boundary. Let $\Lambda = \widehat{\pi_1 L}$ and let A be a subgroup of Λ isomorphic to $\widehat{\mathbb{Z}}^2$. Then A is conjugate into a peripheral subgroup of Λ.*

Proof. This follows from the proof of [WZ17a, Theorem 9.3]. □

Proposition 5.2.8. *The action of $\Gamma = \widehat{\pi_1 M}$ on the standard graph $S(\mathcal{G})$ is 4-acylindrical.*

Proof. Take a path of length 5 consisting of edges $e_0, \ldots, e_4$ joining vertices $v_0, \ldots, v_5$. Let M_i be the manifold $M_{\zeta(v_i)}$, where $\zeta\colon S(\mathcal{G}) \to X$ is the projection. If any of $M_1, \ldots, M_4$ is hyperbolic then the intersection of the two adjacent edge groups is trivial by Lemma 5.2.6. So assume all these M_i are Seifert fibre spaces. There are three cases to consider.

Case 1 Suppose both M_1 and M_2 are major Seifert fibre spaces. Then by Lemma 5.2.4 we have $\Gamma_{e_0} \cap \Gamma_{e_1} = Z_{v_1}$ and $\Gamma_{e_1} \cap \Gamma_{e_2} = Z_{v_2}$; but $Z_{v_1} \cap Z_{v_2}$ is trivial by Lemma 5.2.5. So $\bigcap_{i=0}^{2} \Gamma_{e_i}$ is trivial.

Case 2 Suppose M_1 is a major Seifert fibre space and M_2 is a minor Seifert fibre space. Let γ be an element of $\Gamma_{v_2} \smallsetminus \Gamma_{e_1}$. Then $v_3 = \gamma \cdot v_1$ is major and $Z_{v_3} = Z_{v_1}^{\gamma^{-1}}$. Then acting by γ sends e_1 to e_2 and fixes v_1, hence the intersection of all four edge groups, $Z_{v_1} \cap \Gamma_{e_2} \cap Z_{v_3}$, is a normal subgroup of Γ_2. Moreover, being the intersection of two direct factors of Γ_{e_2}, this intersection is trivial or a maximal copy of $\widehat{\mathbb{Z}}$ in Γ_{e_2}. Hence it is either trivial or is one of the two fibre subgroups of Γ_{v_2}. But the latter case is ruled out as neither of these fibre subgroups intersects Z_{v_1} or Z_{v_3} non-trivially.

Case 3 Suppose v_1 is minor. Then v_2 is major. If v_3 is major then the argument of Case 1 applies. If v_3 is minor then v_4 is major and relabelling by $i \mapsto 5 - i$ we are back in Case 2. $\square$

Corollary 5.2.9. *Let A be an abelian subgroup of Γ. Then $A \cong \mathbb{Z}_\pi$ for some set of primes or A fixes some vertex of $S(\mathcal{G})$.*

Proof. Apply Proposition 3.2.5 and Proposition 5.2.8. $\square$

Having located the $\widehat{\mathbb{Z}}^2$ subgroups, we proceed to distinguish them into two classes: those making up the edge groups between Seifert fibred pieces and those 'internal' to the vertex group in which they are contained or adjacent to a hyperbolic piece. This will be accomplished using the normalisers and centralisers of cyclic subgroups. Recall that the centraliser of a subgroup $\Delta \leq \Gamma$ is denoted $\mathcal{Z}_\Gamma(\Delta)$, and that $\mathcal{N}'_\Gamma(\Theta)$ denotes the restricted normaliser of a procyclic subgroup Θ as defined in Definition 3.2.6.

Definition 5.2.10. A *non-pathological torus* in Γ is a copy $A \leq \Gamma$ of $\widehat{\mathbb{Z}}^2$, not contained in any larger copy of $\widehat{\mathbb{Z}}^2$, with the following property: for every conjugate A^γ of A in Γ, either $A \cap A^\gamma = 1$, or $A \cap A^\gamma$ is a subgroup of $\widehat{\mathbb{Z}}$, or $A = A^\gamma$.

Definition 5.2.11. A procyclic subgroup $\Theta \cong \widehat{\mathbb{Z}}$ of Γ is *major fibre-like* if:

- Θ is a direct factor of some non-pathological torus of Γ;
- $\mathcal{Z}_\Gamma(\Theta)$ is not virtually abelian; and
- Θ is maximal with these properties.

The following result can be deduced from Lemma 5.2.4. However the following proof, being much more elementary, merits inclusion.

Proposition 5.2.12. *Let O be a hyperbolic 2-orbifold, and let c be an element of $\pi_1^{orb}(O)$ representing a boundary component of O. Let $\Delta = \widehat{\pi_1^{orb} O}$. Then the closed subgroup $\Theta \leq \widehat{\pi_1^{orb} O}$ generated by c is not contained in any strictly larger $\widehat{\mathbb{Z}}$-subgroup of Δ. Hence any abelian subgroup of Δ containing Θ is Θ itself.*

Proof. Suppose that $A \cong \widehat{\mathbb{Z}}$ strictly contains Θ. Note that the quotient of $\widehat{\mathbb{Z}}$ by any proper subgroup $\Theta \cong \widehat{\mathbb{Z}}$ is a direct product of finite groups, at least one of which is non-trivial. It follows that there exists a subgroup of A in which Θ is contained with index a prime p. Hence it suffices to show that c cannot be written as a p^{th} power δ^p. If c has this property in some quotient of Δ, then it also has this property in Δ itself. Thus it suffices to find, for each p, a finite quotient of $\pi_1^{\text{orb}} O$ in which the image of c is not a p^{th} power. We split into cases based on the topological type of O. Furthermore, by passing to a quotient it suffices to deal with the cases where c is the only boundary component of O.

Case 1 Suppose first that O either is orientable of genus at least 1 or that it is non-orientable with at least three projective plane summands. Then O has a punctured torus as a boundary-connected summand $O = O' \#_\partial (\mathbb{T}^2 \smallsetminus \{*\})$, and on passing to a quotient we may assume that O is a once-punctured torus. Then $c = [a, b]$ where $\{a, b\}$ is a free generating set of the free group $\pi_1^{\text{orb}} O$.

Suppose $p \neq 2$. Consider the mod-p Heisenberg group

$$\mathcal{H}_3(\mathbb{Z}/p) = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \leq \mathrm{SL}_3(\mathbb{Z}/p)$$

and map Δ to it by

$$a \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

so that $c = [a, b]$ maps to

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now in the Heisenberg group, the formula for an n^{th} power is

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nx & nz + (1 + 2 + \ldots + (n-1))xy \\ 0 & 1 & ny \\ 0 & 0 & 1 \end{pmatrix}$$

so that in particular all p^{th} powers vanish, noting that p is odd so divides $1 + \ldots + (p-1)$. So the image of c cannot possibly be a p^{th} power.

If $p = 2$, instead map to the mod-4 Heisenberg group $\mathcal{H}_3(\mathbb{Z}/4)$ by the same formulae. Then all squares have the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2x & 2z + xy \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{pmatrix}$$

If this were to equal to image of c, then x and y would have to be even, so that $2x + xy$ would be even, and therefore not equal to 1 giving a contradiction. So c cannot be a square in Δ either.

Case 2 If O is a punctured Klein bottle, possibly with cone points, then after factoring out the cone points we may assume that $\pi_1^{\text{orb}} O$ is a free group on two generators a and b with $c = a^2 b^2$. If $p \neq 2$ simply map to $\mathbb{Z}/p$ by $(a \mapsto 1, b \mapsto 1)$ to see that $c \mapsto 4$ is not a multiple of p. If $p = 2$ then map to the mod-4 Heisenberg group $\mathcal{H}_3(\mathbb{Z}/4)$ as above. This time the image of c is

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

If this were to be a square

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2x & 2z + xy \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{pmatrix}$$

then x and y would be odd, hence so would be $2z+xy$ which is thus non-zero. So the image of c is not a square.

Case 3 The remaining cases are either discs with at least two cone points or Möbius bands with at least one cone point. After factoring out any excess cone points, we may assume $\pi_1^{\mathrm{orb}}O$ is either $(\mathbb{Z}/m)*(\mathbb{Z}/n)$ or $(\mathbb{Z}/m)*\mathbb{Z}$, and c is expressed either as ab or ab^2 in these generators. Consider the kernels K of the maps to $(\mathbb{Z}/m)\times(\mathbb{Z}/n)$ or $\mathbb{Z}/m$ respectively. Some power c^k of c is then a boundary component of the cover corresponding to K, which is torsion-free and hence yields one of the 'high genus' cases above. Then c^k is not a p^{th} power in $\widehat{K}$. If p is coprime to m and n then, for $\delta\in\Delta$, we have $\delta^p\in\widehat{K}$ if and only if $\delta\in\widehat{K}$. So c^k is not a p^{th} power in Δ either (hence neither is c itself). Finally if p divides one of m or n (without loss of generality, if p divides m), then mapping Δ to the free factor $\mathbb{Z}/m$ sends c to 1, which is not divisible by p modulo m.

Finally note that since Δ is a free product of cyclic groups its abelian subgroups are (finite or infinite) procyclic (for instance by Theorem 3.1.17 and Proposition 3.1.20). The proof is now complete. $\square$

Proposition 5.2.13. *If v is a major vertex of $S(\mathcal{G})$, then we have $\mathcal{Z}_\Gamma(Z_v)\leq\Gamma_v$ and $\mathcal{N}'_\Gamma(Z_v)=\Gamma_v$.*

Proof. The centraliser of Z_v in Γ contains the non-abelian group $\mathcal{Z}_{\Gamma_v}(Z_v)$, an index 1 or 2 subgroup of Γ_v. By Proposition 3.2.7, the centraliser of Z_v in Γ is contained in a vertex group Γ_w. Any two distinct vertex groups intersect in at most $\widehat{\mathbb{Z}}^2$, so $v=w$ and the centraliser of Z_v in Γ is equal to the centraliser of Z_v in Γ_v. The reduced normaliser is similar, noting that all of Γ_v is contained in $\mathcal{N}'_\Gamma(Z_v)$ to get equality rather than containment. $\square$

Proposition 5.2.14. *If e is an edge of $S(\mathcal{G})$ then Γ_e is a non-pathological torus of Γ.*

Proof. Suppose first that $e=[v,w]$ where v is a major vertex. First note that Γ_e is a maximal copy of $\widehat{\mathbb{Z}}^2$. For if $A=\Gamma_e$ were contained in a larger copy A' of $\widehat{\mathbb{Z}}^2$, then A'

would centralise Z_v and hence be contained in Γ_v. The image of A' in Γ_v/Z_v would be abelian and contain a copy of $\widehat{\mathbb{Z}}$. By Proposition 3.2.5, maximal abelian subgroups of Γ_v/Z_v are finite or projective as it is a free product of procyclic groups. So the image of A' is a copy of $\widehat{\mathbb{Z}}$ properly containing a boundary component. By Proposition 5.2.12, this is impossible. So these edge groups are indeed maximal copies of $\widehat{\mathbb{Z}}^2$ in Γ.

Now suppose $e = [v, w]$ where v is hyperbolic and w hyperbolic or minor, the only cases where e has no major endpoints. Assume A is contained in some strictly larger copy A' of $\widehat{\mathbb{Z}}^2$. Then A' is contained in some vertex group Γ_u. The geodesic $[u, v]$ thus has stabiliser containing A. If $[u, v]$ has one edge then we are done as by Lemma 5.2.6 all edge groups are maximal in adjacent vertex groups. If $[u, v]$ has more than two edges, recall that the peripheral subgroups of Γ_v are malnormal by Lemma 5.2.6; from which we deduce that $[u, v]$ has at most two edges (those adjacent to a minor vertex w) and that A' is contained in these edge groups. Thus $A = A'$ and again A is maximal.

Suppose $\gamma \in \Gamma$ and that $\Gamma_e \cap \Gamma_e^\gamma$ is not a subgroup of $\widehat{\mathbb{Z}}$. By the arguments in Proposition 5.2.8 an intersection of two edge groups is at most $\widehat{\mathbb{Z}}$ unless the edges are equal or are the two edges incident to a minor vertex w. So either $\gamma^{-1} \cdot e = e$, whence $\gamma \in \Gamma_e$ and $\Gamma_e^\gamma = \Gamma_e$, or $\gamma \in \Gamma_w$ and Γ_e is normal in Γ_w so that again $\Gamma_e \cap \Gamma_e^\gamma = \Gamma_e$. $\square$

Proposition 5.2.15. *If v is a major vertex of $S(\mathcal{G})$ then Z_v is a major fibre-like subgroup of Γ.*

Proof. Note that for some edge $e = [v, w] \in S(\mathcal{G})$ the group Z_v is a direct factor of $A = \Gamma_e$, which is a non-pathological torus by Proposition 5.2.14 and has centraliser which is not virtually abelian. It remains to show that Z_v is maximal with these properties. Again, if Z_v were contained in a larger procyclic subgroup C contained in a copy A' of $\widehat{\mathbb{Z}}^2$, this A' would centralise Z_v and hence lie in Γ_v by Proposition 5.2.13. The image of A' in Γ_v/Z_v would then be infinite abelian, hence projective; but killing a non-maximal copy Z_v of $\widehat{\mathbb{Z}}$ inside A' would introduce torsion. This is impossible so Z_v is major fibre-like as claimed. $\square$

Proposition 5.2.16. *Let $\Theta \cong \widehat{\mathbb{Z}}$ be a subgroup of Γ. If Θ is major fibre-like then $\Theta = Z_v$ for some major vertex v. If Θ is contained in an edge group Γ_e then either $\Theta \leq Z_v$ for some (possibly minor) vertex v or $\mathcal{Z}_\Gamma(\Theta) \cong \widehat{\mathbb{Z}}^2$.*

Proof. Whether Θ is major fibre-like or contained in an edge stabiliser Γ_e, there is a non-pathological torus A containing Θ, choosing $A = \Gamma_e$ when $\Theta \leq \Gamma_e$. Note that A is contained in a vertex group by Corollary 5.2.9.

Since Θ is contained in a vertex group, by Proposition 3.2.7 its centraliser $\mathcal{Z}_\Gamma(\Theta)$ lies in some vertex group Γ_v. If v is hyperbolic then A is a peripheral subgroup in Γ_v by Lemma 5.2.7. So if $\gamma \in \mathcal{Z}_\Gamma(\Theta)$ then $A \cap A^\gamma \neq 1$ whence $\gamma \in A$ by malnormality (Lemma 5.2.6). If v is minor, so that A is an index 2 subgroup of Γ_v, the result is easy.

Suppose v is major. Note that since any maximal abelian subgroup of Γ_v/Z_v is procyclic, Z_v is a direct factor of A and hence either $\Theta \leq Z_v$ or $\Theta \cap Z_v = 1$. Assume Θ is not contained in Z_v, so that $\Theta \times Z_v$ is a subgroup of A isomorphic to $\widehat{\mathbb{Z}}^2$. We will show that Θ is not major fibre-like (a major fibre-like subgroup cannot be a proper subgroup of Z_v so this will prove the theorem), and that if Θ was contained in an edge group Γ_e then its centraliser is exactly A.

Consider first the index 1 or 2 subgroup $\mathcal{Z}_\Gamma(Z_v) = \Gamma'_v$ of Γ in which Z_v is central, which is contained in Γ_v with index 1 or 2 by Proposition 5.2.13. If $\gamma \in \Gamma'_v \cap \mathcal{Z}_\Gamma(\Theta)$, then x commutes with both generators of both Θ and Z_v so $A^\gamma \cap A$ is at least the subgroup $\Theta \times Z_v$ of A, which is isomorphic to $\widehat{\mathbb{Z}}^2$. Because A is non-pathological it follows that $A^\gamma = A$. Any action on $A \cong \widehat{\mathbb{Z}}^2$ which is trivial on a subgroup of A isomorphic to $\widehat{\mathbb{Z}}^2$ is also trivial on all of A. This may be seen by factoring $\widehat{\mathbb{Z}}^2$ as a product of the groups $\mathbb{Z}_p^2$ over all primes p and noting that the intersection of any other copy of $\widehat{\mathbb{Z}}^2$ has finite index in each factor $\mathbb{Z}_p^2$, and then applying some elementary algebra. Hence $\overline{\langle \gamma, A \rangle}$ is abelian. As before, taking the quotient by Z_v gives an infinite abelian subgroup of Γ_v/Z_v, which is thus procyclic. Hence $\overline{\langle \gamma, A \rangle}$ is a copy of $\widehat{\mathbb{Z}}^2$. Since A is non-pathological it follows that $\overline{\langle \gamma, A \rangle} = A$, hence $\gamma \in A$ and $\Gamma'_v \cap \mathcal{Z}_\Gamma(\Theta) = A$. So $\mathcal{Z}_\Gamma(\Theta)$ is virtually abelian (with index 1 or 2), and Θ is not major fibre-like.

Let $q\colon \Gamma_v \to \Gamma_v/Z_v$ be the quotient map. If Θ is a direct factor of a boundary component $A = \Gamma_e$ other than Z_v, then the image of Θ in Γ_v/Z_v generates a finite-index subgroup of a peripheral subgroup $\Delta = q(\Gamma_e)$ of the base orbifold. If $\gamma \in \mathcal{Z}_\Gamma(\Theta)$, then from above either $\gamma \in \Gamma_e$ or $\gamma^2 \in \Gamma_e$. Then $q(\gamma)$ commutes with the finite-index subgroup $q(\Theta)$ of Δ. Hence $\Delta^{q(\gamma)}$ intersects Δ non-trivially, and by Proposition 5.2.4 we find $q(\gamma) \in \Delta$ so that $\gamma \in q^{-1}(\Delta) = \Gamma_e$. □

This last proposition shows that the property of being a canonical fibre subgroup Z_v of a major vertex may be defined intrinsically, as a 'major fibre-like' subgroup. We will use this to show invariance of the Seifert-fibred part of the JSJ decomposition.

Proof of Theorem 5.2.2. Let M and N be compact orientable aspherical 3-manifolds with (possibly empty) incompressible toroidal boundary with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively. Let $\mathcal{G} = (X, \Gamma_\bullet)$ and $\mathcal{H} = (Y, \Delta_\bullet)$ be the corresponding graphs of profinite groups and let $S(\mathcal{G})$ and $S(\mathcal{H})$ be the standard graphs for these graphs of profinite groups. Suppose there exists an isomorphism $\Phi\colon \widehat{\pi_1 M} = \Gamma \to \Delta = \widehat{\pi_1 N}$. Let $\zeta\colon S(\mathcal{G}) \to X$ be the projection.

Let A be any maximal copy of $\widehat{\mathbb{Z}}^2$ in Γ. Then A is contained in a vertex stabiliser, and in the centraliser of any of its cyclic subgroups. In particular, if it has two major fibre-like subgroups, then it is contained in two distinct major vertex stabilisers, hence A is some edge stabiliser Γ_e and $\zeta(e) \in X_{\mathrm{SF}}$. Conversely if Γ_e is any edge stabiliser where $\zeta(e) \in X_{\mathrm{SF}}$ then it has two major fibre-like subgroups. If $e = [v, w]$ where v and w are major, then Z_v and Z_w are major fibre-like subgroups of Γ_e. If w is minor, then it has another adjacent vertex v' with $\Gamma_{[v,w]} = \Gamma_{[v',w]}$, and so Z_v and $Z_{v'}$ are major fibre-like subgroups of Γ_e. They are distinct, otherwise as in the proof of Proposition 5.2.8 they coincide with a fibre subgroup of Γ_w, giving a contradiction. Furthermore, the intersection of any three major vertex groups is at most cyclic, so Γ_e cannot contain three distinct major fibre-like subgroups.

Now construct an (unoriented) abstract graph Ξ as follows. The vertices of Ξ are the major fibre-like subgroups (which by Propositions 5.2.15 and 5.2.16 are precisely the fibres Z_v) and the edges of Ξ are those maximal $\widehat{\mathbb{Z}}^2$ subgroups containing two major fibre-like subgroups (i.e. the edge groups Γ_e with $\zeta(e) \in X_{\mathrm{SF}}$). The incidence maps are defined by inclusion. This incidence relation is not quite the same as incidence in $S(\mathcal{G})$; two major vertex groups separated in $S(\mathcal{G})$ by a minor vertex are now adjacent in Ξ. We now rectify this. All maximal cyclic subgroups of an edge group Γ_e have either have centraliser $\widehat{\mathbb{Z}}^2$ or are Z_v for some major or minor vertex v by Proposition 5.2.16; so for each edge group Γ_e with a third maximal procyclic subgroup with centraliser larger than $\widehat{\mathbb{Z}}^2$, subdivide the corresponding edge of Ξ to get a new graph Ξ' with a vertex representing the minor vertex group whose canonical fibre subgroup is contained in Γ_e. Clearly Ξ' is isomorphic to $\zeta^{-1}(X_{\mathrm{SF}})$ as an abstract graph. The Γ-action on Ξ' induced from $S(\mathcal{G})$ via this isomorphism is determined by conjugation of the Z_v. On the other hand, the graph Ξ' and Γ-action so constructed are invariants of the group, so the isomorphism $\Phi\colon \Gamma \cong \Delta$ yields an equivariant isomorphism of Ξ' with the corresponding object for Δ. Hence the quotient graphs X_{SF} and Y_{SF} are isomorphic as claimed. Furthermore corresponding vertex stabilisers (i.e. the profinite completions of the Seifert fibred pieces of M and N) are isomorphic. This completes the first part of the theorem.

Now assume that M and N are graph manifolds. Since $\Xi' \cong S(\mathcal{G})$ as an abstract graph, we get an equivariant isomorphism Ψ between $S(\mathcal{G})$ and $S(\mathcal{H})$, in the sense that

$$\Psi(\gamma \cdot v) = \Phi(\gamma) \cdot \Psi(v)$$

for all $\gamma \in \Gamma$ and $v \in V(S(\mathcal{G}))$. Note that this descends to an isomorphism

$$X = \Gamma \backslash S(\mathcal{G}) \cong \Delta \backslash S(\mathcal{H}) = Y$$

We now check that the morphism of graphs thus constructed is in fact continuous, hence an isomorphism in the category of profinite graphs. For by Lemma 3.1.13, $S(\mathcal{G})$ is the inverse limit of its quotients by finite index normal subgroups of Γ. However, for each such subgroup Λ, we know that Ψ induces a natural morphism of (abstract) graphs

$$\Lambda \backslash S(\mathcal{G}) \cong \Phi(\Lambda) \backslash S(\mathcal{H})$$

But these graphs, being finite covers of X and Y respectively (with covering groups Γ/Λ and $\Delta/\Phi(\Lambda)$ respectively) are finite, hence these morphisms are continuous; that is, we have isomorphisms of inverse systems

$$S(\mathcal{G}) \cong \varprojlim \Lambda\backslash S(\mathcal{G}) \cong \varprojlim \Phi(\Lambda)\backslash S(\mathcal{H}) \cong S(\mathcal{H})$$

so that our morphism Ψ is indeed continuous.

Note that by Proposition 3.2.7, the restricted normaliser of each Z_v is contained in a vertex group Γ_w; as it contains Γ_v which is not contained in any edge group, we find that $v = w$ and $\mathcal{N}'_\Gamma(Z_v) = \Gamma_v$. We now have an equivariant isomorphism $\Psi\colon S(\mathcal{G}) \to S(\mathcal{H})$ of profinite graphs such that

$$\Gamma_v \cong \Phi(\Gamma_v) = \Phi(\mathcal{N}'_\Gamma(Z_v)) = \mathcal{N}'_\Delta(\Phi(Z_v)) = \mathcal{N}'_\Delta(Z_{\Psi(v)}) = \Delta_{\Psi(v)}$$

for $v \in V(S(\mathcal{G}))$ and with each edge group the intersection of the adjacent vertex groups. This descends to an isomorphism

$$X = \Gamma\backslash S(\mathcal{G}) \cong \Delta\backslash S(\mathcal{H}) = Y$$

such that corresponding vertices and edges of X and Y have isomorphic associated groups; that is, we have an isomorphism $\mathcal{G} \cong \mathcal{H}$ of graphs of groups.

The fundamental group of a graph of profinite groups is well-defined independently of any choice of maximal subtree of T and section $T \to S(\mathcal{G})$. Thus, following an automorphism of Δ, we may assume that the isomorphism Φ sends each vertex group of $\mathcal{G}$ to the corresponding vertex group of $\mathcal{H}$. See Section 6.2 of [Rib17] for details. □

5.3 Graph manifolds versus mixed manifolds

The results of the previous section are seen to give very good information about graph manifolds. It is rather more subtle to detect the precise nature of any hyperbolic pieces that may exist by these methods. In this section we show that our analysis of the profinite completion does detect the presence of a hyperbolic piece. This can be seen as an extension of [WZ17a] in that we now know that the profinite completion determines which geometries arise in the geometric decomposition of a 3-manifold. Wilton and Zalesskii [WZ17b] have since published a proof that the profinite completion of a 3-manifold group detects the JSJ decomposition in its entirety.

Theorem 5.3.1. *Let M be a mixed or totally hyperbolic manifold and N be a graph manifold. Then $\pi_1 M$ and $\pi_1 N$ do not have isomorphic profinite completions.*

Proof. Let M and N have JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively and assume $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$. Then by Theorem 5.2.2, $X_{\rm SF} \cong Y_{\rm SF} = Y$. The Seifert graph of the graph manifold N is connected and non-empty. Furthermore any finite-index cover of a graph manifold is a graph manifold, so any finite-index cover of N has a connected, non-empty Seifert graph. Therefore the Seifert graphs of all finite covers of M are also non-empty and connected. We will show that this is impossible if M is not a graph manifold. Note that if M is totally hyperbolic then it has empty Seifert graph by definition and we are done.

So assume M is mixed. Let M_1 be a hyperbolic piece of M adjacent to $X_{\rm SF}$. Take a boundary torus T of M_1 lying between M_1 and a major Seifert fibred piece of M (if M has only minor Seifert fibred pieces then it has a double cover with empty Seifert graph, which is forbidden).

Note that some finite-sheeted cover of M_1 has more than one boundary component which projects to T. The JSJ decomposition of M is efficient by Theorem 2.1.19; therefore some finite-sheeted cover M' of M induces a (possibly deeper) finite-sheeted cover M_1' of M_1, which will still have more than one boundary component projecting to T. One such boundary torus T' is now non-separating in M'. For all preimages of T are adjacent to a Seifert fibred piece of M', so if T' were separating then the Seifert graph of M' would be disconnected, which is forbidden.

Now cut along T', take two copies of the resulting 3-manifold, and glue these together to get a degree two cover $\widetilde{M}$ of M'. Removing the two vertices corresponding to the copies of M_1' from the JSJ graph of $\widetilde{M}$ gives a disconnected graph, each of whose components contains a Seifert fibred vertex. Thus the Seifert graph of $\widetilde{M}$ is disconnected, giving the required contradiction. $\square$

5.4 Totally hyperbolic manifolds

Theorem 5.2.2 does not give any information about those manifolds whose JSJ decomposition has no Seifert fibred pieces at all. Recall that we call such manifolds 'totally hyperbolic'. In this section we show that the analysis in Section 5.2 does allow us to deduce some limited information about the JSJ decomposition of these manifolds, even without a way to detect what the vertex groups are. Specifically we will prove the following theorem.

Theorem 5.4.1. *Let M and N be totally hyperbolic manifolds with $\Gamma = \widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ and with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively. Then the graphs X and Y have equal numbers of vertices and edges and equal first Betti numbers.*

Proof. Note that by Proposition 3.2.5 and Lemma 5.2.7 the maximal copies of $\widehat{\mathbb{Z}}^2$ in Γ are precisely the conjugates of completions of the JSJ tori T of M. Thus immediately the profinite completion determines $|E(X)|$ as the number of such conjugacy classes. The action of conjugation on the homology group $H_2(\Gamma; \widehat{\mathbb{Z}})$ being trivial, each such conjugacy class gives an element of $H_2(\Gamma; \widehat{\mathbb{Z}})$: the image of a generator of $H_2(T; \mathbb{Z})$ under the maps

$$H_2(T; \mathbb{Z}) \to H_2(M; \mathbb{Z}) = H_2(\pi_1 M; \mathbb{Z}) \to H_2(\Gamma; \widehat{\mathbb{Z}})$$

This element η_T is well-defined up to multiplication by an invertible element of $\widehat{\mathbb{Z}}$. Furthermore it is either primitive or zero since the classes in $H_2(M; \mathbb{Z})$ have this property and

$$H_2(\Gamma; \widehat{\mathbb{Z}}) \cong \varprojlim H_2(\Gamma; \mathbb{Z}/n) \cong \varprojlim H_2(\pi_1 M; \mathbb{Z}/n) \cong \widehat{\mathbb{Z}}^{b_2(M)}$$

where the middle isomorphism holds by Proposition 2.2.2. See [RZ00b, Proposition 6.5.7] for the inverse limit.

Notice that the first Betti number of the graph X is equal to the maximal number of edges that can be removed without disconnecting the graph. On the level of the 3-manifold M, this equals the size of a maximal collection of JSJ tori that are together non-separating.

If a collection $\{T_1, \ldots T_r\}$ of tori is separating, then one component of their complement provides a 3-chain giving a homological relation between the various 2-classes η_{T_i}. Conversely, if the tori are non-separating then there exist loops l_i in M which meet the torus T_i exactly once transversely and do not intersect the other T_j; considering intersection numbers with the l_i gives maps $H_2(M; \mathbb{Z}) \to \mathbb{Z}$ which show that the η_{T_i} are independent. Hence a collection of tori is non-separating if and only if the corresponding homology classes $\eta_{T_1}, \ldots, \eta_{T_r} \in H_2(M;\mathbb{Z})$ generate a free abelian subgroup A of $H_2(M;\mathbb{Z})$ of rank exactly r.

The closure of such a subgroup $A = \mathbb{Z}^r$ of $H_2(M;\mathbb{Z}) \cong \mathbb{Z}^{b_2(M)}$ in $H_2(\Gamma;\widehat{\mathbb{Z}})$ is isomorphic to $\widehat{\mathbb{Z}}^r$. This closure $\overline{A}$ is precisely the closed subgroup generated by the η_{T_i}, hence the profinite completion detects whether a collection of tori is non-separating. Thus the Betti numbers of X and Y must be equal.

Finally, since X and Y are connected, the Betti number and the number of edges together determine the number of vertices. This completes the proof. □

5.5 The pro-p JSJ decomposition

Let p be a prime. We have focussed thus far on the full profinite completion of a graph manifold group because of the good separability results which hold for all graph manifolds, and because of the good profinite rigidity properties of Seifert fibre spaces. The pro-p topology on a graph manifold or Seifert fibre space is in general rather poorly behaved—indeed most are not even residually p. In this section we will note that when the pro-p topology is well-behaved, the arguments of the previous section still suffice to prove a pro-p version of Theorem 5.2.2. First let us discuss the pro-p topologies on Seifert fibre space groups. The following lemmas and proposition are well-known but are included for completeness.

Lemma 5.5.1 (Grünberg [Grü57], Lemma 1.5)**.** *Let G be a (discrete) group and suppose H is a subnormal subgroup of G with index a power of p. If H is residually p, so is G.*

Here 'subnormal' means that there is a chain of subgroups

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$$

each normal in the next (but not necessarily normal in G). We extend the correspondence between regular covers of a space and normal subgroups of its fundamental group and say that a cover is *subregular* if it corresponds to a subnormal subgroup of the fundamental group.

Lemma 5.5.2. *Let O be an orientable orbifold with non-positive Euler characteristic and such that each cone point of O has order a power of p. Then O has a subregular cover of degree a power of p which is a surface. Hence $\pi_1^{orb}O$ is residually p.*

It is a well-known fact that orientable surface groups are residually p for all p [Bau62].

Proof. We will suppose that O has no boundary, as this is the more difficult case. We will construct a normal subgroup of $\pi_1^{\text{orb}}O$ which is in some sense simpler than $\pi_1^{\text{orb}}O$; iterating this process will terminate in a subregular subgroup which is torsion-free, hence a surface group. A presentation for $\pi_1^{\text{orb}}O$ is

$$\left\langle a_1, \ldots, a_m, u_1, v_1, \ldots u_g, v_g \,\middle|\, a_i^{p^{n_i}} = 1, a_1 \cdots a_m [u_1, v_1] \cdots [u_g, v_g] = 1 \right\rangle$$

where the cone points have order $p^{n_1} \leq p^{n_2} \leq \ldots \leq p^{n_m}$. Consider the abelian p-group

$$A = \mathbb{Z}/(p^{n_1}) \times \cdots \times \mathbb{Z}/(p^{n_m}) \,/\, (1, 1, \ldots, 1)$$

and the map $\phi\colon \pi_1^{\text{orb}}O \to A$ sending each generator a_i to the i^{th} coordinate vector and sending u_j and v_j to 0. Let K be the kernel of ϕ. Then if there are at least two cone points of maximal order p^{n_m}, no power of any a_i (other than the identity) lies in the kernel of ϕ since no multiple of $(1, 1, \ldots, 1)$ has precisely one non-zero coordinate; hence K is torsion-free as required.

If there is only one cone point, then since the Euler characteristic is non-positive, $g > 0$ and we may pass to a regular cover of degree p with p cone points.

Otherwise, if $n_{m-1} < n_m$ and there are at least two cone points, then any torsion elements in K are conjugates of $a_m^{p^j}$ for some $j > 0$, so that their order is at most $p^{n_m - 1}$; thus the order of the maximal cone point of the cover corresponding to K is strictly smaller than that of O. Iterating this process will thus eliminate all cone points of O, giving the required subnormal subgroup. □

Proposition 5.5.3. *Let p be a prime. Let M be an orientable Seifert fibre space which is not of geometry $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. Then $\pi_1 M$ is residually p if and only if all exceptional fibres of M have order a power of p, and M has orientable base orbifold* *when $p \neq 2$. That is, M has residually p fundamental group precisely when its base orbifold O is $(\mathbb{Z}/p)$-orientable and has residually p fundamental group. Moreover, when this holds the following sequence is exact:*

$$1 \to \mathbb{Z}_p \to \widehat{\pi_1 M}_{(p)} \to \widehat{\pi_1^{orb} O}_{(p)} \to 1$$

where $\mathbb{Z}_p$ is generated by a regular fibre of M.

Proof. ($\Leftarrow$) By the above, such base orbifolds have a subregular cover of degree a power of p which is an orientable surface. Using this subregular cover we need only verify that if the base orbifold of M is an orientable surface Σ then the Seifert fibre space group is residually p. Since surface groups and free groups are p-good, for each n the map

$$H^2(\widehat{\pi_1 \Sigma}_{(p)}; \mathbb{Z}/p^n) \to H^2(\pi_1 \Sigma; \mathbb{Z}/p^n)$$

is an isomorphism. Hence the central extension

$$1 \to \mathbb{Z}/p^n \to \pi_1(M)/\langle h^{p^n} \rangle \to \pi_1 \Sigma \to 1$$

(where h represents a regular fibre of M) is the pullback of a central extension of $\widehat{\pi_1 \Sigma}_{(p)}$ by $\mathbb{Z}/p^n$ (see also the discussion following Proposition 2.2.4). Thus each quotient $\pi_1(M)/\langle h^{p^n} \rangle$ is residually p. This proves that $\pi_1 M$ is residually p and, moreover, that the canonical fibre subgroup is closed in the pro-p topology, so that we have the exact sequence as claimed.

($\Rightarrow$) Recall that a subgroup of a residually p group is residually p. Suppose first that O is non-orientable and $p \neq 2$. Then there exists $g \in \pi_1 M$ such that the subgroup of $\pi_1 M$ generated by g and h is isomorphic to $\mathbb{Z} \rtimes \langle g \rangle$ where the copy of $\mathbb{Z}$ is the fibre subgroup $\langle h \rangle$ and g acts by inversion. Then g^2 acts trivially on $\mathbb{Z} = \langle h \rangle$. In any finite p-group quotient of $\pi_1 M$, the subgroup generated by g^2 contains g, so the image of $[g, h]$ vanishes in any p-group quotient. Hence $\pi_1 M$ is not residually p.

Now let p be arbitrary and suppose some exceptional fibre a has order $p^n m$ where $m \neq 1$ is coprime to p. Let $b = a^{p^n}$. In any finite p-group quotient the image of b is some power of the image of $b^m = h$, which is central. So b is central in any finite p-group quotient of $\pi_1 M$. It follows that if $\pi_1 M$ were residually p then the centre of

$\pi_1 M$ would contain b, which it does not. So all exceptional fibres of M have order p^n for some n. $\square$

The pro-p completion of course does not determine the Seifert fibre space quite as strongly as the profinite completion. However once the fundamental group is residually p the techniques of [BCR16] and Chapter 4, combined with the above proposition, yield the following surprisingly strong results:

Theorem 5.5.4. *Let O_1 and O_2 be 2-orbifolds whose fundamental groups are residually p. If $\widehat{\pi_1^{orb} O_1}_{(p)} \cong \widehat{\pi_1^{orb} O_2}_{(p)}$ then $O_1 \cong O_2$.*

Theorem 5.5.5. *Let M_1 and M_2 be Seifert fibre spaces whose fundamental groups are residually p. If $\widehat{\pi_1 M_1}_{(p)} \cong \widehat{\pi_1 M_2}_{(p)}$ then:*

- *M_1 and M_2 have the same geometry;*
- *M_1 and M_2 have the same base orbifold; and*
- *the Euler numbers of M_1 and M_2 have equal p-adic norm.*

Moving on to graph manifolds, we must define what constitutes a 'nice' pro-p topology for a graph manifold group. A primary tool we used in Section 5.2 was the efficiency of the graph of groups representing a graph manifold group. We recall the precisely analogous property in the pro-p world.

Definition 5.5.6. A graph of discrete groups $(X, G_\bullet)$ is *p-efficient* if $\pi_1(X, G_\bullet)$ is residually p, each group G_x is closed in the pro-p topology on $\pi_1(X, G_\bullet)$, and $\pi_1(X, G_\bullet)$ induces the full pro-p topology on each G_x.

Needless to say, this property does not hold for the majority of graph manifolds; in particular all the Seifert fibred pieces must be of the form specified in Proposition 5.5.3. However, the study of graph manifolds whose JSJ decomposition is p-efficient is by no means vacuous, as shown by the following theorem:

Theorem (Aschenbrenner and Friedl [AF13], Proposition 5.2)**.** *Let M be a closed graph manifold and let p be a prime. Then M has some finite-sheeted cover with p-efficient JSJ decomposition.*

Notice that the profinite properties of graph manifolds used in Section 5.2 to prove Theorem 5.2.2 were:

- efficiency of the JSJ decomposition;
- Lemma 5.2.3 concerning intersections of boundary components; and
- Proposition 5.2.12 concerning maximality of peripheral subgroups.

For p-efficient graph manifolds, observe that in the proofs of the above propositions it suffices to use only regular covers of order a power of p and p-group quotients; and observe that the remainder of the results are applications of the above together with the theory of profinite groups acting on profinite trees, which works just as well (in fact better) in the category of pro-p groups. Hence the same arguments prove the following pro-p version of Theorem 5.2.2:

Theorem 5.5.7. *Let M and N be graph manifolds with p-efficient JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively. Assume that there is an isomorphism $\Phi\colon \widehat{\pi_1 M}_{(p)} \to \widehat{\pi_1 N}_{(p)}$. Then, after possibly performing an automorphism of $\widehat{\pi_1 N}_{(p)}$, the isomorphism Φ induces an isomorphism of JSJ decompositions in the following sense:*

- *there is an (unoriented) graph isomorphism $\phi\colon X \to Y$;*
- *Φ restricts to an isomorphism $\widehat{\pi_1 M_x}_{(p)} \to \widehat{\pi_1 N_{\phi(x)}}_{(p)}$ for every $x \in V(X) \cup E(X)$.*

5.6 Profinite rigidity of graph manifolds

We will now address the question of when two graph manifold groups can have isomorphic profinite completions. First let us recall the results on Seifert fibre spaces from Chapter 4 in a form amenable to our needs.

Definition 5.6.1. Let O be an orientable 2-orbifold with boundary, with fundamental group

$$B = \langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g \mid a_i^{p_i} \rangle$$

where the boundary components of O are represented by the conjugacy classes of the elements $e_1, \ldots, e_s$ together with

$$e_0 = (a_1 \cdots a_r e_1 \cdots e_s [u_1, v_1] \cdots [u_g, v_g])^{-1}$$

Then an *exotic automorphism of O of type μ* is an automorphism $\psi\colon \widehat{B} \to \widehat{B}$ such that $\psi(a_i) \sim a_i^\mu$ and $\psi(e_i) \sim e_i^\mu$ for all i, where $\sim$ denotes conjugacy in $\widehat{B}$. Similarly, let O' be a non-orientable 2-orbifold with boundary, with fundamental group

$$B' = \langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, \ldots, u_g \mid a_i^{p_i} \rangle$$

where the boundary components of O' are represented by the conjugacy classes of the elements $e_1, \ldots, e_s$ together with

$$e_0 = \left(a_1 \cdots a_r e_1 \cdots e_s u_1^2 \cdots u_g^2\right)^{-1}$$

Let $o\colon \widehat{B}' \to \{\pm 1\}$ be the orientation homomorphism of O'. Let $\sigma_0, \ldots, \sigma_s \in \{\pm 1\}$. Then an *exotic automorphism of O' of type μ with signs $\sigma_0, \ldots, \sigma_s$* is an automorphism $\psi\colon \widehat{B}' \to \widehat{B}'$ such that $\psi(a_i) \sim a_i^\mu$ and $\psi(e_i) = (e_i^{\sigma_i \mu})^{g_i}$ for all i where $o(g_i) = \sigma_i$ for all i.

Remark. The reader may find the term 'exotic automorphism of O' a little jarring as the automorphism really acts on $\widehat{B}$. This name was chosen to emphasize that this notion depends on an identification of the group as the fundamental group of a specific orbifold, with specific elements representing boundary components. For example, an exotic automorphism of a three-times punctured sphere is a rather different thing from an exotic automorphism of a once-punctured torus: even though both groups are free of rank 2, there are different numbers of boundary components to consider.

For the same reason, the canonical map to a cyclic group of order 2 giving the orientation homomorphism is well-defined: this is not a characteristic quotient of a free group, but is uniquely defined when an identification with an orbifold group is chosen.

Let us now recall for convenience the characterisation of Seifert fibre spaces with isomorphic profinite completions in the form that we require.

Theorem 5.6.2. *Let M and N be Seifert fibre spaces with non-empty boundary. Suppose $\Phi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$ is an isomorphism which preserves peripheral systems (as defined in Definition 4.3.7). Then:*

- *If M is a minor Seifert fibre space, then $M \cong N$.*
- *If M is a major Seifert fibre space, then the base orbifolds of M and N may be identified with the same orbifold O such that Φ splits as an isomorphism of short exact sequences*

$$\begin{tikzcd} 1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] \arrow[d, "\cdot\lambda"] & \widehat{\pi_1 M} \arrow[r] \arrow[d, "\Phi"] & \widehat{\pi_1^{orb} O} \arrow[r] \arrow[d, "\phi"] & 1 \\ 1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & \widehat{\pi_1 N} \arrow[r] & \widehat{\pi_1^{orb} O} \arrow[r] & 1 \end{tikzcd}$$

where λ is some invertible element of $\widehat{\mathbb{Z}}$ and ϕ is an exotic automorphism of O of type μ.

Hence if M has fibre invariants (p_i, q_i) and N has fibre invariants (p_i, q_i') with respect to some presentation of the base orbifold group B, then

$$q_i' \cong \kappa q_i \text{ modulo } p_i$$

for all i where $\kappa = \lambda/\mu$.

Remark. This theorem statement does not appear in Chapter 4 in quite this form. First we remark that this theorem follows immediately from the case of orientable base orbifold. Note that Φ preserves the canonical index 2 subgroup given by the centraliser of the fibre, so the corresponding 2-fold covers $\widetilde{M}$ and $\widetilde{N}$ are related in the above fashion, for some labelling of the boundary components of each. The result then follows, noting that an automorphism of the base which induces an exotic automorphism on the index 2 orientation subgroup is itself an exotic automorphism. The statement for orientable base follows immediately from Theorem 4.3.8 and its proof.

Definition 5.6.3. If M and N are as in the latter case of the above theorem, we say that (M, N) is a *Hempel pair of scale factor κ*, where $\kappa = \mu^{-1}\lambda$. Note that κ is only well-defined modulo the order of ϕ, which may be taken to be the lowest common multiple of the orders of the cone points of M. Note that a Hempel pair of scale factor ± 1 is a pair of homeomorphic Seifert fibre spaces.

We will first restrict attention to those graph manifolds whose vertex groups have orientable base space. Let us recall from the standard theory of graph manifolds how one obtains numerical invariants of a graph manifold M. Let the JSJ decomposition be $(X, M_\bullet)$. Fix an orientation on M and hence on each vertex space M_x. In this section we shall adopt the convention (from Serre) that each 'geometric edge' of a finite graph is a pair $\{e, \bar{e}\}$ of oriented edges, with $\bar{e}$ being the 'reverse' of e. Fix presentations in the standard form

$$\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g, h \mid a_i^{p_i} h^{q_i}, h \text{ central} \rangle$$

for each M_x ($x \in VX$). This determines an ordered basis $\{h, e_i\}$ for the fundamental group of each boundary torus of M_x, where the final boundary component is described by

$$e_0 = (a_1 \cdots a_r e_1 \cdots e_s [u_1, v_1] \cdots [u_g, v_g])^{-1}$$

The gluing map along an edge e (from the boundary component of $M_{d_0(e)}$ to that of $M_{d_1(e)}$) now takes the form of a matrix (acting on the left of a column vector)

$$\begin{pmatrix} \alpha(e) & \beta(e) \\ \gamma(e) & \delta(e) \end{pmatrix}$$

where $\gamma(e)$, the intersection number of the fibre of $d_0(e)$ with that of $d_1(e)$, is non-zero by the definition of a graph manifold. The number $\gamma(e)$ is well-defined up to a choice of orientation of the fibres of the two vertex groups. This matrix has determinant -1 because of the requirement that the graph manifold be orientable. Once an orientation of the fibre and base are fixed, the number $\delta(e)$ becomes independent of the choice of presentation, modulo $\gamma(e)$. Changing the orientations of fibre and base multiplies the matrix by -1. The 'section' of the fibre space determining $\delta(e)$ may be changed by a Dehn twist along an annulus which either joins two boundary components or joins the boundary component to a regular fibre. This however leaves the *total slope*

$$\tau(x) = \sum_{e: d_0(e) = x} \frac{\delta(e)}{\gamma(e)} - \sum \frac{q_i}{p_i}$$

of the space invariant. Note also that these quantities are all invariant under the conjugation action of the group on itself, i.e. it does not matter which conjugate of each edge group we consider.

In the following theorem we analyse the consequences of two graph manifold groups having isomorphic profinite completions. Later (Theorem 5.6.14) we will see that the conditions obtained are sufficient as well as necessary.

Theorem 5.6.4. *Let M and N be closed orientable graph manifolds with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$, where all major vertex spaces have orientable base orbifold. Suppose $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ and let $\phi\colon X \to Y$ be the induced graph isomorphism from Theorem 5.2.2.*

1. *If X is not a bipartite graph, then M is homeomorphic to N via a homeomorphism inducing the graph isomorphism ϕ.*

2. *Suppose X is bipartite on vertex sets R and B and assume that M and N are not homeomorphic. Then there exists an element $\kappa \in \widehat{\mathbb{Z}}^\times \smallsetminus \{\pm 1\}$ such that the following conditions hold, for some choices of orientations of the fibres of each major vertex group of M and N:*

   (a) For all $e \in EX$, we have $\gamma(e) = \gamma(\phi(e))$

   (b) For every vertex space of both M and N the total slope of that vertex is zero.

   (c1) For every vertex $r \in R$, $(M_r, N_{\phi(r)})$ is a Hempel pair of scale factor κ and for every edge e with $d_0(e) = r$ we have $\delta(\phi(e)) \equiv \kappa\delta(e)$ modulo $\gamma(e)$.

   (c2) For every vertex $b \in B$, $(M_b, N_{\phi(b)})$ is a Hempel pair of scale factor κ^{-1} and for every edge e with $d_0(e) = b$ we have $\delta(\phi(e)) \equiv \kappa^{-1}\delta(e)$ modulo $\gamma(e)$.

Remark. The conditions (a)–(c1) (or (c2)) in the above theorem have a rather neat interpretation in terms of the following object.

Definition 5.6.5. Let $M = (X, M_\bullet)$ be a closed graph manifold with the given JSJ decomposition, and let $x \in VX$. Define the *filled vertex space* $\overline{M_x}$ of M_x as follows.

For every edge e with $d_0(e) = x$, the fibre of $M_{d_1(x)}$ describes a meridian on the relevant boundary torus T_e of M_v. There is a unique way to glue in a solid torus along T_e with this meridian such that the Seifert fibration on M_v extends over the solid torus. Gluing in a solid torus in this way gives a closed Seifert fibre space $\overline{M_x}$.

Suppose that $(M_r, N_{\phi(r)})$ is a Hempel pair for scale factor κ and for simplicity of notation suppose that M_r and $N_{\phi(r)}$ have the same base orbifold O. Now the filled vertex space $\overline{M_r}$ has an exceptional fibre with invariants $(\gamma(e), -\delta(e))$ for each boundary torus e of M and has Euler number $\tau(e)$. Perform the same operation on $N_{\phi(r)}$ to obtain $\overline{N_{\phi(r)}}$. Then (a) becomes the statement that $\overline{M_r}$ and $\overline{N_{\phi(r)}}$ still have the same base orbifold $\overline{O}$, (b) states that both have Euler number zero, and (c1) states that they are still a Hempel pair!

We will break off some of the analysis in to the following lemmas for easier reference later.

Lemma 5.6.6. *Let M and N be compact orientable graph manifolds with (possibly empty) toroidal boundary. Let the JSJ decompositions be $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively, where all major vertex spaces have orientable base orbifold. Suppose $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ and let $\phi\colon X \to Y$ be the induced graph isomorphism from Theorem 5.2.2. For an edge e of X choose bases of $\pi_1 M_e$ and $\pi_1 N_{\phi(e)}$ whose first element is the class of a regular fibre of the adjacent vertex space $\pi_1 M_x$ and $\pi_1 N_{\phi(x)}$. Then the maps Φ_x induce a map*

$$\begin{pmatrix} \lambda_x & \rho_e \\ 0 & \mu_e \end{pmatrix}$$

from $\widehat{\pi_1 M_e}$ to $\widehat{\pi_1 N_{\phi(e)}}$. Furthermore there exists a sign $\pm_e$ for each e such that $\gamma(e) = \pm_e \gamma(\phi(e))$, $\lambda_x = \pm_e \mu_{\bar{e}}$ and $\mu_e = \pm_e \lambda_y$ where $y = d_1(e)$.

Proof. The maps on $\widehat{\mathbb{Z}}^2$ do indeed have this form since the class of a regular fibre is preserved by Φ (due to Theorem 4.3.4). Since the first element of each basis is common over all e, the upper left entry λ_x does not depend on e. Let $e = \overrightarrow{xy}$ be an edge of X. The fact that the gluing maps along e commute with Φ forces the equation

$$\begin{pmatrix} \alpha(\phi(e)) & \beta(\phi(e)) \\ \gamma(\phi(e)) & \delta(\phi(e)) \end{pmatrix} = \begin{pmatrix} \lambda_y & \rho_{\bar{e}} \\ 0 & \mu_{\bar{e}} \end{pmatrix} \begin{pmatrix} \alpha(e) & \beta(e) \\ \gamma(e) & \delta(e) \end{pmatrix} \begin{pmatrix} \lambda_x & \rho_e \\ 0 & \mu_e \end{pmatrix}^{-1} \tag{5.1}$$

to hold. To simplify notation, write $\alpha'(e) = \alpha(\phi(e))$ and so on, and suppress the edge e for the time being. In this notation we have

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \lambda_x^{-1}\lambda_y\alpha + \lambda_x^{-1}\bar{\rho}\gamma & [\cdots] \\ \lambda_e^{-1}\mu_{\bar{e}}\gamma & \mu_e^{-1}\mu_{\bar{e}}\delta - \lambda_x^{-1}\mu_e^{-1}\mu_{\bar{e}}\rho\gamma \end{pmatrix} \tag{5.2}$$

where the upper right entry is complicated and unimportant. The worried reader should note that both sides have determinant -1 and that the lower-left entry is not a zero-divisor in $\widehat{\mathbb{Z}}$; these considerations determine the last entry.

Now we have $\lambda_x^{-1}\mu_{\bar{e}}\gamma = \gamma'$. Since γ and γ' are non-zero elements of $\mathbb{Z}$, this (by Lemma 2.1.16) forces $\lambda_x = \pm_e\mu_{\bar{e}}$ for some sign $\pm_e$. Consideration of the determinant yields $\lambda_y = \pm_e\mu_e$; that is, $\pm_{\bar{e}} = \pm_e$. $\square$

Lemma 5.6.7. *Let M and N be compact orientable graph manifolds with (possibly empty) toroidal boundary. Let the JSJ decompositions be $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively, where all major vertex spaces have orientable base orbifold. Suppose $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ and let $\phi\colon X \to Y$ be the induced graph isomorphism from Theorem 5.2.2. Let $Z \subseteq X$ denote the full subgraph of X spanned by the vertices x such that neither M_x nor $N_{\phi(x)}$ has free boundary components—that is, do not contain any boundary components of M or N. Then for each $x \in Z$ there exist $\lambda_x, \mu_x \in \widehat{\mathbb{Z}}^2$ such that the isomorphism $\Phi_x\colon \widehat{\pi_1 M_x} \cong \widehat{\pi_1 N_{\phi(x)}}$ takes the form specified in Theorem 5.6.2 for $\lambda = \lambda_x$ and $\mu = \mu_x$. Let $\kappa_x = \lambda_x/\mu_x$. Furthermore there exist signs $\pm_e$ for each edge e of X with both endpoints in Z such that the following conditions hold:*

(a) For all $e \in EX$, we have $\gamma(e) = \pm_e\gamma(\phi(e))$

(b) For all $x \in VZ$, either $\kappa_x = \pm 1$ or the total slopes $\tau(x)$ and $\tau(\phi(x))$ vanish.

(c) For every vertex $x \in X$, $(M_x, N_{\phi(x)})$ is a Hempel pair of scale factor κ_x and for every edge $e \in EZ$ with $d_0(e) = x$ we have $\delta(\phi(e)) \equiv \pm_e\kappa_x\delta(e)$ modulo $\gamma(e)$.

(d) For all edges $e \in EZ$ with endpoints x and y we have $\lambda_x = \pm_e\mu_y$ and $\mu_x = \pm_e\lambda_y$.

Proof. Let $\Phi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$ be an isomorphism. By Theorem 5.2.2 we have an isomorphism $\phi\colon X \to Y$ and isomorphisms $\Phi_x\colon \widehat{\pi_1 M_x} \to \widehat{\pi_1 N_{\phi(x)}}$ for every $x \in X$. As in

Lemma 5.6.6 there are $\lambda_x, \mu_e, \rho_e \in \widehat{\mathbb{Z}}$ such that the induced map on the $\widehat{\mathbb{Z}}^2$-subgroup of $\widehat{\pi_1 M_x}$ corresponding to the edge e is

$$\begin{pmatrix} \lambda_x & \rho_e \\ 0 & \mu_e \end{pmatrix}$$

Now let $x \in VZ$. Because M_x and $N_{\phi(x)}$ have no free boundary components, the preservation of the edge groups of the JSJ decomposition implies that Φ_x preserves the peripheral structures of the groups. Choose some orientation of the fibres (i.e. generators of the fibre subgroups), giving an identification of the fibre subgroup with $\widehat{\mathbb{Z}}$. Then by Theorem 5.6.2 we may identify the base orbifolds of M_x and $N_{\phi(x)}$ in such a way that Φ_x splits as an isomorphism of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] \arrow[d, "\cdot\lambda_x"] & \widehat{\pi_1 M_x} \arrow[r] \arrow[d, "\Phi"] & \widehat{\pi_1^{\text{orb}} O_x} \arrow[r] \arrow[d, "\psi_x"] & 1 \\
1 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & \widehat{\pi_1 N_{\phi(x)}} \arrow[r] & \widehat{\pi_1^{\text{orb}} O_x} \arrow[r] & 1
\end{tikzcd}$$

where λ_x is some invertible element of $\widehat{\mathbb{Z}}$ and ψ_x is an automorphism of $B = \widehat{\pi_1^{\text{orb}} O}$ which sends each boundary component e_i to a conjugate of $e_i^{\mu_x}$ and sends each cone-point a_i to a conjugate of $a_i^{\mu_x}$, for some $\mu \in \widehat{\mathbb{Z}}^\times$. That is, for all e with $d_0(e) = x$ we have $\mu_e = \mu_x$. Furthermore $(M_x, N_{\phi(x)})$ is a Hempel pair with scale factor λ_x/μ_x. Note that, given an orientation on M and of each fibre, the base orbifolds also inherit an orientation.

Conditions (a), (c) and (d) now follow immediately from Lemma 5.6.6 and equation (5.2). Let us consider (b).

First consider a major vertex x of Z. As before, pick standard presentations of every major vertex group of M and N. The constants ρ_e may now be specified as follows. Let $\pi_1 M_x$ have presentation

$$\left\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g, h \mid a_i^{p_i} h^{q_i}, h \text{ central} \right\rangle$$

and $\pi_1 N_{\phi(x)}$ have presentation

$$\left\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g, h \mid a_i^{p'_i} h^{q'_i}, h \text{ central} \right\rangle$$

where we use the same letters to denote various generators using the identification of the base orbifolds. Then we have

$$\Phi_x(e_i) \sim e_i{}^{\mu_x} h^{\rho_{e_i}}$$

for each edge e_i, where $\sim$ denotes conjugacy. For the remaining edge

$$e_0 = (a_1 \cdots a_r e_1 \cdots e_s [u_1, v_1] \cdots [u_g, v_g])^{-1}$$

we have of necessity

$$\Phi_x(e_0) \sim e_0{}^{\mu_x} h^{-(\rho_{e_1} + \cdots + \rho_{e_s} + \nu_1 + \cdots \nu_r)}$$

where the ν_i are the unique elements of $\widehat{\mathbb{Z}}$ such that

$$\mu_x q_i' = \lambda_x q_i + \nu_i p_i$$

or, in terms of the map Φ_x, given by $\Phi_x(a_i) \sim a_i{}^{\mu_x} h^{\nu_i}$. These two definitions are equivalent due to the fact that Φ_x is an isomorphism. Hence we find

$$\sum_{d_0(e)=x} \rho_e = \sum \lambda_x \frac{q_i}{p_i} - \sum \mu_x \frac{q_i'}{p_i} \tag{5.3}$$

an analogue of the classic restrictions on Dehn twists taking one Seifert fibre space presentation to another.

For minor pieces the isomorphism also induces a map on the single boundary component, this time represented by a diagonal matrix as the boundary component of the minor piece has a canonical basis (up to multiplication by ± 1), where the second basis element is given by the unique maximal cyclic subgroup which is normal but not central. Equation (5.3) is still satisfied as all terms are zero.

Finally assume that $x \in VZ$. We have

$$\begin{aligned}
\tau(\phi(x)) &= \sum_{d_0(e)=x} \frac{\delta(\phi(e))}{\gamma(\phi(e))} - \sum \frac{q_i'}{p_i'} \\
&= \sum_{d_0(e)=x} \frac{\pm_e \kappa_x \delta(e) - \pm_e \mu_x^{-1} \rho_e \gamma(e)}{\pm_e \gamma(e)} - \sum \frac{q_i'}{p_i'} \\
&= \sum_{d_0(e)=x} \frac{\kappa_x \delta(e)}{\gamma(e)} - \sum \kappa \frac{q_i}{p_i} = \kappa_x \tau(x)
\end{aligned}$$

where the deduction of the second line uses equation (5.2) and Lemma 5.6.6 and the last line uses equation (5.3). Now (again by Lemma 2.1.16) this forces either $\kappa = \pm 1$ or $\tau(x) = \tau(\phi(x)) = 0$. Hence condition (b) holds. $\square$

Proof of Theorem 5.6.4. In this theorem the manifolds M and N are closed so the graph Z from the previous lemma coincides with X.

Suppose that X admits a cycle of odd length. The relations (d) of Lemma 5.6.7, tracked around this cycle, force $\lambda_x = \pm\mu_x$ for some (and hence, by connectedness of X and relations (d) of Lemma 5.6.7, every) vertex x. Here the $\pm$ sign is independent of x, so after changing the orientation on M we may assume it is 1. But then $(M_x, N_{\phi(x)})$ is a Hempel pair of stretch factor $\lambda_x/\mu_x = 1$—that is, they are isomorphic Seifert fibre spaces. It follows that if we can find integers $r_e \in \mathbb{Z}$ for every edge such that

$$\begin{pmatrix} \alpha(\phi(e)) & \beta(\phi(e)) \\ \gamma(\phi(e)) & \delta(\phi(e)) \end{pmatrix} = \begin{pmatrix} 1 & r_{\bar{e}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha(e) & \beta(e) \\ \gamma(e) & \delta(e) \end{pmatrix} \begin{pmatrix} 1 & r_e \\ 0 & 1 \end{pmatrix}^{-1}$$

and with these r_e realisable by Dehn twists—that is, with

$$\sum_{d_0(e)=x} r_e = \sum \frac{q_i}{p_i} - \sum \frac{q'_i}{p'_i}$$

then $\pi_1 M$ and $\pi_1 N$ are isomorphic. But since for each edge we have

$$\delta + \mu_x^{-1} \rho\gamma = \delta'$$

and γ, δ and δ' are in $\mathbb{Z}$, the numbers $r_e = \mu_x^{-1}\rho_e$ are in fact integers. Equation (5.3) now states that these r_e are actually realisable by Dehn twists. Hence M and N are homeomorphic (by a homeomorphism covering ϕ) and we have proved the theorem in the case when X is not bipartite.

Now suppose X is bipartite on two sets of vertices R and B. We will first take care of the signs $\pm_e$ by changing the fibre orientations of certain vertex groups of N. Pick some basepoint $x_0 \in R$ and a maximal subtree T of X and, moving outward from the basepoint along T, change such fibre orientations as are required to force $\pm_e = +$ for every edge e of T. Now define $\lambda = \lambda_{x_0}$ and $\mu = \mu_{x_0}$. We now have

$$\lambda_r = \lambda, \quad \mu_r = \mu, \quad \lambda_b = \mu, \quad \mu_b = \lambda$$

for all vertices $r \in R, b \in B$. For every remaining edge e (with $d_0(e) = r \in R$ and $d_1(e) = b \in B$, say) we have $\lambda = \lambda_r = \pm_e \mu_b = \pm_e \lambda$, whence $\pm_e = +$ for all e.

Now define $\kappa = \lambda/\mu$. For all edges $e = \overrightarrow{xy}$ we have $\gamma(\phi(e)) = \lambda_x^{-1}\mu_y\gamma(e) = \gamma(e)$, so condition (a) of the theorem holds. By construction of κ and the matrix equation (5.2), properties (c1) and (c2) also hold. If any total slope is non-zero then by (b) of Lemma 5.6.7 we find $\kappa = \pm 1$ and, as in the non-bipartite case, M and N are homeomorphic. So we have condition (b) of the theorem. $\square$

The above arguments make it clear that, unless graph manifolds are to be rigid, we must be able to realise non-trivial values of the quantity 'μ_x', else κ would have to be ± 1. That is, we must be able to find exotic automorphisms of type μ. For the term 'exotic automorphism' the reader is referred to Definition 5.6.1.

Theorem 5.6.8 ([Bel80], Section 1; see also [GDJZ15], Lemma 9)**.** *A sphere with three discs removed admits an exotic automorphism of type μ for every $\mu \in \widehat{\mathbb{Z}}^\times$.*

We will now extend this to all 2-orbifolds with boundary, and then describe the automorphisms of the profinite completions of Seifert fibre space groups that we will use.

Proposition 5.6.9. *For every $n \geq 3$, a sphere with n discs removed admits an exotic automorphism of type μ, for every $\mu \in \widehat{\mathbb{Z}}^\times$.*

Proof. The proof is by induction on n, starting at $n = 3$. Suppose that ψ_{n-1} is an exotic automorphism of an $(n-1)$-holed sphere S_{n-1}. By applying an inner automorphism we may assume that some boundary loop e_0 is sent by ψ_{n-1} to precisely e_0^μ. Let ψ_3 be an exotic automorphism of type μ of a 3-holed sphere S_3. Let the fundamental group of S_3 be generated by a and b; again we may assume that for the third boundary component $e = (ab)^{-1}$ we have $\psi_3(e) = e^\mu$.

Now gluing e_0 to e produces an n-holed sphere S_n with fundamental group

$$G = \left\langle \pi_1 S_{n-1}, \pi_1 S_3 \mid e = e_0^{-1} \right\rangle$$

By construction, defining $\psi_n \colon \widehat{G} \to \widehat{G}$ by ψ_{n-1} and ψ_3 on the two vertex groups of the amalgam gives a well-defined map from $\widehat{G}$ to $\widehat{G}$ to itself. This map is an isomorphism by the universal property of amalgamated free products (or because $\widehat{G}$ is Hopfian). We now have an exotic automorphism ψ_n as required. $\square$

Proposition 5.6.10. *Let O be an orientable 2-orbifold with boundary. Then O admits an exotic automorphism of type μ for any $\mu \in \widehat{\mathbb{Z}}$. Moreover this automorphism may be induced by an exotic automorphism of the orbifold $\mathring{O}$ obtained from O by removing a small disc about each cone point.*

Proof. Let the fundamental group of O have presentation

$$B = \langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g \mid a_i^{p_i} \rangle$$

with notation as in Definition 5.6.1 and let F be a discrete free group of rank $r+s+2g$ on a generating set

$$\{a_1, \ldots, a_r, e_1, \ldots, e_s, v'_1, v_1, \ldots, v'_g, v_g\}$$

realised as the fundamental group of an appropriately punctured sphere S. Let ψ be an exotic automorphism of S of type μ and let

$$\psi(v_i) = (v_i^\mu)^{g_i}, \quad \psi(v'_i) = (v_i'^\mu)^{g'_i}$$

Let G be the fundamental group of $\mathring{O}$ and write G as an iterated HNN extension

$$\left\langle F, u_1, \ldots, u_g \mid v'_i = (v_i^{-1})^{u_i} \right\rangle$$

Then we may extend ψ over the iterated HNN extension $\widehat{G}$ by setting

$$\psi(u_i) = g_i^{-1} u_i g'_i$$

Note that the map ψ so defined is actually a surjection, by the standard criterion for HNN extensions—we have a surjection on the vertex group and, on factoring out the vertex group, an epimorphism of free groups (in this case the identity). Since all finitely generated profinite groups are Hopfian, we do have an isomorphism witnessing the fact that $\mathring{O}$ admits an exotic automorphism of type μ. Note that u_i does not represent a boundary component of O, so it is of no concern that its conjugacy class is not preserved.

Note that ψ preserves the normal subgroup generated by the $a_i^{p_i}$, so descends to an automorphism of the quotient group of $\widehat{G}$ by the normal subgroup generated by these elements. This quotient group is precisely the profinite completion of the fundamental group of O. The proof is complete. $\square$

Proposition 5.6.11. *Let O be a non-orientable 2-orbifold with $s+1$ boundary components. Let $\mu \in \widehat{\mathbb{Z}}^\times$ and let $\sigma_0, \ldots, \sigma_s \in \{\pm 1\}$. Then O admits an exotic automorphism of type μ with signs $\sigma_0, \ldots, \sigma_s$. Moreover this automorphism may be induced by an automorphism of the orbifold $\mathring{O}$ obtained from O by removing a small disc about each cone point.*

Proof. Let the fundamental group of O' be

$$B' = \langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, \ldots, u_g \mid a_i^{p_i} \rangle$$

with notation as in Definition 5.6.1. Write B' as an amalgamated free product

$$\left\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g \mid a_i^{p_i}, u_i^2 = v_i \right\rangle$$

Let F be a free group on $r+s+g$ generators with generating set

$$\{a_1, \ldots, a_r, e_1, \ldots, e_s, v_1, \ldots, v_g\}$$

considered as the fundamental group of a $(r+s+g+1)$-punctured sphere S. Take an exotic automorphism of S of type μ and extend over the amalgam

$$\widehat{F}' = \left\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, v_1, \ldots, u_g, v_g \mid u_i^2 = v_i \right\rangle$$

by sending each u_i to $(u_i^\mu)^{g_i}$ where $\psi(v_i) = (v_i^\mu)^{g_i}$. Note that all conjugating elements g_i lie in the completion of the fundamental group of the orientable surface S, which is contained in the kernel of the orientation homomorphism.

To introduce the signs σ_i we may compose with automorphisms induced by automorphisms of the discrete group F'. Of course it suffices to change the signs one at a time. For instance to realise $\sigma_0 = -1$ we may compose with the map given by

$$v_g \mapsto (a_1 \cdots a_r e_1 \cdots e_s v_1^2 \cdots v_{g-1}^2)^{-1} v_g^{-1}$$

and by the identity on all other generators. This sends

$$\begin{aligned} e_0 &= (a_1 \cdots a_r e_1 \cdots e_s v_1^2 \cdots v_g^2)^{-1} \\ &\mapsto \left(v_g^{-1}(a_1 \cdots a_r e_1 \cdots e_s v_1^2 \cdots v_{g-1}^2)^{-1} v_g^{-1}\right)^{-1} \\ &= v_g a_1 \cdots a_r e_1 \cdots e_s v_1^2 \cdots v_{g-1}^2 v_g \\ &= v_g e_0^{-1} v_g^{-1} \end{aligned}$$

as required. Finally factoring out the relations $a_i^{p_i}$ gives the required exotic automorphism of O'. $\square$

Theorem 5.6.12. *Let M and N be orientable Seifert fibre spaces with $s+1$ boundary components. Suppose that (M, N) is a Hempel pair with scale factor $\kappa = \lambda/\mu$ for some $\lambda, \mu \in \widehat{\mathbb{Z}}^\times$. Then we may identify the base orbifolds of M and N with the same O and choose 'standard form' presentations*

$$G = \pi_1 M = \left\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, \ldots, h \mid a_i^{p_i} h^{q_i}, h^g = h^{o(g)} \right\rangle$$

and

$$H = \pi_1 N = \left\langle a_1, \ldots, a_r, e_1, \ldots, e_s, u_1, \ldots, h \mid a_i{}^{p_i} h^{q_i'}, h^g = h^{o(g)} \right\rangle$$

for the fundamental groups where q_i' is congruent to $\lambda\mu^{-1} q_i$ modulo p_i and where o is the orientation homomorphism (see Definition 5.6.13 below). Let $\rho_0, \ldots, \rho_s$ be any elements of $\widehat{\mathbb{Z}}$ such that

$$\sum \rho_i = \lambda \sum \frac{q_i}{p_i} - \mu \sum \frac{q_i'}{p_i} \tag{5.4}$$

and let $\sigma_0, \ldots, \sigma_s \in \{\pm 1\}$. Suppose that all σ_i are $+1$ if the base orbifold is orientable.

Let $B = \pi_1^{orb} O$. Then there exists an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{B} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \cdot\lambda} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} & & \\ 1 & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & \widehat{H} & \longrightarrow & \widehat{B} & \longrightarrow & 1 \end{array}$$

where ψ is an exotic automorphism of type μ and where the map on each boundary component is described by

$$\Psi(h) = h^\lambda, \quad \Psi(e_i) = (e_i^{\sigma_i \mu})^{g_i} h^{\rho_i}$$

for some $g_i \in \widehat{H}$ mapping to σ_i under the orientation homomorphism on $\widehat{B}$.

Proof. Define θ_i to be the unique element of $\widehat{\mathbb{Z}}$ such that

$$\mu q_i' = \lambda q_i + \theta_i p_i$$

and note that the hypothesis of the theorem forces

$$\sum \rho_i + \sum \theta_j = 0$$

Let $\mathring{O}$ be the orbifold obtained from O by removing a small disc about each cone point. Let $\tilde{\psi}$ be an exotic automorphism of $\mathring{O}$ of type μ as in Proposition 5.6.10 or 5.6.11 such that

$$\tilde{\psi}(a_i) = ({a_i}^{\mu})^{f_i}, \quad \tilde{\psi}(e_i) = ({e_i}^{\sigma_i \mu})^{g_i}$$

Let π be the natural homomorphism from the free profinite group on the generators $\{a_i, e_i, u_i, v_i\}$ to $\widehat{H}$. As in Propositions 5.6.10 and 5.6.11 the map $\tilde{\psi}$ induces an exotic automorphism of O. Now define $\Psi \colon \widehat{G} \to \widehat{H}$ as follows:

$$\Psi(h) = h^{\lambda}, \quad \Psi(a_i) = \pi\tilde{\psi}(a_i)h^{\theta_i}, \quad \Psi(e_i) = \pi\tilde{\psi}(e_i)h^{\rho_i}$$

and by $\pi\tilde{\psi}$ on the remaining generators. This map Ψ is well-defined by the definition of the θ_i and is an isomorphism as it induces isomorphisms on both fibre and base. It has all the advertised properties by construction, except possibly the condition on $\Psi(e_0)$. For O non-orientable we have

$$\begin{aligned}
\Psi(e_0) &= \Psi(a_1 \cdots a_r e_1 \cdots e_s u_1^2 \cdots u_g^2)^{-1} \\
&= \left(\pi\tilde{\psi}(a_1 \cdots u_g^2)h^\wedge\{\sum \theta_i + \sum_{i\neq 0} \rho_i\}\right)^{-1} \\
&= \pi\tilde{\psi}(e_0)h^{\rho_0} = (e_0^{\sigma_0\mu})^{g_0} h^{\rho_0}
\end{aligned}$$

as required, noting that all elements $\pi\tilde{\psi}(u_i^2)$ et cetera commute with h. The case of orientable O is similar. $\square$

Definition 5.6.13. If M_v is a closed Seifert fibre space, consider the *orientation homomorphism* from $\widehat{\pi_1 M_v}$ to $\{\pm 1\}$ whose kernel is the centraliser of the canonical fibre subgroup. More precisely if h represents a regular fibre then for all $g \in \widehat{\pi_1 M_v}$ we have $g^{-1}hg = h^{o(g)}$ where $o(g) = \pm 1$. This function o is the orientation homomorphism.

Let M_v^{or} be the *orientation cover* of M_v—that is, the cover of M_v corresponding to the kernel of o.

If M is a graph manifold with JSJ decomposition $(X, M_\bullet)$, define a regular degree 2 cover M^{or} of M as follows. We have orientation homomorphisms $o_v \colon \pi_1 M_v \to \mathbb{Z}/2$ for each major vertex $v \in VX$. Since all boundary tori contain a regular fibre, their fundamental groups lie in the centraliser of that fibre, and hence are in the kernel of the orientation homomorphism o_v. So we may define a map $\pi_1(M) \to \mathbb{Z}/2$ by first using the homomorphisms o_v to map $\pi_1 M$ to the fundamental group of the 'constant' graph of groups $(X, \mathbb{Z}/2)$ (with all vertex groups $\mathbb{Z}/2$ and all edge groups trivial) and then taking the obvious map to $\mathbb{Z}/2$. The kernel of this homomorphism defines the cover M^{or} and induces the orientation cover of all major vertex spaces with non-orientable base orbifold.

Denote the JSJ decomposition of M^{or} by $(X^{\mathrm{or}}, M_\bullet^{\mathrm{or}})$. Note that, since every loop in X may be realised by a loop in M which lifts to M^{or}, the graph X is bipartite if and only if X^{or} is bipartite, and the bipartition on X lifts to that on X^{or}. Also note that $(\overline{M_v})^{\mathrm{or}} = \overline{(M_v^{\mathrm{or}})}$ where the bar denotes the filled vertex space.

Suppose that we have two graph manifolds M and N with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$, and a graph isomorphism $\phi \colon X \to Y$ taking vertices with non-orientable base orbifold to vertices with non-orientable base orbifold, and major vertices to major vertices. There is also a graph isomorphism $\phi^{\mathrm{or}} \colon X^{\mathrm{or}} \to Y^{\mathrm{or}}$ covering ϕ. If there is a homeomorphism from M^{or} to N^{or} covering the map ϕ^{or}, then M and N are homeomorphic. For the fact that the graph isomorphism covers ϕ and that the different sorts of vertices match up correctly implies that the $\mathbb{Z}/2$ actions on each vertex space of M^{or} and N^{or} are matched up by this homeomorphism (or at least, one isotopic to it).

We will now move towards the final theorem governing the profinite completions of graph manifold groups. As in the discussion preceding Theorem 5.6.4, choosing a generator of the fibre subgroup of each vertex group allows us to define a matrix representing the gluing maps on each edge e. Before this matrix was independent under conjugacy of the edge group. When base orbifolds can be non-orientable, all these invariants are independent of conjugacy only up to sign (unless both adjacent vertex groups have orientable base). The reader should not then be unduly surprised by the presence of sign indeterminacy in the following theorem—it is a consequence of ambiguities in the graph manifolds themselves, rather than anything mysterious concerning the profinite completions. Also note that the definition of 'total slope' only involved the ratios δ/γ and so is well-defined.

Theorem 5.6.14. *Let M and N be closed orientable graph manifolds with JSJ decompositions $(X, M_\bullet)$ and $(Y, N_\bullet)$ respectively. Suppose M and N are not homeomorphic.*

1. *If X is not bipartite then $\pi_1 M$ is profinitely rigid.*
2. *If X is bipartite on two sets R and B, then $\pi_1 M$ and $\pi_1 N$ have isomorphic profinite completions if and only if, for some choices of generators of fibre subgroups, there is a graph isomorphism $\phi\colon X \to Y$ and some $\kappa \in \widehat{\mathbb{Z}}^\times \smallsetminus \{\pm 1\}$ such that:*
    - *(a) For each edge e of X, $\gamma(\phi(e)) = \pm_e \gamma(e)$, where the sign is positive if both end vertices of e have orientable base.*
    - *(b) The total slope of every vertex space of M or N vanishes*
    - *(c1) If $d_0(e) = r \in R$ then $\delta(\phi(e)) = \pm_e \kappa \delta(e)$ modulo $\gamma(e)$, and $(M_r, N_{\phi(r)})$ is a Hempel pair of scale factor κ.*
    - *(c2) If $d_0(e) = b \in B$ then $\delta(\phi(e)) = \pm_e \kappa^{-1} \delta(e)$ modulo $\gamma(e)$, and $(M_b, N_{\phi(b)})$ is a Hempel pair of scale factor κ^{-1}.*

Remark. As in Theorem 5.6.4, these conditions (a)–(c1) are almost equivalent to the requirement that the filled manifolds $\overline{M_r}$ and $\overline{N_{\phi(r)}}$ form a Hempel pair. The only difference is that forming the filled manifold forgets the orientation of the fibre of the adjacent manifold, hence it would not guarantee that the signs may be fixed as in the final part of (a). This is a necessary condition, as was seen in Theorem 5.6.4.

Corollary 5.6.15. *If M is a closed orientable graph manifold then there are only finitely many graph manifolds N (up to homeomorphism) whose fundamental groups have the same profinite completion as $\pi_1 M$.*

Proof. We note that a closed graph manifold is determined by: the JSJ graph and homeomorphism type of the Seifert fibred pieces; the numbers $\gamma(e)$; the total slope at each vertex; and the numbers $\delta(e)$. Note also that by Dehn twists the values of $\delta(e)$ may be changed by any multiple of $\gamma(e)$, subject only to the condition that the total slope is fixed. It follows that the only choices to be made in the above theorem which affect N are the signs $\pm_e$ and the number κ modulo the lowest common multiple of all the $\gamma(e)$ and the orders of the cone points of the Seifert fibre spaces: if κ is congruent to 1 modulo this lowest common multiple then N is homeomorphic to M. It follows that there are indeed only finitely many choices for N up to homeomorphism. $\square$

Proof of Theorem 5.6.14. We will first deduce the 'only if' direction from Theorem 5.6.4. Let Ψ be an isomorphism from $\widehat{\pi_1 M}$ to $\widehat{\pi_1 N}$ and let $\phi\colon X \to Y$ be the induced graph isomorphism. Since the isomorphism Ψ preserves centralisers, there is an induced isomorphism

$$\Psi\colon \widehat{\pi_1 M^{\rm or}} \to \widehat{\pi_1 N^{\rm or}}$$

and an induced isomorphism $\phi^{\rm or}\colon X^{\rm or} \to Y^{\rm or}$ which covers ϕ.

Suppose first that X is not bipartite. Then neither is $X^{\rm or}$, so by Theorem 5.6.4 there is a homeomorphism $M^{\rm or} \to N^{\rm or}$ covering $\phi^{\rm or}$. By the comments following Definition 5.6.13, it follows that M is homeomorphic to N.

Now suppose that X is bipartite. We claim that the conclusion of Theorem 5.6.4 applied to $M^{\rm or}$ and $N^{\rm or}$ immediately forces the equations in (a)–(c2) to hold. For some choices of fibre orientations of the vertex spaces in the covers give the equations in the theorem statement for every lift of an edge e of X, and different lifts of e have values of γ, δ et cetera equal to those of e, up to a choice of sign. Hence all the equations of the statement hold up to sign. It only remains to show that we may fix the signs on edges between two spaces of orientable base as in (a). Consider the graphs A_M and $A_M^{\rm or}$ obtained from X and $X^{\rm or}$ by removing all major vertices with non-orientable base, and all preimages of those vertices. Of course, A_M contains all the edges we are concerned with. The components of $A_M^{\rm or}$ are homeomorphic copies of the components of A. Similarly define A_N and $A_N^{\rm or}$. So choosing orientations on the fibres of $M^{\rm or}$ and $N^{\rm or}$ as in Theorem 5.6.4 gives a consistent choice of fibre orientations on the vertex spaces each component of A_M or A_N satisfying (a), simply by choosing some lift of the component to (for example) A_M^{or} and inheriting orientations from there. Hence the theorem is true in this case as well.

Now let us turn our attention to the 'if' direction of (2). Suppose that M and N are as in the conditions of that theorem. We will build a more or less explicit isomorphism of profinite groups, defined with respect to a presentation of $\pi_1 M$. We say 'more or less' explicit as we cannot describe precisely the conjugating elements in exotic automorphisms of orbifolds. This unfortunately requires a small hurricane of notation.

First let us describe the presentations that we will use. We will use primes to denote the invariants $\gamma(e), \delta(e)$ et cetera deriving coming from N as opposed to M—that is, $\gamma'(e) = \gamma(\phi(e))$. We may as well identify X and Y using the isomorphism ϕ. Choose a maximal subtree T in X. We may also identify the base orbifolds of M_x and N_x. Fix some presentation for each such orbifold. We will use the same letters to denote the generating sets of $\pi_1 M_x$ and $\pi_1 N_x$ in a presentation coming from some presentation for the orbifold group, with h_x denoting the fibre subgroup (with the choice of generator in the theorem statement). For the edge e (with $d_0(e) = x$) a meridian on the relevant boundary torus of x will also be denoted with the letter e. The meridian on the other vertex group adjacent to e will then be denoted $\bar{e}$, being e with the opposite orientation. Then the conditions of the theorem, and the definitions of the invariants involved, specify the relations in $\pi_1 M$ and $\pi_1 N$ coming from each edge e. For instance, for an edge e from x to y we have relations

$$h_x^{(t_e)} = h_y^{\alpha(e)} \bar{e}^{\gamma(e)}, \quad e^{(t_e)} = h_y^{\beta(e)} \bar{e}^{\delta(e)}$$

in $\pi_1 M$, where (t_e) is either the identity if $e \in T$ or a stable letter for the HNN extension over e if $e \notin T$. Similarly in $\pi_1 N$ we have

$$h_x^{(t_e)} = h_y^{\alpha'(e)} \bar{e}^{\gamma'(e)}, \quad e^{(t_e)} = h_y^{\beta'(e)} \bar{e}^{\delta'(e)}$$

Now that we have fixed all the fibre orientations and a presentation, all these numbers become well-defined. To unify treatment of vertices in R and B, define $\lambda_r = \kappa$ and

$\mu_r = 1$ for $r \in R$ and $\lambda_b = 1, \mu_b = \kappa$ for $b \in B$. We therefore also have a well-defined sign $\pm_e$ and well-defined $\rho(e) \in \widehat{\mathbb{Z}}$ for each edge e such that

$$\gamma' = \pm_e \gamma, \quad \mu_x \delta' = \pm_e(\lambda_x \delta - \rho\gamma) \tag{5.5}$$

when $d_0(e) = x$. The existence of ρ is guaranteed by (c1) or (c2), and its uniqueness comes from the fact that γ is not a zero-divisor in $\widehat{\mathbb{Z}}$. Note that, by inverting the relations above, we find $\pm_{\bar{e}} = \pm_e$ and $\alpha(e) = -\delta(\bar{e})$ so that

$$\mu_y \alpha' = \pm_e(\lambda_y \alpha + \bar{\rho}\gamma) \tag{5.6}$$

for $d_1(e) = y$, where $\bar{\rho}(e) = \rho(\bar{e})$.

Finally for each edge e define $\sigma_e \in \{\pm 1\}$ as follows. If $x = d_0(e)$ has orientable base, set $\sigma_e = +1$. If $x \in R$ has non-orientable base, set $\sigma_e = \pm_e 1$. If $x \in B$ has non-orientable base, set $\sigma_e = \pm_e 1$ if $d_1(e)$ has orientable base, and $+1$ otherwise. Note that $\sigma_e \sigma_{\bar{e}} = \pm_e 1$.

Now define an isomorphism $\Psi_x \colon \widehat{\pi_1 M_x} \to \widehat{\pi_1 N_x}$ using Theorem 5.6.12 with input values $\lambda_x, \mu_x, \{\rho(e)\}$ and $\{\sigma_e\}$. The condition (b) guarantees that the hypothesis (5.4) of Theorem 5.6.12 is satisfied because:

$$\begin{aligned}
\sum \rho_i &= \sum \frac{\lambda_x \delta(e)}{\gamma(e)} - \sum \frac{\pm_e \mu_x \delta'(e)}{\gamma(e)} \\
&= \lambda_x \sum \frac{\delta(e)}{\gamma(e)} - \mu_x \sum \frac{\delta'(e)}{\gamma'(e)} \\
&= \lambda_x \sum \frac{q_i}{p_i} - \mu_x \sum \frac{q_i'}{p_i'}
\end{aligned}$$

We are at long last in a position to build the promised isomorphism Ω from $\widehat{\pi_1 M}$ to $\widehat{\pi_1 N}$. First we will build a map defined on the vertex groups on the tree T, then deal with HNN extensions. Let $G_\bullet = \pi_1 M_\bullet$ and $H_\bullet = \pi_1 N_\bullet$. Choose some basepoint $x_1 \in T$ and define $\Omega = \Psi_t$ on $\widehat{G}_t$. Order the vertices of X as $\{x_1, \ldots, x_n\}$ such that each $\{x_1, \ldots, x_m\}$ spans a subtree T_m of T. Suppose inductively that we have defined Ω coherently on the tree of groups $(T_m, \widehat{G}_\bullet)$ such that Ω is defined on $\widehat{G}_{x_i}$ by $\Omega(g) = \Psi_{x_i}(g)^{f_i}$ where f_i is an element of $\widehat{H}$ of the form

$$f_i = k_i k_{i-1} \cdots k_1$$

with $k_j \in \widehat{H}_j$ for $j \leq i$. There is an edge e of T with $d_0(e) = x = x_j \in \{x_1, \ldots, x_m\}$ and with $d_1(e) = y = x_{m+1}$. By construction of Ψ_x there is $g_e \in \widehat{H}_x$ such that g_e evaluates to σ_e under the orientation homomorphism on $\widehat{H}_x$ and

$$\Psi_x(h_x) = h_x^{\lambda_x}, \quad \Psi_x(e) = (e^{\sigma_e \mu_x})^{g_e} h_x^{\rho_e} = (e^{\sigma_e \mu_x} h_x^{\sigma_e \rho_e})^{g_e}$$

Similarly there is $g_{\bar{e}} \in \widehat{H}_y$ evaluating to $\sigma_{\bar{e}}$ under the orientation homomorphism, so that

$$\Psi_y(h_y) = h_y^{\lambda_y}, \quad \Psi_y(\bar{e}) = (\bar{e}^{\sigma_{\bar{e}} \mu_y} h_y^{\sigma_{\bar{e}} \bar{\rho}_e})^{g_{\bar{e}}}$$

Extend the definition of Ω to $\widehat{G}_y$ by setting $\Omega(g) = \Psi_y(g)^{g_{\bar{e}}^{-1} g_e f}$. Note that the conjugating element $f_{m+1} = g_{\bar{e}}^{-1} g_e f$ is of the form specified in the inductive hypothesis. We must now check that this map is well-defined—that is, we must check that the relations in $\pi_1 M$ given by this edge are mapped to the trivial element of $\widehat{H} = \widehat{\pi_1 N}$ under Ω. This calculation is essentially the same as the matrix calculations leading to equation (5.2). To reassure the reader that all the signs check out, and to atone for my sins, I will give the computations anyway. The reader should note that throughout we will be implicitly using various facts about conjugations, for instance the fact that every conjugate of e commutes with h_x. We will drop the subscript e from much of the notation, adding bars when necessary (e.g. $\sigma_{\bar{e}}$ will be written $\bar{\sigma}$). In the following equation arrays we will sometimes use the notation '$h^\wedge\{n\}$' to denote exponentiation when we wish to place more emphasis on the exponent and increase its readability.

$$\begin{array}{rcl}
\Omega(h_x^{-1} h_y^{\alpha} \bar{e}^{\gamma}) & = & \Psi_x(h_x^{-1})^{f_j} \Psi_y(h_y^{\alpha} \bar{e}^{\gamma})^{g_{\bar{e}}^{-1} g_e f_j} \\
& \sim_{f^{-1}} & h_x^{-\lambda_x} \left(h_y^{\alpha\lambda_y} (\bar{e}^{\gamma\bar{\sigma}\mu_y} h_y^{\bar{\sigma}\bar{\rho}\gamma})^{g_{\bar{e}}}\right)^{g_{\bar{e}}^{-1} g_e} \\
& \sim_{g_e^{-1}} & h_x^{-\sigma\lambda_x} h_y^{\bar{\sigma}\alpha\lambda_y + \bar{\sigma}\bar{\rho}\gamma} \bar{e}^{\gamma\bar{\sigma}\mu_y} \\
& = & (h_y^{\alpha'} \bar{e}^{\gamma'})^{-\sigma\lambda_x} h_y^{\bar{\sigma}\alpha\lambda_y + \bar{\sigma}\bar{\rho}\gamma} \bar{e}^{\gamma\bar{\sigma}\mu_y} \\
& = & h_y{}^\wedge\{-\sigma\alpha'\lambda_x + \bar{\sigma}\alpha\lambda_y + \bar{\sigma}\bar{\rho}\gamma\} \cdot \bar{e}^\wedge\{-\sigma\lambda_x\gamma' + \bar{\sigma}\mu_y\gamma\} \\
& = & 1
\end{array}$$

where $\sim_g$ denotes conjugation by g. In the last line we use the definitions (5.5) and

(5.6) of ρ and $\bar\rho$ and the equalities $\sigma\bar\sigma = \pm_e 1$ and $\lambda_x = \mu_y$. Secondly, we have

$$\begin{array}{rcl}
\Omega(e^{-1}h_y^{\beta}\bar e^{\delta}) & = & \Psi_x(e^{-1})^{f_j}\Psi_y(h_y^{\beta}\bar e^{\delta})^{g_{\bar e}^{-1}g_e f_j} \\
& \sim_{f^{-1}} & \left(e^{-\sigma\mu_x}h_x^{-\sigma\rho}\right)^{g_e}\left(h_y^{\beta\lambda_y}(\bar e^{\delta\bar\sigma\mu_y}h_y^{\bar\sigma\bar\rho\delta})^{g_{\bar e}}\right)^{g_{\bar e}^{-1}g_e} \\
& \sim_{g_e^{-1}} & \left(h_y^{\beta'}\bar e^{\delta'}\right)^{-\sigma\mu_x}\left(h_y^{\alpha'}\bar e^{\gamma'}\right)^{-\sigma\rho} h_y^{\bar\sigma\beta\lambda_y+\bar\sigma\bar\rho\delta}\bar e^{\delta\bar\sigma\mu_y} \\
& = & h_y{}^\wedge\{-\beta'\sigma\mu_x - \alpha'\sigma\rho + \bar\sigma\beta\lambda_y + \bar\sigma\bar\rho\delta\}\cdot \\
& & \quad \bar e^\wedge\{-\sigma\gamma'\rho - \sigma\mu_x\delta' + \delta\bar\sigma\mu_y\}
\end{array}$$

Now the exponent of $\bar e$ in this formula vanishes by (5.5). To deal with the exponent E of h_y, recall that because M and N are orientable the maps on boundary tori have determinant -1. Hence $\beta\gamma = 1+\alpha\delta$, and $\beta'\gamma' = 1+\alpha'\delta'$. Since $-\sigma\gamma'$ is not a zero-divisor in $\widehat{\mathbb{Z}}$, it suffices to check that the exponent vanishes when multiplied by $-\sigma\gamma'$.

$$\begin{array}{rcl}
-\sigma\gamma' E & = & \beta'\gamma'\mu_x + \alpha'\gamma'\rho - \beta\gamma'\sigma\bar\sigma\lambda_y - \delta\gamma'\sigma\bar\sigma\bar\rho \\
& = & (1+\alpha'\delta')\mu_x - (1+\alpha\delta)\lambda_y + \alpha'\gamma'\rho - \delta\gamma\bar\rho \\
& = & \pm_e\alpha'(\lambda_x\delta - \rho\gamma) - \alpha\delta\lambda_y \pm_e \alpha'\gamma\rho - \delta\gamma\bar\rho \\
& = & \pm_e\delta\left(\lambda_x\alpha' \pm_e (-\alpha\lambda_y - \gamma\bar\rho)\right) \\
& = & 0
\end{array}$$

where we have freely used (5.5) and (5.6), as well as $\mu_x = \lambda_y$. Thus all relations are satisfied and we have defined Ω on the tree of groups $(T, \widehat{G}_\bullet)$.

It remains to define Ω on the stable letters t_e for the HNN extensions over the remaining edges of X. Let e be such an edge. By construction there are elements $g_e \in H_x$ and $g_{\bar e} \in H_y$ and elements f_x and f_y in the subgroup generated by the vertex groups such that

$$\Omega|_{G_e} = (\Psi_x)^{f_j}, \quad \Psi_x(e) = (e^{\sigma_e\mu_x}h_x^{\sigma_e\rho})^{g_e}$$

and similarly for y. Then set

$$\Omega(t_e) = f_x^{-1}g_e^{-1}t_e g_{\bar e} f_y$$

We must of course check that the relations on the edge e are satisfied. The verification of this is, up to conjugacy, almost identical with the verifications above and we will not punish the reader by writing it a second time.

We now have a well-defined homomorphism $\Omega\colon \widehat{G} \to \widehat{H}$. We may check that is an isomorphism by a suitable application of the universal property of a graph of profinite groups. However, since Ω is not quite a morphism of graphs of groups on the nose—having various conjugating elements involved—we instead give another argument. Recall that $\widehat{H}$ is generated by its vertex groups and the stable letters t_e. By induction, each vertex group $\widehat{H}_{x_j}$ lies in the image of Ω: by construction the subgroup $\widehat{H}_{x_j}^{g_e f_j}$ lies in this image where $g_e \in \widehat{H}_j$ and where f_j is an element of $\widehat{H}$ of the form

$$f_j = k_j k_{j-1} \cdots k_1$$

with $k_i \in \widehat{H}_i$ for $i \leq j$. By induction all the $k_i\,(i < j)$ are in the image of Ω, hence so $\widehat{H}_{x_j}^{g_e k_j} = \widehat{H}_{x_j}$. Finally, since we have $\Omega(t_e) = l' t_e l$ where l, l' are in the subgroup generated by the $\widehat{H}_i$, it follows that each stable letter t_e is in the image of Ω. So Ω is surjective.

Finally note that by symmetry we may also construct a surjective homomorphism $\Omega'\colon \widehat{H} \to \widehat{G}$. Both groups being Hopfian, this forces both Ω and Ω' to be isomorphisms. The proof is complete. $\square$

5.7 Examples

We now give some simple illustrative examples to demonstrate some of the phenomena that may occur in consequence of Theorem 5.6.14. For some of the examples we will also explicitly describe the isomorphisms of profinite completions.

Example 5.7.1 (Changing a vertex space). Fix a positive integer p and let $0 < q < p/2$ be coprime to p. Consider the following family of graph manifold groups, whose vertex groups for different values of q are not isomorphic relative to their peripheral structures.

$$\begin{aligned} G_q &= \big\langle a_1, a_2, u, v, h, u', v', h' \,\big|\, a_1^p h^q, a_2^p h^{-q}, e' = h, h' = e, \\ &\qquad [h, a_1], [h, a_2], [h, u], [h, v], [h', u'], [h', v'] \big\rangle \\ &= \langle a_1, a_2, u, v, h \rangle *_{\mathbb{Z}^2} \langle u', v', h' \rangle \end{aligned}$$

where $e = (a_1 a_2 [u,v])^{-1}$ and $e' = [u',v']^{-1}$. The manifolds themselves are schematically illustrated in Figure 5.1.

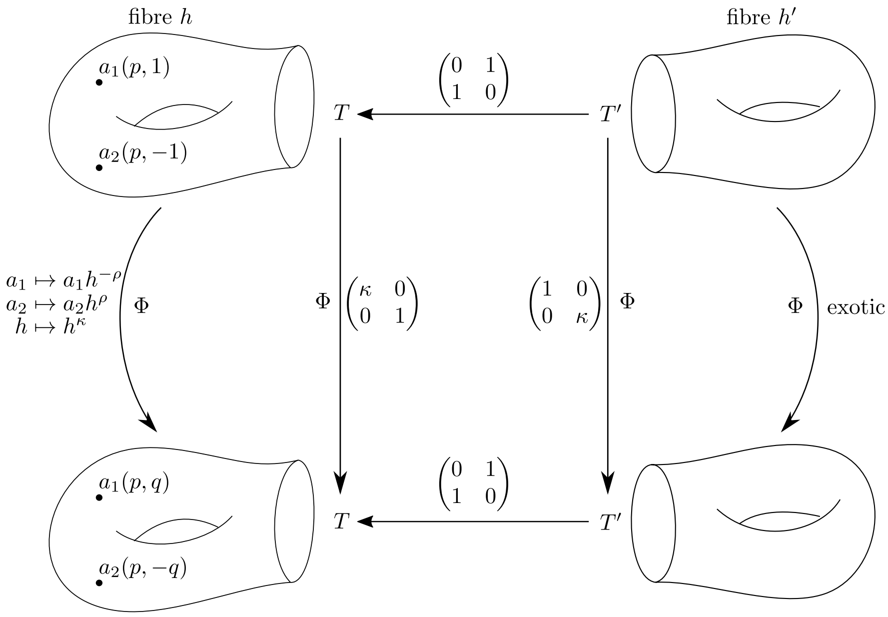


Figure 5.1: A schematic depiction of the graph manifolds considered in Example 5.7.1. Here we represent each Seifert fibred piece as a surface with some marked points, where the marked points represent exceptional fibres and are labelled with the Seifert invariants of that fibre. Each boundary subgroup $T^{(\prime)}$ is given the ordered basis $(h^{(\prime)}, e^{(\prime)})$.

One may readily check that the conditions of Theorem 5.6.14 are satisfied so that these groups have isomorphic profinite completions for all q. The isomorphism from $\widehat{G}_1$ to $\widehat{G}_q$ defined in the theorem may be described as follows. Let $\kappa = q + p\rho$ be an element of $\widehat{\mathbb{Z}}^{\times}$ congruent to q modulo p. Let ϕ be an automorphism of the free profinite group on the generators u' and v' such that

$$\phi([u',v']) = [u',v']^{\kappa}$$

i.e. an exotic automorphism of type κ of a once-punctured torus. Such a ϕ exists by Proposition 5.6.10. Then define $\Phi\colon \widehat{G}_1 \to \widehat{G}_q$ by sending u' and v' to their images under ϕ, by mapping

$$h \mapsto h^{\kappa}, \quad a_1 \mapsto a_1 h^{-\rho}, \quad a_2 \mapsto a_2 h^{\rho}$$

and by the 'identity' on all other generators. The reader may readily verify that this gives a well-defined surjection of profinite groups. As argued in the proof of the theorem it is in fact an isomorphism. This may also be seen from the fact that the map so given is an isomorphism of graphs of profinite groups.

Example 5.7.2 (Changing a gluing map). Consider the two graph manifolds depicted schematically in Figure 5.2. Each is composed of two product Seifert fibre spaces $S \times \mathbb{S}^1$ and $S' \times \mathbb{S}^1$ glued together, where S and S' are copies of a torus with two open discs removed. One readily verifies that the conditions of Theorem 5.6.14 hold, so the fundamental groups have isomorphic profinite completions. As in the previous

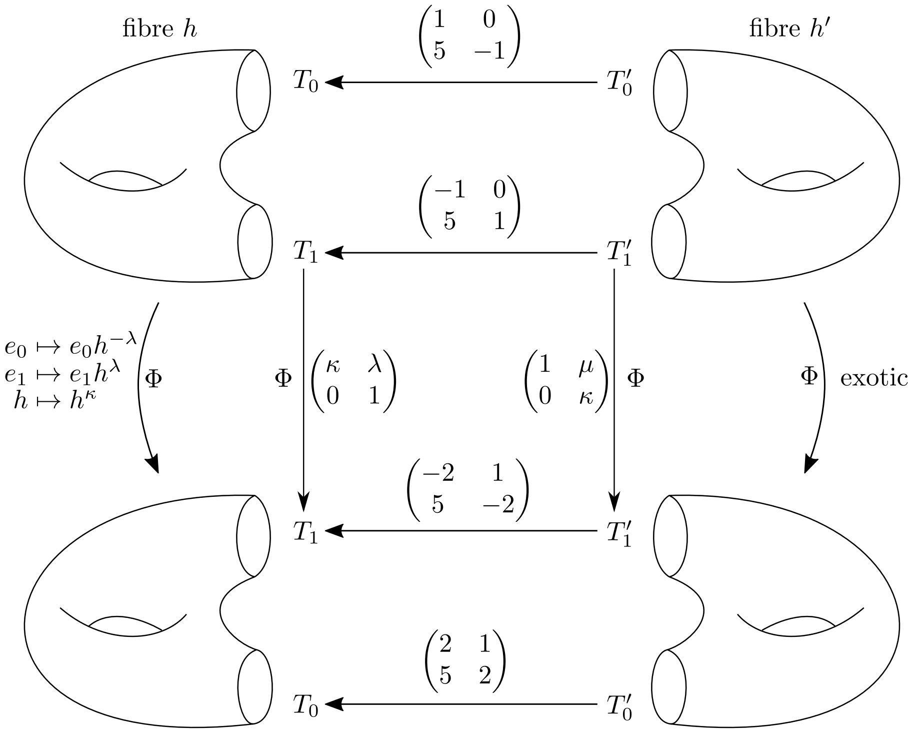


Figure 5.2: The graph manifolds considered in Example 5.7.2. Here each Seifert fibred piece is product of a surface with $\mathbb{S}^1$. Each boundary subgroup $T_i^{(\prime)}$ is given an ordered basis $(h^{(\prime)}, e_i^{(\prime)})$. Please note that for clarity we have swapped the roles of T_0 and T_1 in the lower diagram and omitted the maps induced by Φ on T_0 and T_0'.

example, we will write down the isomorphism explicitly. The fundamental group of the first manifold has presentation

$$\begin{aligned} G_1 = \big\langle e_1, u, v, h, e_1', u', v', h', t \,\big|\, & [h, e_1], [h, u], [h, v], [h', e_1'], [h', u'], [h', v'] \\ & e_1' = e_1, h' = h^{-1}e_1^5, (e_0')^t = e_0^{-1}, (h')^t = he_0^5 \big\rangle \end{aligned}$$

where $e_0 = (e_1[u,v])^{-1}$ and similarly for e_0'. The second group has presentation

$$G_2 = \big\langle e_1, u, v, h, e_1', u', v', h', t \,\big|\, [h, e_1], [h, u], [h, v], [h', e_1'], [h', u'], [h', v'] \\ e_1' = h e_1^{-2}, h' = h^{-2} e_1^5, (e_0')^t = h e_0^2, (h')^t = h^2 e_0^5 \big\rangle$$

Let $\kappa \in \widehat{\mathbb{Z}}^\times$ be congruent to 2 modulo 5, so that -2κ is congruent to 1 modulo 5. Take $\lambda, \mu \in \widehat{\mathbb{Z}}$ such that

$$\kappa = 2 + 5\lambda, \quad 1 = -2\kappa + 5\mu$$

Let ϕ be an exotic automorphism of S' of type κ, such that

$$\phi(e_1') = (e_1')^\kappa, \quad \phi(e_0') = [(e_0')^\kappa]^g$$

for some g in the subgroup of $\widehat{G}_2$ generated by u', v' and e_1'. Now define $\Phi\colon \widehat{G}_1 \to \widehat{G}_2$ as follows:

$$\begin{aligned} h &\mapsto h^\kappa, & e_1 &\mapsto e_1 h^\lambda, & u &\mapsto u,\ v \mapsto v,\ t \mapsto g^{-1} t \\ h' &\mapsto h', & e_1' &\mapsto \phi(e_1')(h')^\mu, & u' &\mapsto \phi(u'),\ v' \mapsto \phi(v') \end{aligned}$$

The reader is left to verify that this map is well-defined. The only real issue is whether the maps on the edge tori match up correctly. As indicated in Figure 5.2, this amounts to checking a matrix equation

$$\begin{aligned} \begin{pmatrix} -2 & 1 \\ 5 & -2 \end{pmatrix} &= \begin{pmatrix} \kappa & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & \kappa \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -\kappa + 5\lambda & \mu - 5\lambda\kappa^{-1}\mu + \lambda\kappa^{-1} \\ 5 & \kappa^{-1}(1 - 5\mu) \end{pmatrix} \end{aligned}$$

on the 'e_1 edge' (and a similar one on the 'e_0 edge'). These equations hold by the definitions of λ and μ.

Example 5.7.3 (Fibred examples). Consider the surface S formed from a sphere by removing 10 small discs spaced equidistantly along the equator. This surface has an order 5 self-homeomorphism φ given by a rotation. The surface bundle $M_q = S \rtimes_{\varphi^q} \mathbb{S}^1$ with monodromy φ^q (for q coprime to 5) is a Seifert fibre space whose base orbifold has genus 0, two boundary components and two exceptional fibres of order 5 with

Seifert invariants $(5, q)$ and $(5, -q)$. The surface S describes a family of parallel circles on each boundary torus; choose one such curve to give the second basis element of the fundamental group of the boundary. Take two such Seifert fibre spaces M_{q_1} and M_{q_2}. Glue the 'e_0 boundary' of M_{q_2} to the 'e_0 boundary' of M_{q_1} by a map

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

(interpreted as a map from the boundary of M_{q_2} to the boundary of M_{q_1}) and glue the 'e_1 boundary' of M_{q_2} to the 'e_1 boundary' of M_{q_1} by a map

$$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

These identifications give a graph manifold L_{q_1,q_2} (see Figure 5.3). The choice of the second column guarantees that the glued-up manifold is still fibred, since in each case the fibre surfaces, each running exactly five times over the boundary components in curves isotopic to e_0 or e_1, may be matched by this gluing homeomorphism. By construction the total slope at each vertex space M_{q_i} is zero. We may now apply Theorem 5.6.14 to conclude that the distinct fibred graph manifolds $L_{1,1}$ and $L_{2,-2}$ have isomorphic profinite completions of fundamental groups.

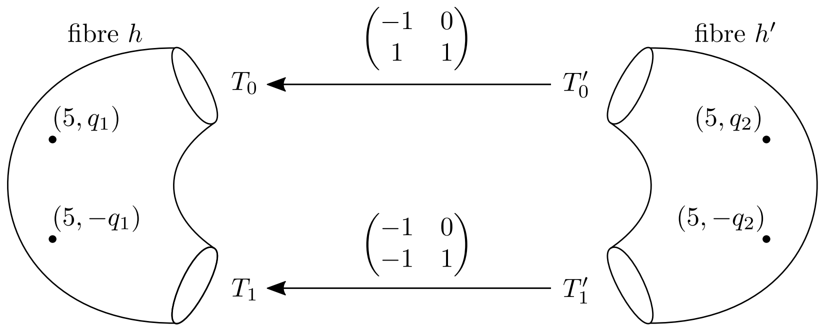


Figure 5.3: Schematic picture of the graph manifold L_{q_1,q_2} considered in Example 5.7.3

Remark. The study of fibred manifolds is closely connected with the study of mapping class groups. This will be discussed in Section 5.10.

5.8 Commensurability of graph manifolds

The following proposition answers a question asked of me by Michel Boileau at a conference in Marseille. All other known pairs of 3-manifold groups with the same profinite completions are commensurable—both Seifert fibred examples and examples with Sol geometry. The graph manifolds given by Theorem 5.6.14 also fit in this pattern.

Proposition 5.8.1. *Every closed orientable graph manifold has a finite-sheeted cover with profinitely rigid fundamental group. Hence if two graph manifold groups have isomorphic profinite completions, then they are commensurable.*

Proof. The second statement follows easily from the first. For the first, we will consider the following class of graph manifolds. We will say that a graph manifold M with JSJ decomposition $(X, M_\bullet)$ is *right-angled* if the following conditions hold:

(RA1) the total slope at every vertex space of M is zero;

(RA2) for every edge space the intersection number of the fibres of the adjacent vertex spaces is ± 1; and

(RA3) every vertex space is of the form $S \times \mathbb{S}^1$ for S an orientable surface with genus at least 2 (that is, if all boundary components are filled in with discs, the result has genus at least 2).

Conditions (RA1) and (RA2) together imply that, after performing suitable Dehn twists, the gluing maps on every torus are simply $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. That is, the fibres of adjacent pieces 'are at right angles to each other'. A right-angled graph manifold is thus determined completely by its underlying graph and the first Betti numbers of the vertex groups and the above signs. This graph, and the first Betti numbers of the vertex spaces, are profinite invariants by Theorem 5.2.2. The signs $\gamma(e) = \pm 1$ are profinite invariants by Theorem 5.6.14.2(a). Therefore to show that right-angled graph manifolds are indeed profinitely rigid it only remains to show that the property of being right-angled is a profinite invariant.

Theorem 5.6.14.2(b) immediately shows that having all total slopes zero is a profinite invariant. Theorem 5.6.14.2(a) shows that property (RA2) is preserved by profinite completions. Finally (RA3) is a profinite invariant by Theorems 5.6.2 and 5.2.2. Hence right-angled graph manifolds are indeed profinitely rigid.

Now consider a closed orientable graph manifold M. If the total slope of some vertex is non-zero then we have rigidity by Theorem 5.6.14 so we will ignore this case for the rest of the proof. Also notice that the vanishing of a total slope is equivalent to the vanishing of Euler numbers of the relevant filled Seifert fibre spaces (see Definition 5.6.5). So the vanishing of total slopes is preserved by taking finite-sheeted covers of M. We will show that every graph manifold with zero total slope at each vertex has a finite-sheeted cover which is right-angled.

First pass to a suitable index 1 or 2 cover to eliminate all minor pieces. Every base orbifold of a major Seifert fibre space has a finite cover which is a surface S of genus at least two, hence every Seifert fibred piece has a finite cover of the form $S \times \mathbb{S}^1$. The JSJ decomposition is efficient in the profinite topology (Theorem 2.1.19) so some finite-sheeted cover of the graph manifold induces such a cover on every Seifert-fibred piece (it may be necessary to pass to a deeper cover on each vertex space than the one specified—but this is still of the required form $S \times \mathbb{S}^1$).

We must now aim to satisfy the condition on intersection numbers. Consider some edge group $T_e = \mathbb{Z}^2$. The fibres of the adjacent vertex groups are primitive elements generating an index γ subgroup of T_e, so there is a quotient map $T_e \twoheadrightarrow \mathbb{Z}/\gamma\mathbb{Z}$ sending both fibres to zero. The fibres therefore lift to the corresponding degree γ cover of the torus, and the intersection number of any choice of such lifts is ± 1. So if we can find a finite-sheeted cover of the graph manifold inducing this precise cover of each boundary torus, we are done.

For each vertex space M_v we may, as previously noted, fill in each boundary torus T_e by gluing in a solid torus whose meridian is the fibre of the adjacent piece. This gives a Seifert fibre space whose base orbifold is the base orbifold of M_v with each boundary component collapsed and replaced by a cone point of order $\gamma(e)$, the intersection number of the two fibres. The fundamental group of this orbifold is residually finite, so there is a finite quotient into which all the isotropy groups of cone points inject. That is, we have a quotient $G_v \twoheadrightarrow Q_v$ such that the map on each boundary subgroup T_e is precisely the map $T_e \twoheadrightarrow \mathbb{Z}/\gamma(e)\mathbb{Z}$ discussed above.

We may now piece all these quotient groups together into a quotient graph of groups

$$(X, G_\bullet) \twoheadrightarrow (X, Q_\bullet)$$

where each Q_v is as above and each edge group Q_e is a copy of $\mathbb{Z}/\gamma(e)\mathbb{Z}$. This is a graph of finite groups, hence has residually finite fundamental group (see for example [Ser03, Section II.2.6, Proposition 11]). So there is some finite quotient into which all the finite groups Q_x inject. The corresponding finite-sheeted cover of the graph manifold is the required rigid cover. □

Remark. The 'right-angled' graph manifolds used above are also known as 'flip manifolds' [KL98].

5.9 Behaviour of knot complements

In this section we will prove the following theorem. A *graph knot* is a knot in $\mathbb{S}^3$ whose exterior is a graph manifold. This includes the class of 'hose knots' studied in, for example, [Ste71].

Theorem 5.9.1. *Let M_K be the exterior of a graph knot K. Let N be another compact orientable 3-manifold and assume that $\widehat{\pi_1 M_K} \cong \widehat{\pi_1 N}$. Then $\pi_1 M_K \cong \pi_1 N$. In particular if K is prime and N is also a knot exterior then N is homeomorphic to M_K.*

It is not necessarily true that an isomorphism of the profinite completions of 3-manifold groups preserves the peripheral structures of the groups (even when the numbers of boundary components are equal). Indeed this is not even true for the discrete fundamental group—manifolds with boundary are not uniquely determined by their discrete fundamental groups. However we shall see that the only ambiguities in determining the manifold M_K from its profinite fundamental group come from ambiguities in the discrete fundamental group.

Before proving the theorem we will need to discuss in more detail the Seifert fibre spaces arising in the JSJ decomposition of a graph knot exterior. Statements made in this discussion will be used without explicit reference in the following proofs.

Since a loop in the JSJ graph would give rise to a non-trivial element of $H^1(M_K;\mathbb{Z})$ vanishing on the boundary component of M_K, which is impossible by standard properties of knots, the JSJ graph is a rooted tree with root given by the single boundary component. It follows from Section 7 and Corollary 9.3 of [EN85] that the only possible vertex spaces are those described in the following list. A paraphrase of these results would be that all graph knots are built up from torus knots by the operations of cabling and connected-summation. See also Proposition 3.2 of [Bud05]. Note that there are additional possibilities when considering exteriors of graph *links*. We do not consider this issue here.

Torus knot exteriors

The exterior $E_{p,q}$ of a (p,q)-torus knot, for $|p|,|q| \geq 2$. This is a Seifert fibre space with two exceptional fibres of orders p and q. Since p and q are coprime there exist $\bar{p}$ and $\bar{q}$ such that $\bar{p}p + \bar{q}q = 1$. Then a presentation for the fundamental group is

$$\left\langle a, b, h \,\middle|\, h \text{ central}, a^p h^{\bar{q}}, b^q h^{\bar{p}} \right\rangle$$

where h is the homotopy class of a regular fibre and ab is a meridian curve of the knot. We remark that replacing $\bar{q}$ and $\bar{p}$ by any integers coprime to p and q respectively does not change this group up to isomorphism.

The profinite completion of this group has centre $\widehat{\mathbb{Z}}$ generated by h, and the quotient by this centre is $\mathbb{Z}/p \amalg \mathbb{Z}/q$, where $\amalg$ denotes the free profinite product. There is a unique 2-orbifold (a disc with two cone points of orders p and q) with this profinite fundamental group (see also [GZ11, Section 4]). So by Theorem 4.3.4 any Seifert fibre space with the same profinite fundamental group as $E_{p,q}$ has the same base orbifold, and will therefore have the same discrete fundamental group by the remark in the previous paragraph.

Theorem 5.6.2 implies that the Seifert fibre spaces with the same profinite fundamental group as $E_{p,q}$ *preserving peripheral structures* are precisely those with the

Seifert invariants $\bar{q}$ and $\bar{p}$ replaced by $k\bar{q}$ and $k\bar{p}$ for any k coprime to pq. Note that the requirement $\bar{p}p + \bar{q}q = 1$ additionally shows that, for k not congruent to 1 modulo pq, this Seifert fibre space is not a knot exterior.

If a torus knot exterior arises as a JSJ piece of a graph knot exterior then this piece must of course be a leaf of the JSJ tree. As the other possibilities in the list will show, the converse also holds: any leaf (except the root) is a torus knot exterior.

Products

Pieces of the form $S \times \mathbb{S}^1$, where S is a sphere with at least $k + 1 \geq 3$ open discs removed. The only Seifert fibre spaces with the same profinite fundamental group as $S \times \mathbb{S}^1$ also have the same discrete fundamental group (see Theorem 4.3.4) and are therefore also of the form $F \times \mathbb{S}^1$ for some surface F with $\pi_1 F \cong \pi_1 S$. Note that if F is orientable then either $F = S$ or F has at most $k - 1$ boundary components.

The presence of such a piece in the JSJ decomposition of a graph knot exterior represents the procedure of taking the connected sum of several graph knots. Note that under this operation, the *meridian* of each summand becomes a *fibre* of the product piece. Such a piece is either of valence $k + 1$ in the JSJ graph, or k if it happens to be the root piece.

Cable spaces

A cable space $C_{s,t}$ of type (s, t) for $|s| \geq 2$ consists of the space formed from a fibred solid torus T with Seifert invariants (s, t) by removing a neighbourhood of a regular fibre. Equivalently this is the orientable Seifert fibre space with base orbifold an annulus with a single cone point of Seifert invariants (s, t) where s is coprime to t. A presentation for the fundamental group is

$$\pi_1 C_{s,t} = \left\langle c, e, j \,\middle|\, j \text{ central}, \, c^s j^t \right\rangle$$

where the regular fibre is j and the boundary components are given by the conjugacy classes of the subgroups $\langle j, e\rangle$ and $\langle j, (ce)^{-1}\rangle$. Note also that, if the boundary component of T is represented in $\pi_1 C_{s,t}$ by $\langle j, e\rangle$ then the boundary of a meridian disc of T is given by $m = j^t e^{-s}$.

Notice that these groups $\pi_1 C_{s,t}$ as t varies are abstractly isomorphic to each other, and that such isomorphisms can be chosen to fix one (but not both) boundary components. Theorem 5.6.2 shows that the profinite fundamental groups of $C_{s,t}$ as t varies are isomorphic while preserving all of the given peripheral structure.

Furthermore Theorem 4.3.4 shows that the only orientable Seifert fibre spaces with the same profinite fundamental group (ignoring the peripheral structure) as a cable space have orientable base orbifolds with fundamental group $\mathbb{Z} * \mathbb{Z}/s$. There is only one such orbifold, so the Seifert fibre space in question is again a cable space (with the same invariant s).

The presence of a cable space piece in the JSJ decomposition of a graph knot exterior represents the cabling operation on knots. The corresponding vertex of the JSJ graph has valence two, or one if it happens to be the root. Note that in this case the meridian curve of the initial knot becomes the element m given above (i.e. the meridian curve of T). A longitude of the initial knot will therefore be given by $l = j^{\bar{s}} e^{\bar{t}}$ for some integers $\bar{s}$ and $\bar{t}$ such that $s\bar{s} + t\bar{t} = 1$. The ambiguity in the choice of these integers reflects the ambiguity in the choice of longitude.

Lemma 5.9.2. *Let K be a graph knot with exterior M_K. Let $(X, M_\bullet)$ be the JSJ decomposition of M_K, where X is viewed as a rooted tree. Let x be a leaf of X that is not the root. Then the total slope at x is non-zero.*

Proof. Since x is a leaf, the corresponding manifold M_x is the exterior of a (p, q) torus knot for some coprime integers p and q. As above choose a presentation

$$\pi_1 M_x = \left\langle a, b, h \,\middle|\, h \text{ central}, a^p h^{\bar{q}}, b^q h^{\bar{p}} \right\rangle$$

where $\bar{p}p + \bar{q}q = 1$ and the meridian of the torus knot is ab. Let y be the unique vertex of X adjacent to x. There are two cases to consider: whether M_y is a cable space or a product.

Suppose first that M_y is a product. Then by the discussions above the meridian ab of the torus knot is isotopic to the fibre j of the product. Thus the gluing matrix along the edge joining x and y (oriented from x to y) has the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}$$

Hence the total slope at x is

$$\tau(x) = 0 - \frac{\bar{q}}{p} - \frac{\bar{p}}{q} = -\frac{1}{pq} \neq 0$$

Next we consider the case when M_y is an (s,t)-cable space whose fundamental group has presentation

$$\pi_1 C_{s,t} = \left\langle c, e, j \,\middle|\, j \text{ central}, c^s j^t \right\rangle$$

where the regular fibre is j and the boundary components are given by the conjugacy classes of the subgroups $\langle j, e\rangle$ and $\langle j, (ce)^{-1}\rangle$. Without loss of generality let the boundary component glued to M_x be $\langle j, e\rangle$. As discussed above the meridian ab of the torus knot is given by $ab = j^t e^{-s}$ and the fibre h—which is a longitude of the torus knot—is given by $h = j^{\bar{s}} e^{\bar{t}}$ for some integers $\bar{s}, \bar{t}$ such that $s\bar{s} + t\bar{t} = 1$.

Thus the gluing matrix along the edge joining x and y, again oriented from x to y, has the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \bar{s} & t \\ \bar{t} & -s \end{pmatrix}$$

And the vanishing of the total slope would imply

$$0 = pq\bar{t}\tau(x) = pq(-s) - \bar{t}(\bar{q}q + \bar{p}p) = -pqs - \bar{t}$$

and hence s would divide $\bar{t}$, so $s = 1$ giving a contradiction. So the total slope is non-zero as required. $\square$

Proof of Theorem 5.9.1. Let $(X, M_\bullet)$ and $(Y, N_\bullet)$ be the JSJ decompositions of M_K and N respectively. Let $G = \pi_1 M_K$, $G_\bullet = \pi_1 M_\bullet$, $H = \pi_1 N$ and $H_\bullet = \pi_1 N_\bullet$. Let $\Phi\colon \widehat{G} \to \widehat{H}$ be an isomorphism. By Theorem 5.2.2 and Theorem 5.3.1 we find that N is a graph manifold and, possibly after post-composing Φ with some automorphism of $\widehat{H}$, there is a graph isomorphism $\phi\colon X \to Y$ such that Φ restricts to an isomorphism $\widehat{G}_x \cong \widehat{H}_{\phi(x)}$ for each $x \in X$. By the discussions above, this implies that G_x and $H_{\phi(x)}$ are abstractly isomorphic.

Now X is a rooted tree with root r distinguished by the single boundary component of M_K. For standard cohomological reasons (Corollary 2.2.10) the manifold N cannot be closed. We claim that N has exactly one boundary component, located in the piece $N_{\phi(r)}$. We do not claim—indeed it may not be true even for isomorphisms of discrete fundamental groups—that this boundary component ∂N satisfies

$$\widehat{\partial N} = \Phi(\widehat{\pi_1(\partial M_K)})$$

even up to conjugacy. However its position in the JSJ decomposition is fixed because of the following argument. Here a 'free boundary component' of a JSJ piece of a 3-manifold will mean a boundary torus not glued to any JSJ piece—that is, those boundary components which survive in the boundary of the ambient manifold.

Every Seifert fibre space with fundamental group isomorphic to that of a torus knot exterior has exactly one boundary component—and every such piece of N has valence 1 in Y because of the isomorphism Φ, hence has no free boundary components. Similarly every Seifert fibre space with fundamental group isomorphic to that of a cable space has two boundary components, and all such pieces N_y have valence 2 in Y unless $y = \phi(r)$, when there is one free boundary component ∂N. Comparing X and Y, every piece N_y with fundamental group isomorphic to $F_k \times \mathbb{Z}$ has valence $k+1$ in Y unless $y = \phi(r)$, when it has valence r. Now any orientable Seifert fibre space with fundamental group $F_k \times \mathbb{Z}$ has either $k+1$ boundary components or has strictly fewer than k—hence all these pieces have no free boundary components except if $y = \phi(r)$ when it has exactly one. So we see that N may have at most one boundary component, and it is located in the piece $N_{\phi(r)}$.

Now consider the isomorphisms $\Phi_x \colon \widehat{G}_x \to \widehat{H}_{\phi(x)}$ for $x \neq r$. By Lemma 5.6.7 the isomorphism Φ_x determines constants $\lambda_x, \mu_x \in \widehat{\mathbb{Z}}^{\times}$ such that λ_x gives the map from the fibre subgroup of M_x to the fibre subgroup of $N_{\phi(x)}$ and the map on base orbifolds is an exotic isomorphism of type μ_x. If x' is adjacent to x (and neither is the root) we have $\lambda_x = \mu_{x'}$ and $\mu_x = \lambda_{x'}$, perhaps up to choosing appropriate orientations on fibre subgroups to eliminate minus signs. Furthermore either the total slope $\tau(x)$ vanishes or $\lambda_x = \pm\mu_x$.

However the total slope $\tau(x)$ of any piece M_x which is a torus knot exterior is non-zero by Lemma 5.9.2. So $\lambda_x = \mu_x$ (up to changes in orientations). Every connected component of $X \smallsetminus \{r\}$ contains a leaf of X, hence has a vertex space which is a torus knot exterior. It follows that $\lambda_\bullet$ and $\mu_\bullet$ are constant on connected components of $X \smallsetminus \{r\}$.

Now on the root pieces of M and N, the fibre subgroup is the unique maximal central subgroup of the relevant profinite fundamental group by Theorem 4.3.4 and so is preserved by Φ_r. Let the map on fibre subgroups be multiplication by λ_r (as usual, identifying this fibre subgroup with $\widehat{\mathbb{Z}}$ via a generator in the discrete fundamental group). By Lemma 5.6.6 if x is adjacent to r then $\pm\mu_x = \lambda_r$. Therefore we find that $\lambda_\bullet$ and $\mu_\bullet$ are constants over all of X (up to reversing all the orientations on components of $X \smallsetminus \{r\}$ to fix the signs).

By Theorem 5.6.2, the fact that $\lambda_x/\mu_x = 1$ (together with the fact that away from the root all boundary tori of JSJ pieces are edge groups in the JSJ decomposition, hence are preserved by Φ) implies that M_x and $N_{\phi(x)}$ are homeomorphic for $x \neq r$.

Now, the fundamental group of the graph manifold N is determined up to isomorphism by the following data: the JSJ graph; the isomorphism types of group pairs given by each vertex group and its adjacent edge groups; the intersection numbers $\gamma(e)$ of adjacent fibres, for each edge e of the JSJ graph; the invariants $\delta(e)$ modulo $\gamma(e)$; and the total slope of each vertex space that is not the root. These allow us to reconstruct the total group uniquely—as explained in the preliminaries in Section 5.6, the indeterminacy of $\delta(e)$ modulo $\gamma(e)$ may be resolved using Dehn twists given that the total slope is fixed. For the root one may use the free boundary component for Dehn twists, hence no sort of total slope condition is required.

Now the analysis in the proof of Theorem 5.6.4 shows that the fact that λ_x/μ_x is always equal to 1 fixes all of these data for N to be equal to those for M. Hence the manifolds have the same fundamental group.

In the case when K is prime (that is, the root piece is a cable space) and N is also a knot exterior, then the fundamental group determines the homeomorphism type and final statement of the theorem also follows. See Section 7 and Corollary 9.3 of [EN85] for the proof of the for graph knots; more generally this is [GL89, Corollary 2.1]. □

Remark. We comment that the properties deriving from the fact that M_K was a knot exterior were crucial to the rigidity in this theorem. In particular the fact that every leaf of the JSJ tree had non-zero total slope provides strong rigidity to all complements of the root piece, leaving little flexibility in what remained. If one had some complement of the root piece which was not profinitely rigid (relative to the boundary component joining it to the root) one can easily extend this to further non-rigid examples. The free boundary component in the root means that results such as Theorem 5.6.2 which force all the boundary components of the root to behave in roughly the same way simply do not apply. Possibly one could impose extra constraints on the boundary to extend the analysis of Theorem 5.6.14. However a large part of the interest of Theorem 5.9.1 is that no such boundary condition is needed.

Remark. One could ask about graph manifolds with boundary other than knot exteriors, which we have not treated in this chapter or Theorem 5.6.14. The methods given in this chapter would almost certainly suffice to answer this question, albeit with a more complicated classification. In this thesis we will not deal with these questions, primarily for reasons of space.

5.10 Relation to mapping class groups

In this section, we will only discuss closed orientable manifolds, to avoid worries on the boundary. We will view a fibred 3-manifold (M, ζ) as a 3-manifold M equipped with a choice of homomorphism $\zeta\colon \pi_1 M \twoheadrightarrow \mathbb{Z}$ with finitely generated kernel $\pi_1 S$, where S is a closed orientable surface. By Stallings' theorem on fibred 3-manifolds [Sta62, Theorem 2], this is equivalent to the topological definition. For such a fibred manifold, $\pi_1 M$ has many expressions as a semidirect product $\pi_1 S \rtimes_\phi \mathbb{Z}$ each given by a section of ζ. The different maps ϕ differ by composition with an inner automorphism of $\pi_1 S$, hence give a well-defined element of $\mathrm{Out}(\pi_1 S)$. If we have two automorphisms ϕ_1 and ϕ_2 of $\pi_1 S$ then these are conjugate in $\mathrm{Aut}(\pi_1 S)$ by some automorphism ψ if and only if there is an 'isomorphism of semidirect products'

$$(\psi, \mathrm{id})\colon \pi_1 S \rtimes_{\phi_1} \mathbb{Z} \to \pi_1 S \rtimes_{\phi_2} \mathbb{Z}$$

By an 'isomorphism of semidirect products' we mean an isomorphism of the form

$$(g, n) \mapsto (\psi(g), n) \quad (g \in \pi_1 S, n \in \mathbb{Z})$$

where ψ is an automorphism of $\pi_1 S$.

Allowing for a change in section, we find that ϕ_1 and ϕ_2 are conjugate in $\mathrm{Out}(\pi_1 S)$ if and only if there is a commuting diagram

$$\begin{tikzcd}
\pi_1 M_1 \arrow[r, two heads, "\zeta_1"] \arrow[d, "\Psi"] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\pi_1 M_2 \arrow[r, two heads, "\zeta_2"] & \mathbb{Z}
\end{tikzcd}$$

where (M_1, ζ_1) and (M_2, ζ_2) are the fibred manifolds corresponding to ϕ_1 and ϕ_2.

All the above equivalences still hold when one replaces all manifold groups with their profinite completions and $\mathrm{Aut}(\pi_1 S)$ with $\mathrm{Aut}(\widehat{\pi_1 S})$. There is a canonical injection

$$\mathrm{Aut}(\pi_1 S) \hookrightarrow \mathrm{Aut}(\widehat{\pi_1 S})$$

and we will abuse notation by identifying an automorphism of $\pi_1 S$ with the induced automorphism of the profinite completion. The canonical map

$$\mathrm{Out}(\pi_1 S) \to \mathrm{Out}(\widehat{\pi_1 S})$$

is also an injection by [Gro74, Theorem 1] (this is not explicitly stated but is implicit in the proof). As was noted by Boileau and Friedl [BF15a, Corollary 3.6], Theorems 4.3.2 and 4.4.1 give another proof of this: an element of the kernel of this map would correspond to a fibred 3-manifold, which is not a product, with the same profinite fundamental group as the trivial product bundle $S \times \mathbb{S}^1$.

Thus related to the question of whether two fibred manifolds can have isomorphic profinite fundamental groups we have a question concerning automorphisms of surface groups.

Question 5.10.1. *Do there exist automorphisms ϕ_1 and ϕ_2 of a surface group $\pi_1 S$ which are conjugate in $\mathrm{Out}(\widehat{\pi_1 S})$ but not in $\mathrm{Out}(\pi_1 S)$?*

When S is a once-punctured torus, this question has been proven to have a negative answer by Bridson, Reid and Wilton [BRW17, Theorem A].

While a positive answer to Question 5.10.1 would give examples of fibred manifold groups with isomorphic profinite completions, the converse does not always hold; one could conceivably have profinite isomorphisms of the manifold groups which do not in any sense preserve any fibrations. For instance Example 5.7.3 above does not give a positive solution to Question 5.10.1. As we will see below, no other graph manifold does either. Let us make this precise.

Definition 5.10.2. Let (M, ζ_M) and (N, ζ_N) be fibred graph manifolds. Suppose

$$\Psi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$$

is an isomorphism. We say that Ψ *weakly preserves the fibration* if there is a commuting diagram

$$\begin{tikzcd} \widehat{\pi_1 M} \arrow[r, two heads, "\hat{\zeta}_M"] \arrow[d, "\Psi"] & \widehat{\mathbb{Z}} \arrow[d, "\cdot\kappa"] \\ \widehat{\pi_1 N} \arrow[r, two heads, "\hat{\zeta}_N"] & \widehat{\mathbb{Z}} \end{tikzcd}$$

for some $\kappa \in \widehat{\mathbb{Z}}^{\times}$. We say that Ψ *strongly preserves the fibration* if there exists such a diagram with, additionally, $\kappa = +1$.

Relating this definition to the discussion above, we see that strong fibre preservation yields conjugacy in $\mathrm{Out}(\widehat{\pi_1 S})$. Weak fibre preservation says that ϕ_1 is conjugate to ϕ_2^{κ} in $\mathrm{Out}(\widehat{\pi_1 S})$.

Finite order automorphisms give rise to Seifert fibre spaces of geometry $\mathbb{H}^2 \times \mathbb{R}$. Hempel's original paper giving examples of Seifert fibre spaces which are not profinitely rigid does not give an explicit isomorphism of profinite groups, so does not say anything about mapping classes. The isomorphisms constructed in Chapter 4 only preserve the fibre weakly. For a suitable choices of sections, the mapping classes involved are ϕ^k and ϕ for some k which is coprime to n equal to the order of ϕ. The weak fibre preservation then says that ϕ^k is conjugate to ϕ^{κ} in $\mathrm{Out}(\pi_1 S)$, where κ is the factor by which we stretch the homotopy class h of a regular (Seifert) fibre. This is not exactly surprising, since in fact these automorphisms are equal. Indeed this gives yet another way to see that the corresponding Seifert fibre space groups must have isomorphic profinite completions.

We can however say more. The mapping classes ϕ^k and ϕ are genuinely conjugate in $\mathrm{Out}(\widehat{\pi_1 S})$—that is, there exists an isomorphism which *strongly* preserves the fibration. The isomorphisms in Chapter 4 do not do this, but armed with the exotic automorphisms of surface groups from Proposition 5.6.10 we can build new isomorphisms. The case with non-empty boundary was covered by Theorem 5.6.12 as the focus was then on constructing graph manifolds. We now deal with the case of closed Seifert fibre spaces.

Theorem 5.10.3. *Let S be a closed hyperbolic surface and let φ be a periodic self-homeomorphism of S. Let k be coprime to the order of φ. Let (M, ζ) be the surface bundle with fibre S and monodromy φ and let (M', ζ') be the surface bundle with fibre S and monodromy φ^k. Then there is an isomorphism*

$$\Psi\colon \widehat{\pi_1 M} \to \widehat{\pi_1 M'}$$

which strongly preserves the fibration.

Proof. We may choose presentations for these two Seifert fibre spaces in the standard form. Note that since M is fibred over the circle, the base orbifold is orientable and the geometry is $\mathbb{H}^2 \times \mathbb{R}$. So let

$$\pi_1 M = \Big\langle a_1, \ldots, a_r, u_1, v_1, \ldots, u_g, v_g, h \,\Big|\, h \in Z(\pi_1 M),\ a_i^{p_i} h^{q_i},\ a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g] = h^b \Big\rangle$$

where $b = -\sum q_i/p_i$ since the Euler number vanishes. The map ζ is given by

$$h \mapsto \prod_j p_j, \quad a_i \mapsto -q_i \prod_{j \neq i} p_j$$

and by sending each u_i and v_i to zero. Similarly define

$$\pi_1 M = \Big\langle a_1, \ldots, a_r, u_1, v_1, \ldots, u_g, v_g, h \,\Big|\, h \in Z(\pi_1 M),\ a_i^{p_i} h^{q_i'},\ a_1 \ldots a_r [u_1, v_1] \ldots [u_g, v_g] = h^{b'} \Big\rangle$$

where $b' = -\sum q_i'/p_i$. Again the map ζ' is given by

$$h \mapsto \prod_j p_j, \quad a_i \mapsto -q_i' \prod_{j \neq i} p_j$$

Note that by construction we have $q_i \equiv \kappa q_i'$ modulo p_i for every i, for $\kappa \in \widehat{\mathbb{Z}}^\times$ congruent to k modulo the order of φ. See [Hem14, Proposition 5.1] for the translation of data from surface bundles to Seifert invariants. Let $\rho_i \in \widehat{\mathbb{Z}}$ be such that $q_i = \kappa q_i' + \rho_i p_i$. Now by Proposition 5.6.10 there is an automorphism ψ of the free group on the generators $\{a_i, u_i, v_i\}$ which sends each a_i to a conjugate of a_i^κ and sends

$$a_1 \dots a_r[u_1, v_1] \dots [u_g, v_g] \mapsto (a_1 \dots a_r[u_1, v_1] \dots [u_g, v_g])^\kappa$$

Now define $\Psi \colon \widehat{\pi_1 M} \to \widehat{\pi_1 M}$ by

$$h \mapsto h, \quad a_i \mapsto \psi(a_i) h^{-\rho_i}, \quad u_i \mapsto \psi(u_i), \quad v_i \mapsto \psi(v_i)$$

The reader may readily check that this map Ψ is a well-defined isomorphism of profinite groups and that $\hat{\zeta} = \hat{\zeta}'\Psi$ as required. $\square$

Theorem 5.10.4. *If M and N are non-homeomorphic closed fibred graph manifolds and $\Psi \colon \widehat{\pi_1 M} \to \widehat{\pi_1 N}$ is any isomorphism, then Ψ does not weakly preserve any fibrations of M and N. Hence if ϕ_1 and ϕ_2 are automorphisms of a closed surface group $\pi_1 S$ which are piecewise periodic but not periodic, and which are not conjugate in $\mathrm{Out}(\pi_1 S)$, then ϕ_1 is not conjugate to any power of ϕ_2 in $\mathrm{Out}(\widehat{\pi_1 S})$.*

Remark. Here the 'piecewise periodic' means that the corresponding fibred manifold is a graph manifold or Seifert fibre space. Excluding the periodic maps ensures that the manifolds in this theorem are *bona fide* graph manifolds rather than Seifert fibre spaces. The author is not aware of a standard term for this. Another suggestion by Ric Wade was 'root of a multitwist'.

Proof. For suppose (M, ζ_M) and (N, ζ_N) are fibred graph manifolds and Ψ is an isomorphism of the profinite completions of their fundamental groups weakly preserving the fibration. Suppose M and N are not homeomorphic. Let the JSJ decompositions be $(X, M_\bullet)$ and $(Y, N_\bullet)$ and denote the regular fibre of a vertex group $\pi_1 M_x$ or $\pi_1 N_y$ by h_x or h_y. Note that any fibre surface in a graph manifold must intersect the Seifert fibres of all vertex spaces transversely, so every Seifert fibre survives as a non-trivial element of $\mathbb{Z}$ under the map ζ_M (or ζ_N). Also, for an orientable fibred graph manifold all the base orbifolds of major pieces must be orientable. We may therefore apply the analysis in the proof of Theorem 5.6.4 to conclude that there is a graph isomorphism $\psi\colon X \to Y$ and numbers $\lambda, \mu \in \widehat{\mathbb{Z}}^{\times}$ such that for adjacent vertices x and y are of X, we have

$$\Psi(h_x) \sim h^{\lambda}_{\psi(x)}, \quad \Psi(h_y) \sim h^{\mu}_{\psi(y)}$$

(or vice versa), where $\sim$ denotes conjugacy in $\widehat{\pi_1 N}$. Since M and N are not homeomorphic the ratio λ/μ is not equal to ± 1. Now if Ψ weakly preserves the fibre, then we have equations

$$\kappa\zeta_M(h_x) = \zeta_N(\Psi(h_x)) = \zeta_N(h^{\lambda}_{\psi(x)}) = \lambda\zeta_N(h_{\psi(x)})$$

so that λ/κ, when multiplied by a non-zero element of $\mathbb{Z}$, remains in $\mathbb{Z}$. Thus by Lemma 2.1.16, $\kappa = \pm\lambda$. Applying the same argument to y gives $\kappa = \pm\mu$, so that $\lambda/\mu = \pm 1$, giving a contradiction. □

Corollary 5.10.5. *A closed fibred graph manifold with first Betti number one is profinitely rigid.*

Proof. By Theorem 5.3.1 and [JZ17, Theorem 1.1] any other manifold with the same profinite fundamental group as the given manifold is also a closed fibred graph manifold. If the first Betti number is 1 then there is a unique map to $\widehat{\mathbb{Z}}$ so that any isomorphism weakly preserves the fibration. Theorem 5.10.4 now gives the result. □

Remark. There is another proof of this corollary, which we will now sketch, which is more closely related to Theorem 5.6.14 and does not rely on [JZ17]. Consider a closed fibred graph manifold M of first Betti number one. Then there is an essentially unique homomorphism $\zeta\colon \pi_1 M \to \mathbb{Q}$ which, as noted above, does not vanish on the regular fibre of any Seifert-fibred piece of M. This in turn implies that the JSJ graph of M is a tree and that the base orbifolds of all pieces are spheres with cone points with discs removed.

Consider a piece M_l corresponding to a leaf l in the JSJ graph. We will show that the total slope at M_l is non-zero, so that M is profinitely rigid by Theorem 5.6.14. Let e be the edge emanating from l, let h be the homotopy class of a regular fibre of M_l, and let h' be the regular fibre of $d_1(e)$. If $\pi_1 M_l$ has a standard form presentation

$$\left\langle a_1, \ldots, a_r, h \,\middle|\, a_i^{p_i} h^{q_i}, h \text{ central} \right\rangle$$

then, if $e_0 = (a_1 \cdots a_r)^{-1}$ we have

$$h' = e_0^{-\gamma(e)} h^{\delta(e)} = (a_1 \cdots a_r)^{\gamma} h^{\delta}$$

where $\gamma \neq 0$. Without loss of generality suppose $\zeta(h) = 1$. Then $\zeta(a_i) = -q_i/p_i$ and

$$\zeta(h') = -\gamma \sum \frac{q_i}{p_i} + \delta = \gamma\tau(l)$$

Since $\zeta(h') \neq 0$ we find that the total slope at l is non-zero as claimed.

Theorem 5.10.4 shows that Theorem 5.6.14, while finding examples of non-rigid fibred manifolds, leaves open the possibility that Question 5.10.1 could have a negative answer for all infinite order mapping classes. It also raises the possibility that even if profinite rigidity for hyperbolic 3-manifolds fails, the weaker statement about mapping class groups could still hold.

Remark. It is curious to compare the directions of the proofs in this section and in [BRW17]. In the latter paper, strong properties of $\mathrm{Out}(F_2)$ ('congruence omnipotence for elements of infinite order') were used to deduce that independent mapping classes $\phi_1, \phi_2 \in \mathrm{Out}(F_2)$ were not conjugate to any power of each other in $\mathrm{Out}(\widehat{F}_2)$, so that no isomorphism of the profinite completions of once-punctured torus bundles could (in our terminology) weakly preserve the fibration. Meanwhile the assumption that the first Betti number is 1 showed that any such isomorphism must weakly preserve the fibration. In this way [BRW17] obtained a profinite rigidity theorem for once-punctured torus bundles.

In our situation, the direction is quite different: we investigated profinite completions of graph manifolds, and in doing so learnt about conjugacy of certain elements in $\mathrm{Out}(\widehat{\pi_1 S})$. One rather suspects that the results about the mapping class group should come first, but these seem to be lacking except in the case described above.

Chapter 6

Virtual Pro-p Properties

6.1 Virtual p-efficiency

As mentioned in Section 5.5, every graph manifold was shown to have a finite cover with p-efficient JSJ decomposition by [AF13, Proposition 5.2]. In that paper it was asked whether all 3-manifolds have this property. We use the fact that every aspherical 3-manifold that is not a graph manifold is virtually fibred [Ago13, PW18] to answer this question in the affirmative. First we need various preliminary lemmas about the pro-p topology on a surface group.

Proposition 6.1.1. *Let l be an essential simple closed curve on an orientable compact surface Σ. Then for any r, there is some p-group quotient of $G = \pi_1\Sigma$ in which the image of l is p^r-torsion. In particular, G induces the full pro-p topology on $L = \pi_1 l$.*

Proof. We will find, for each integer r, a finite p-group quotient of G such that the image of l has order p^r. If l is non-separating, then it represents a primitive class in $H_1(\Sigma; \mathbb{Z})$, so a suitable map $G \twoheadrightarrow \mathbb{Z}/p^r$ induces the map $L \twoheadrightarrow \mathbb{Z}/p^r$. If l is separating, let G_1 and G_2 be the respective fundamental groups of the two components Σ_1 and Σ_2 of $\Sigma \smallsetminus l$. If Σ_i has l as its only boundary component, then G_i is a free group with generators a_i, b_i $(1 \leq i \leq g)$ in which l is the product of the commutators $[a_i, b_i]$; now define a map from G_i to the 'mod-p^r Heisenberg group'

$$\mathcal{H}_3(\mathbb{Z}/p^r) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z}/p^r \right\}$$

by mapping

$$a_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and mapping the other generators to the identity matrix, so that the image of l is the order p^r element

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If Σ_i has another boundary component, then l is again a primitive element in the homology of Σ_i, so a suitable map to $\mathbb{Z}/p^r$ induces a surjection $L \twoheadrightarrow \mathbb{Z}/p^r$. We may now exhibit the required quotients of G, according to the division of cases above. When both Σ_i have another boundary component, take a suitable map $G \to \mathbb{Z}/p^r$, for in this case l is primitive in the homology of Σ. When Σ_1 has no boundary component other than l, map

$$G = G_1 *_L G_2 \to \mathcal{H}_3(\mathbb{Z}/p^r) *_{\mathbb{Z}/p^r} \mathbb{Z}/p^r = \mathcal{H}_3(\mathbb{Z}/p^r)$$

and when both Σ_i have this property, map

$$G = G_1 *_L G_2 \to \mathcal{H}_3(\mathbb{Z}/p^r) *_{\mathbb{Z}/p^r} \mathcal{H}_3(\mathbb{Z}/p^r) \to \mathcal{H}_3(\mathbb{Z}/p^r)$$

where the final homomorphism identifies the two copies of the Heisenberg group. □

It will follow from the next sequence of propositions that L is also p-separable in G—for L will be p-separable in each G_i by the next proposition and the splitting along L will be p-efficient by Propositions 6.1.5 and 6.1.7.

Proposition 6.1.2. *Let Σ be a compact orientable surface with non-empty boundary that is not a disc. Let l be a boundary component. Then $L = \pi_1 l$ is p-separable in $G = \pi_1\Sigma$.*

Proof. If Σ has only one boundary component then it has positive genus, so we may pass to a regular abelian p-cover with more than one boundary component to which all boundary components of Σ lift. In particular l lifts to this cover, and it suffices to prove that L is p-separable when Σ has more than one boundary component. In this case, L is a free factor of G; that is, $G = L * F$ for some free group F. Let $g \in G \smallsetminus L$ and write g as a reduced word

$$g = l^{m_1} f_1 l^{m_2} f_2 \dots f_n$$

where the $m_i \in \mathbb{Z}$, $f_i \in F$ are all non-trivial except possibly f_n (when $n > 1$) and m_1. Then there is a finite p-group quotient $F \twoheadrightarrow P$ in which no non-trivial f_i is mapped to the identity. Taking r larger than all m_i, the image of g under the quotient map

$$\phi\colon G = L * F \to \mathbb{Z}/p^r * P$$

is a reduced word with some letter in $P \smallsetminus \{1\}$ so that $\phi(g) \notin \phi(L)$. Since $\mathbb{Z}/p^r * P$ is residually p, we can pass to a finite p-group quotient distinguishing $\phi(g)$ from the (finitely many) elements of $\phi(L)$; this quotient p-group separates g from L. Hence L is p-separable in G. □

Definition 6.1.3. Let P be a finite p-group. A *chief series* for P is a sequence

$$1 = P_n \leq P_{n-1} \leq \cdots \leq P_2 \leq P_1 = P$$

of normal subgroups of P such that each quotient P_i/P_{i+1} is either trivial or isomorphic to $\mathbb{Z}/p$.

Theorem 6.1.4 (Higman [Hig64])**.** *Let A and B be finite p-groups with common subgroup $A \cap B = U$. Then $A *_U B$ is residually p if and only if there are chief series $\{A_i\}$ and $\{B_i\}$ of A and B respectively such that $\{U \cap A_i\} = \{U \cap B_i\}$. In particular, $A *_U B$ is residually p when U is cyclic.*

Proposition 6.1.5. *Let Σ be a compact orientable surface and let l be an essential separating simple closed curve on Σ. Let Σ_1 and Σ_2 be the closures of the two components of $\Sigma \smallsetminus l$. Let $G = \pi_1\Sigma$, $G_i = \pi_1\Sigma_i$, and $L = \pi_1 l$. Then $G = G_1 *_L G_2$ is a p-efficient splitting.*

Proof. Let $H \lhd_p G_1$ and $P_1 = G_1/H$, and suppose $LH/H \cong \mathbb{Z}/p^r$. By Proposition 6.1.1 there is a p-group quotient $G_2 \twoheadrightarrow P_2$ such that the image of L is again isomorphic to $\mathbb{Z}/p^r$. The quotient $P_1 *_{\mathbb{Z}/p^r} P_2$ thus obtained is residually p, so admits a p-group quotient Q distinguishing all the (finitely many) elements of P_1. The kernel of the composite map

$$G_1 \to G = G_1 *_L G_2 \to P_1 *_{\mathbb{Z}/p^r} P_2 \to Q$$

is H, so G induces the full pro-p topology on G_1. Similarly G induces the full pro-p topology on G_2.

We now show that G_1 is p-separable in G. Let $g \in G \smallsetminus G_1$ and write

$$g = a_1 b_1 \cdots a_n b_n$$

where all $a_i \in G_1$ and $b_i \in G_2$ are not in L (except possibly $b_n = 1$ if $n > 1$, or possibly $a_1 \in L$)—that is, write g as a reduced word in the amalgamated free product. Note that $b_1 \neq 1$. By Proposition 6.1.2 we may find $H_i \lhd_p G_i$ such that the image of every non-trivial b_j in $P_2 = G_2/H_2$ does not lie in the image of L (and similarly for $P_1 = G_1/H_1$). Suppose that the image of l in P_i has order p^{r_i}, and take $r = \max\{r_1, r_2\}$. By Proposition 6.1.1 we may find $K_i \lhd_p G_i$ such that $K_i \cap L = p^r L$. Replace H_i by the deeper subgroup $H_i \cap K_i$. In this way we ensure that the image of L is $\mathbb{Z}/p^r$ in both P_1 and P_2, and we may form the amalgamated free product $P_1 *_{\mathbb{Z}/p^r} P_2$. By construction the image $\phi(g)$ of g under the quotient $\phi\colon G = G_1 *_L G_2 \to P_1 *_{\mathbb{Z}/p^r} P_2$ is given by a reduced word which has a non-trivial letter in P_2. Hence $\phi(g)$ does not lie in P_1. Since $P_1 *_{\mathbb{Z}/p^r} P_2$ is residually p by Theorem 6.1.4, we may find a p-group quotient $P_1 *_{\mathbb{Z}/p^r} P_2 \twoheadrightarrow Q$ distinguishing $\phi(g)$ from P_1. Then $G \to Q$ distinguishes g from G_1 and so G_1 is p-separable in G. $\square$

Theorem 6.1.6 (Chatzidakis [Cha94])**.** *Let P be a finite p-group, let A and B be subgroups of P, and let $f\colon A \to B$ be an isomorphism. Suppose that P has a chief series $\{P_i\}$ such that $f(A \cap P_i) = B \cap P_i$ for all i and the induced map*

$$f_i\colon AP_i \cap P_{i-1}/P_i \to BP_i \cap P_{i-1}/P_i$$

is the identity for all i. Then P embeds in a finite p-group T in which f is induced by conjugation. Hence the HNN extension $P*_A$ is residually p.

The reader may be surprised by the use of the word 'identity' for the map f_i above, being a map between two apparently distinct groups. However both of these groups are subgroups of P_{i-1}/P_i (which is trivial or $\mathbb{Z}/p$) and the hypothesis is intended to state that these two subgroups coincide and that the map f_i is the identity on this subgroup.

Proposition 6.1.7. *Let Σ be a compact orientable surface and let l be a non-separating simple closed curve on Σ. Choose a regular neighbourhood $l \times [-1, 1]$ of l in Σ and let $\Sigma_1 = \Sigma \smallsetminus (l \times (-1, 1))$. Let $G = \pi_1\Sigma$, $H = \pi_1\Sigma_1$, $A = \pi_1(l \times \{-1\})$, and $B = \pi_1(l \times \{+1\})$. Let $f\colon A \to B$ be the natural isomorphism. Then the HNN extension $G = H*_A$ is a p-efficient splitting.*

Proof. Choose an orientation of l so that A, B and their quotients are given fixed generators. First we will prove that G induces the full pro-p topology on H. Let $P = H/\gamma_n^{(p)}(H)$ be one of the lower central p-quotients of H, and let $\phi\colon H \to P$ be the quotient map. Let a and b denote the given generators of $\phi(A)$ and $\phi(B)$ respectively. Note that since commutator subgroups and terms of the lower central p-series are verbal subgroups, there is a commuting diagram:

$$\begin{tikzcd}
 & H \arrow[dl] \arrow[dr] & \\
H/\gamma_n^{(p)}(H) = P \arrow[d] & & H/[H,H] = H_{\mathrm{ab}} \arrow[d] \\
P/[P,P] \arrow[rr, "\cong"] & & H_{ab}/\gamma_n^{(p)}(H_{\mathrm{ab}})
\end{tikzcd}$$

so that $P_{\mathrm{ab}} = H_{\mathrm{ab}} \otimes \mathbb{Z}/p^{n-1}$. The image of a in P thus has order at least p^{n-1}; by definition of the lower central series any element of P has order at most p^{n-1}. Hence the image of A in P injects into P_{ab}. Since P is a characteristic quotient of H and there is an automorphism of H taking a to b, the order of b will also be p^{n-1}. Furthermore a and b are mapped to the same element of P_{ab}. Now construct a chief series (P_i) for P whose first n terms are the preimages of the terms of a chief series for P_{ab} which intersects to a chief series on the subgroup of P_{ab} generated by the image of a. Then for $i \geq n$, we have $\phi(A) \cap P_i = \phi(B) \cap P_i = 1$ and for $i < n$ the conditions of Theorem 6.1.6 hold by construction. Hence $P*_{\phi(A)}$ is residually p, and we may take a p-group quotient $P*_{\phi(A)} \to Q$ in which no element of P is killed. The kernel of the composite map

$$H \to G = H*_A \to P*_{\phi(A)} \to Q$$

is $\gamma_n^{(p)}(H)$ as required.

To show that H is p-separable in G, proceed as in the proof of Proposition 6.1.5: write $g \in G \smallsetminus H$ as a reduced word in the sense of HNN extensions, and take a sufficiently deep lower central p-quotient $P = H/\gamma_n^{(p)}(H)$ so that the image of g in $P*_{\phi(A)}$ is again a reduced word not in P. As shown above, $P*_{\phi(A)}$ is residually p, so admits a p-group quotient Q distinguishing the image of g from the image of P. This quotient Q of G exhibits that H is p-separable in G. $\square$

Propositions 6.1.5 and 6.1.7 together give the following more general result.

Proposition 6.1.8. *Let Σ be a compact orientable surface and let $l_1, \ldots, l_n$ be a collection of pairwise disjoint, non-isotopic, essential simple closed curves in Σ. Then the splitting of Σ along the l_i gives a p-efficient graph of groups decomposition of $\pi_1\Sigma$.*

The following proposition is an easy consequence of [DdSMS03, Propositions 0.8 and 0.10].

Lemma 6.1.9. *Let P be a finite p-group and suppose $\psi \in \mathrm{Aut}(P)$ acts unipotently on the $\mathbb{F}_p$-vector space $H_1(P; \mathbb{Z}/p)$. Then ψ has p-power order.*

The reader is reminded that ψ 'acts unipotently' if some power of $(\psi - \mathrm{id})$ is the zero map. Lemma 6.1.9 was used in [Kob13] to prove that certain semidirect products are residually p. The reader is warned that in the arXiv version [Kob09] of that paper, Lemma 6.1.9 is stated in the context of finite nilpotent groups, where it is false.

Lemma 6.1.9 allows us to give a complete characterisation of the pro-p topology on certain semidirect products. First we fix some notation. Let G and C be finitely generated groups, and let $\Phi\colon C \to \mathrm{Aut}(G)$ be a homomorphism. Denote the automorphism $\Phi(c)$ by Φ_c and define the semidirect product $G \rtimes C$ to be the set $G \times C$ equipped with group operation

$$(g_1, c_1) \star (g_2, c_2) = (g_1 \Phi_{c_1}(g_2), c_1 c_2)$$

Identify G with $\{(g,1) : g \in G\}$ and C with $\{(1,c) : c \in C\}$. There is a function (not a homomorphism of course) u from the semidirect product $G \rtimes C$ to the direct product $G \times C$ 'forgetting the map Φ', which is the identity on the underlying sets of the two groups. Note that if N is a characteristic normal subgroup of G and D is a subgroup of C, then $N \rtimes D$ is a subgroup of $G \rtimes C$ and $u(N \rtimes D) = N \times D$.

Proposition 6.1.10. *Let G and C be finitely generated groups and let $\Phi\colon C \to \mathrm{Aut}(G)$ be a homomorphism. Suppose that the automorphism Φ_c acts unipotently on $H_1(G;\mathbb{F}_p)$ for each $c \in C$. Then the forgetful function $u\colon G \rtimes C \to G \times C$ is a homeomorphism, where both groups are given their pro-p topology.*

Proof. We first claim that it suffices to prove the following two statements:

(i) For each $U \lhd_p G \rtimes C$ (i.e. for each basic open neighbourhood of 1), there exists $V \subseteq G \times C$ open such that $1 \in V \subseteq u(U)$.

(ii) For each $U \lhd_p G \times C$ a basic open neighbourhood of 1, there exists $V \subseteq G \rtimes C$ open such that $1 \in V \subseteq u^{-1}(U)$.

That is, the neighbourhood bases at 1 match up. For left-multiplication by $(g,1)$ and right-multiplication by $(1,c)$ are continuous both as maps on $G \times C$ and on $G \rtimes C$, and commute with u. Thus if $U \subseteq G \rtimes C$ is a basic open neighbourhood of (g,c), then finding $V \subseteq G \times C$ such that $1 \in V \subseteq (g^{-1},1)u(U)(1,c^{-1})$ gives a $(G \times C)$-open set $(g,1)V(1,c)$ exhibiting that $u(U)$ is a $G \times C$-neighbourhood of (g,c). Hence (i) implies that u is an open map. Similarly (ii) implies that u is continuous.

Let us prove statement (i). If $U \lhd_p G \rtimes C$ is a basic open neighbourhood of 1, then $U \cap G \lhd_p G$ and $U \cap C \lhd_p C$, so $V = (U \cap G) \times (U \cap C)$ is a normal p-power index subgroup of $G \times C$, so is $(G \times C)$-open. Also $1 \in V \subseteq u(U)$ since if $(g,1),(1,c) \in U$ then $(g,c) = u((g,1) \star (1,c)) \in U$. So (i) holds and u is an open mapping.

The more difficult statement is (ii). Let $U = N \times D$ be a basic open neighbourhood of 1 in $G \times C$, where $N \lhd_p G$, $D \lhd_p C$ and N is characteristic in G. Then $N \rtimes D = u^{-1}(U)$ is a subgroup of $G \rtimes C$ with index a power of p; however it need not be normal. We will find a deeper subgroup that is normal in $G \rtimes C$, still with index a power of p.

Now $H_1(G/N;\mathbb{F}_p)$ is a quotient of $H_1(G;\mathbb{F}_p)$, on which Φ_c acts unipotently for every $c \in C$; so by Lemma 6.1.9, the map induced by Φ_c on G/N has order a power of p. Thus every element of the image of $C \to \mathrm{Aut}(G/N)$ has order a power of p and the image of C is a finite p-group. Let $K \lhd_p C$ be the kernel of this map. Each element of $D \cap K$ acts trivially on G/N, so we have a quotient map

$$G \rtimes C \to (G/N) \rtimes C \to (G/N) \rtimes (C/D \cap K)$$

whose kernel $V = N \rtimes (D \cap K)$ is thus a normal subgroup of $G \rtimes C$ with index a power of p, and $1 \in V \subseteq u^{-1}(U)$. Thus (ii) holds as required. $\square$

Remark. Note that the pro-p topology on the group $G \times C$ is the product of the pro-p topologies of G and C. In particular if both G and C are residually p then so is $G \times C$, hence under the conditions of the above proposition $G \rtimes C$ is also residually p.

Theorem 6.1.11. *Let M be a compact fibred 3-manifold with fibre Σ and monodromy ϕ, where Σ is a surface of negative Euler characteristic. Let p be a prime. Then M has a finite-sheeted cover with p-efficient JSJ decomposition.*

Proof. Without loss of generality both M and Σ are orientable. Then, possibly after performing an isotopy of the monodromy, the JSJ tori of M intersect Σ in a collection of disjoint non-isotopic essential simple closed curves $\{l_1, \ldots, l_n\}$ which are permuted by the monodromy ϕ. The l_i divide Σ into a number of subsurfaces $\Sigma_1, \ldots, \Sigma_m$. The monodromy acts on the set of subsurfaces $\{\Sigma_j\}$. Each piece of the JSJ decomposition corresponds to an orbit of this action, and is fibred over any element of that orbit. If n_j is the size of the orbit of Σ_j then ϕ^{n_j} acts on Σ_j either periodically or as a pseudo-Anosov. Let k be the order of ϕ in

$$\mathrm{Sym}(\{\Sigma_1, \ldots, \Sigma_m\}) \times \mathrm{Sym}(\{l_1, \ldots, l_n\})$$

The map ϕ^k also acts on each $H_1(\Sigma_j;\mathbb{F}_p)$. Take some multiple k' of k such that $\phi^{k'}$ acts by the identity on each $H_1(\Sigma_j;\mathbb{F}_p)$. Let $\psi = \phi^{k'}$ and let $\widetilde{M}$ be the surface bundle over Σ with monodromy ψ, an index k' cover of M. We claim that $\widetilde{M}$ has p-efficient JSJ decomposition.

Now ψ fixes each Σ_j and l_i and acts on each Σ_j periodically or as a pseudo-Anosov, so that the JSJ tori of $\widetilde{M}$ are precisely the tori $l_i \times \mathbb{S}^1$ and the pieces of the JSJ decomposition are the mapping tori $\widetilde{M_j} = \Sigma_j \rtimes_\psi \mathbb{S}^1$. We must show that each vertex (respectively edge group) $\pi_1(\Sigma_j \rtimes_\psi \mathbb{S}^1)$ (respectively edge group $\pi_1(l_i \times \mathbb{S}^1)$) is p-separable and inherits the full pro-p topology from $\pi_1 \widetilde{M}$. We prove this statement for the vertex groups.

Choose a basepoint $x \in \Sigma_j$ and a loop γ lying in $\widetilde{M_j}$ transverse to the fibres and passing through x. The homotopy class of γ gives a splitting of the quotient map to $\mathbb{Z}$ coming from the fibration, hence gives an identification of $\pi_1 \widetilde{M}$ with $\pi_1\Sigma \rtimes_\psi \mathbb{Z}$ in which the vertex group $\pi_1 \widetilde{M_j}$ is embedded as $\pi_1\Sigma_j \rtimes_\psi \mathbb{Z}$. The forgetful function $u\colon \pi_1\Sigma \rtimes_\psi \mathbb{Z} \to \pi_1\Sigma \times \mathbb{Z}$ now sends $\pi_1\Sigma_j \rtimes_\psi \mathbb{Z}$ to $\pi_1\Sigma_j \times \mathbb{Z}$. The action of ψ on each $H_1(\Sigma_i; \mathbb{F}_p)$ is unipotent by construction, hence also is the action on $H_1(\Sigma; \mathbb{F}_p)$. Hence by Proposition 6.1.10, u is a homeomorphism of pairs

$$(\pi_1 \widetilde{M}, \pi_1 \widetilde{M_j}) = (\pi_1\Sigma \rtimes_\psi \mathbb{Z}, \pi_1\Sigma_j \rtimes_\psi \mathbb{Z}) \to (\pi_1\Sigma \times \mathbb{Z}, \pi_1\Sigma_j \times \mathbb{Z})$$

By Proposition 6.1.8, $\pi_1\Sigma_j$ is p-separable in $\pi_1\Sigma$ and inherits its full pro-p topology; The same is thus true of $\pi_1\Sigma_j \times \mathbb{Z}$ in the product topology. The homeomorphism u now yields the result.

The proof for edge groups is almost identical; one only need sure that x and γ lie in the appropriate torus rather than in a vertex space. $\square$

6.2 Conjugacy p-separability

The efficiency of the JSJ decomposition of a 3-manifold played a key role in the proof that 3-manifold groups are conjugacy separable [WZ10, HWZ12]. In [AF13] it was asked whether p-efficiency plays the same role for conjugacy p-separability. In this section we answer that question in the affirmative for graph manifolds.

In [WZ10, Theorem 5.2] Wilton and Zalesskii proved a combination theorem for conjugacy separability. The proof of this uses the theory of profinite groups acting on profinite trees. The parallel theory for pro-p groups yields the following theorem.

Theorem 6.2.1. *Let $\mathcal{G} = (X, G_\bullet)$ be a graph of groups with conjugacy p-separable vertex groups G_v. Let $G = \pi_1(\mathcal{G})$ and suppose that the graph of groups $\mathcal{G}$ is p-efficient and that the action of $\widehat{G}_{(p)}$ on the standard tree of $\widehat{\mathcal{G}}_{(p)} = (X, \widehat{G_\bullet}_{(p)})$ is 2-acylindrical. Suppose that the following conditions hold for any vertex v of X and any edges e and f of X incident to v:*

1. *for any $g \in G_v$ the double coset $G_e g G_f$ is p-separable in G_v;*
2. *the edge group G_e is conjugacy p-distinguished in G_v;*
3. *the intersection of the closures of G_e and G_f in the pro-p completion is equal to the pro-p completion of their intersection, i.e. $\overline{G}_e \cap \overline{G}_f = \widehat{G_e \cap G_f}_{(p)}$.*

Then G is conjugacy p-separable.

The proof is in all respects a repetition of the argument in [WZ10], and we shall not reproduce it here. The difficulty lies in applying Theorem 6.2.1 in the absence of any sledgehammer properties such as subgroup separability or double coset separability in the pro-p world. Instead we must verify these properties for the specific cases involved in a particular application, and resist the temptation to attempt to prove too broad a result.

As an immediate consequence, when all the conditions on edge groups are trivial, we have the following corollary.

Corollary 6.2.2. *A free product of conjugacy p-separable groups is conjugacy p-separable.*

We now prove a series of lemmas directed towards showing that the conditions of Theorem 6.2.1 hold in the cases of Fuchsian groups and p-efficient graph manifolds. Many of the lemmas follow closely the analogous results for the profinite topology; where this is wholly or partly the case the result will be cited in brackets.

In [Nib92] Niblo uses the following 'doubling trick' to deduce double-coset separability. The proof works just as well for the pro-p topology, so we will use it to check condition 1 of Theorem 6.2.1.

Theorem 6.2.3 (Niblo [Nib92]). *Let K and L be subgroups of G. Let τ denote the involution which swaps the two factors of $G *_L G$. If $\langle K, K^\tau \rangle$ is p-separable in $G *_L G$ then the double coset LK is p-separable in G.*

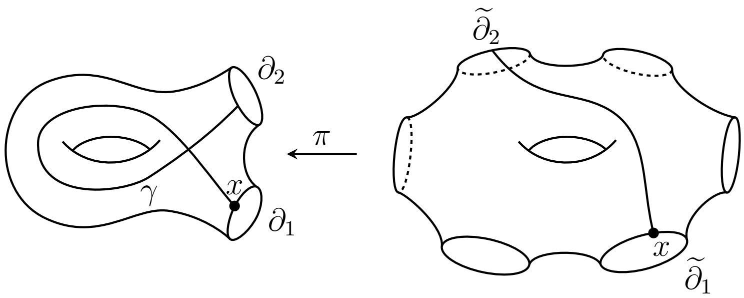
Proof. Identical with the proof of [Nib92, Theorem 3.2]. □

Lemma 6.2.4. *Let Σ be an orientable surface, let $G = \pi_1\Sigma$ and let D_1 and D_2 be maximal peripheral subgroups of G. Then the double coset D_1D_2 is p-separable in G.*

Proof. By Proposition 6.1.2 we may assume $D_1 \neq D_2$. Suppose that D_1 and D_2 arise from boundary components ∂_1 and ∂_2 of Σ (possibly $\partial_1 = \partial_2$). Choose a basepoint x on ∂_1. Performing a conjugation we may assume that D_1 is generated by the homotopy class of the loop running around ∂_1 based at x. Choose an immersed arc γ joining x to a point on ∂_2 such that D_2 is generated by the homotopy class of the loop based at x which runs along γ to ∂_2, once around ∂_2, then back to x along γ. In the case that $\partial_2 = \partial_1$ choose γ to be a loop based at x.

The finitely many self-intersections of γ with itself give a finite collection of unbased loops in Σ. Pass to a regular p-power degree cover $\pi\colon \widetilde{\Sigma} \to \Sigma$ so that none of these loops lifts; such a cover exists since $\pi_1\Sigma$ is residually p. Furthermore, in the case when $\partial_1 = \partial_2$, we can use the p-separability of $D_1 = \pi_1(\partial_1, x)$ to choose $\widetilde{\Sigma}$ such that γ is not congruent to any element of D_1 modulo $\pi_1\widetilde{\Sigma}$. Let $H \lhd_p G$ be the corresponding subgroup of G. Choose a lift $\tilde{x}$ of x to $\widetilde{\Sigma}$ to serve as a new basepoint. Then by construction γ lifts to an *embedded* arc $\tilde{\gamma}$ in $\widetilde{\Sigma}$ starting at $\tilde{x}$. Let the component of $\pi^{-1}(\partial_1)$ containing $\tilde{x}$ be denoted $\widetilde{\partial}_1$, and the component of $\pi^{-1}(\partial_2)$ containing the other endpoint of $\tilde{\gamma}$ be $\widetilde{\partial}_2$. Note that $\widetilde{\partial}_1 \neq \widetilde{\partial}_2$ since if γ is a loop, $\widetilde{\Sigma}$ was constructed so that γ does not lift either to a loop or to an arc with both endpoints on $\widetilde{\partial}_1$ (since such a lift would imply that γ was congruent to D_1 modulo H). Then $D_1 \cap H$ is generated by the loop $\widetilde{\partial}_1$ based at $\tilde{x}$, and $D_2 \cap H$ is generated by the homotopy class of the loop based at $\tilde{x}$ which runs along $\tilde{\gamma}$ to $\widetilde{\partial}_2$, once around $\widetilde{\partial}_2$, then back to $\tilde{x}$ along $\tilde{\gamma}$.

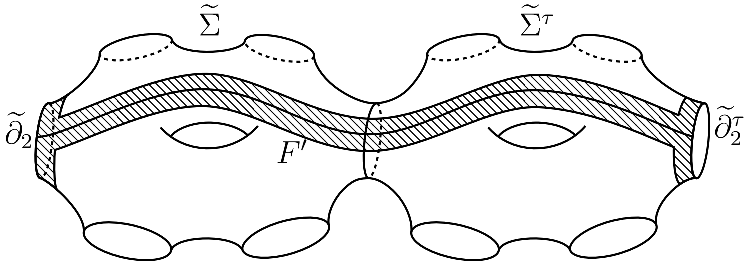
Note that p-separability of $(D_1 \cap H)(D_2 \cap H)$ in H implies p-separability of D_1D_2 in G; for the latter double coset is the union of finitely many translates of the former. We may now apply the 'doubling trick'. Glue two copies $\widetilde{\Sigma}$ and $\widetilde{\Sigma}^\tau$ of $\widetilde{\Sigma}$ along $\widetilde{\partial}_1$ to



obtain a surface F. The subgroup $\langle (D_2 \cap H), (D_2 \cap H)^\tau \rangle$ of $\pi_1(F, \tilde{x}) = H *_{D_1 \cap H} H$ is now the fundamental group of a certain subsurface F' of F whose boundary is an essential curve in F—specifically, take F' to be a regular neighbourhood

$$\mathcal{N}(\gamma \cup \widetilde{\partial}_2 \cup \gamma^\tau \cup \widetilde{\partial}_2^\tau)$$

Now $\langle (D_2 \cap H), (D_2 \cap H)^\tau \rangle = \pi_1(F', \tilde{x})$ is p-separable in H by Proposition 6.1.5, so by Theorem 6.2.3 the double coset $(D_1 \cap H)(D_2 \cap H)$ is p-separable in H and the proof is complete.



□

Corollary 6.2.5. *Let G be the fundamental group of a 2-orbifold O. Assume G is residually p and that O is orientable if $p \neq 2$. Let D_1 and D_2 be maximal peripheral subgroups of G. Then the double coset D_1D_2 is p-separable in G.*

Proof. The given assumptions imply that there is a regular index p cover of O which is an orientable surface Σ. If $H = \pi_1\Sigma$ then $(D_1 \cap H)(D_2 \cap H)$ is p-separable in H. As noted above, this implies that D_1D_2 is p-separable in G. □

Corollary 6.2.6. *Let G be the fundamental group of a Seifert fibre space M with non-empty boundary. Assume G is residually p and let D_1 and D_2 be maximal peripheral subgroups of G. Then the double coset D_1D_2 is p-separable in G.*

Proof. Again it suffices to pass to a regular p-cover. Because G is residually p, G admits a regular p-cover of the form $\Sigma \times \mathbb{S}^1$, where Σ is an orientable surface. If $\pi\colon \Sigma \times \mathbb{S}^1 \to \Sigma$ is the projection then

$$D_1 D_2 = \pi_*(D_1)\pi_*(D_2) \times \mathbb{Z}$$

so the result follows. $\square$

Recall for the following discussions that the boundary of a 2-orbifold is not necessarily the same as the boundary ∂_{top} of the underlying surface. An orbifold with boundary is locally modelled on quotients of open subsets of the upper half-plane by group actions, and boundary points of the orbifold come from boundary points of the upper half-plane. Some portions of ∂_{top} may indeed be part of the orbifold boundary; however some of ∂_{top} may be included in the singular locus as 'reflector' curves. The isotropy group of an interior point of a reflector curve is $\mathbb{Z}/2$. The endpoints of a reflector curve may have 'corner reflector' points whose isotropy subgroup is dihedral. Alternatively an endpoint of a reflector curve may again have isotropy group $\mathbb{Z}/2$, the local model for such a point being the upper half-plane modulo a reflection in the y-axis. Since reflections are order 2, when $p \neq 2$ reflector curves do not arise in an orbifold with residually p fundamental group. When they do arise, there is a canonical 'reflectorless' index 2 cover of the orbifold with no reflector curves. Corner reflectors become cone points in this cover. An orbifold is said to be orientable if its underlying surface is orientable. We recall some key points from Section 5.2, stated in their pro-p variants.

Remark. The reader may be wondering why these reflector curves did not appear in Chapters 4 or 5 where 2-orbifolds featured heavily. The reason is that reflector curves never arise as base orbifolds of the orientable Seifert fibre spaces considered there—reflector curves would correspond to 'fibred solid Klein bottles' in Seifert fibre spaces, which cannot appear in a non-orientable manifold. The reader who is only interested in the results showing conjugacy p-separability for fundamental groups of orientable Seifert fibre spaces may therefore ignore the issues associated to reflector curves.

Lemma 6.2.7. *Let O be a hyperbolic 2-orbifold with non-empty boundary and no reflector curves. Let ∂_1 and ∂_2 be curves representing components of ∂O. Suppose $\pi_1^{orb}O$ is residually p. Let $\Gamma = \widehat{\pi_1^{orb}O}_{(p)}$ and let Θ_i be the closure in Γ of $\pi_1\partial_i$. Then for $\gamma_i \in \Gamma$, either $\Theta_1^{\gamma_1} \cap \Theta_2^{\gamma_2} = 1$ or $\partial_1 = \partial_2$ and $\gamma_2\gamma_1^{-1} \in \Theta_1$.*

Proof. See Lemma 5.2.3. □

Lemma 6.2.8. *Let L be a Seifert fibre space with non-empty boundary with hyperbolic base orbifold O. Suppose that $\pi_1 L$ is residually p. Let $\Lambda = \widehat{\pi_1 L}_{(p)}$ and Z be the subgroup of $\pi_1 L$ generated by a regular fibre. Let Δ_1 and Δ_2 be peripheral subgroups of H—that is, conjugates in H of the closure of peripheral subgroups of $\pi_1 L$. Then $\Delta_1 \cap \Delta_2 = \overline{Z}$ unless $\Delta_1 = \Delta_2$, where $\overline{Z}$ is the closure of Z in Λ.*

Proof. See Lemma 5.2.4. □

For the next two propositions we use the following notation. Let G be the fundamental group of a p-efficient graph manifold, with graph of groups decomposition $(X, G_\bullet)$. Let $\Gamma = \Pi_1(\widehat{\mathcal{G}}_{(p)})$ be the pro-p completion of G. Let $S(\widehat{\mathcal{G}}_{(p)})$ be the standard graph for this graph of pro-p groups. For a vertex group G_v of $\mathcal{G}$, let Z_v be the subgroup generated by its regular fibre (the 'canonical fibre subgroup'). Let $\overline{Z}_v$ be the closure in $\Gamma_v = \widehat{G_v}_{(p)}$ and extend this notation to all vertex groups of $S(\widehat{\mathcal{G}}_{(p)})$ by the conjugation action.

Lemma 6.2.9. *Let $e = [v, w]$ be an edge of $S(\widehat{\mathcal{G}}_{(p)})$. Let Z_v and Z_w be the canonical fibre subgroups of G_v and G_w respectively. Then $\langle \overline{Z}_v, \overline{Z}_w \rangle \lhd_p \Gamma_e$, and so $\overline{Z}_v \cap \overline{Z}_w = 1$.*

Proof. After a conjugation in Γ, we may assume that e is an edge in the standard graph of the abstract fundamental group G, i.e. Γ_e is the closure in Γ of a peripheral subgroup of some G_v. Elementary calculations show that if two elements of $\mathbb{Z}^2$ generate an index subgroup $p^r m$ subgroup of $\mathbb{Z}^2$, where m is coprime to p, then they generate a subgroup of any p-group quotient of $\mathbb{Z}^2$ of index dividing p^r and hence generate an index p^r subgroup of $\mathbb{Z}_p^2$. The result follows. □

Proposition 6.2.10. *Let M be a p-efficient graph manifold in which all Seifert fibre spaces have hyperbolic base orbifold. Then the action of $\Gamma = \widehat{\pi_1 M}_{(p)}$ on the standard graph $S(\widehat{\mathcal{G}}_{(p)})$ is 2-acylindrical.*

Remark. The condition on the base orbifolds is automatic when $p \neq 2$. In general it may be achieved by passing to an index 2 cover.

Proof. Essentially identical to the proof of Proposition 5.2.8, simplified for this special case. $\square$

Proposition 6.2.11. *Let O be a hyperbolic 2-orbifold with non-empty boundary. Let $G = \pi_1^{orb} O$ and suppose G is residually p. Let $D = \langle l \rangle$ be the fundamental group of a boundary component of O. Then D is conjugacy p-distinguished in G.*

Proof. First suppose that D is a free factor of G, say $G = D * G'$. Suppose that $g \in G$ is not conjugate in G to any power of l. Write g as a reduced word

$$g = g_1 d_1 g_2 \dots g_n d_n$$

where $g_i \in G'$, $d_i \in D$ are all non-trivial except perhaps g_1 or d_n. We may ensure $g_1 \neq 1$ by conjugating by d_1. Since g is not conjugate into D, at least one of the following occurs:

- n is odd
- $d_n \neq 1$
- $g_i \neq g_{n+1-i}^{-1}$ for some i
- $d_i \neq d_{n-i}^{-1}$ for some $i \neq n/2$

since if all the above fail then we have expressed g as a conjugate of $d_{n/2}$. By uniqueness of reduced forms, no element whose reduced form has any of the above properties can be conjugate into D: for writing any $h \in G$ as a reduced word, $h^{-1} d h$ is already written as a reduced word having none of the above properties.

Now G is residually p, so we may find finite p-group quotients $D \to P_1$ and $G' \to P_2$ such that no non-trivial d_i or g_i vanishes under the quotient map, and so that any of the properties from the above list are preserved in the quotient. Then if $\phi\colon G \to P_1 * P_2$ is the quotient map, $\phi(g)$ is written as a reduced word in $P_1 * P_2$ which has one of the above properties and hence is not conjugate into P_1. Since P_1 is finite and $P_1 * P_2$ is conjugacy p-separable, there is a p-group quotient $\psi\colon P_1 * P_2 \to Q$ such that $\psi\phi(g)$ is not conjugate into $\psi\phi(D) = \psi(P_1)$. Hence D is conjugacy p-distinguished in G.

We now deal with the general case. To this end, let $g \in G$ and suppose that $\gamma^{-1} g \gamma = l^\alpha \in \overline{D}$ for some $\gamma \in \widehat{G}_{(p)}$ and $\alpha \in \mathbb{Z}_p$. Note that g is infinite order. Let $F \lhd_p G$ represent a regular p-power degree cover of O with more than one boundary component, so that $D \cap F$ is a free factor of F. Note that $\gamma = h\delta$ for some $h \in G$ and $\delta \in \overline{F}$. For some $n = p^r$ we have $g^n \in F$. Then

$$\delta^{-1}(h^{-1} g^n h)\delta = \gamma^{-1} g^n \gamma = l^{n\alpha} \in \overline{F \cap D}$$

By the first part, since $F \cap D$ is conjugacy p-distinguished in F and $\delta \in \overline{F}$, there exists some $f \in F$ such that $f^{-1}(h^{-1} g^n h) f \in F \cap D$. Thus $g' = (hf)^{-1} g (hf)$ is a parabolic element of G, some power of which lies in D; and since parabolic subgroups of a Fuchsian group either intersect trivially or are equal, it follows that $g' \in D$. Hence g is conjugate into D as required. □

Lemma 6.2.12. *Let O be a hyperbolic 2-orbifold with residually 2 fundamental group G and let ρ be a reflector curve of O with isotropy group $\mathbb{Z}/2 = \langle \tau \rangle$. Then τ is conjugacy 2-distinguished in G.*

Proof. First consider the degree 2 reflectorless cover O' of O obtained by doubling along reflector curves. Let the corresponding index 2 subgroup of G be G'. The order 2 elements of G which do not lie in G' are precisely the conjugates of reflector elements: cone points in O lift to O', and the intersection of each isotropy group of a corner reflector with G' is precisely its rotation subgroup. It thus suffices to distinguish τ from the other reflector elements. So let ρ' be a different reflector curve of O, with isotropy group $\langle \tau' \rangle$.

Now collapse the complement of a neighbourhood of the boundary component of $\partial_{\rm top}(O)$ containing ρ, and consider the induced map $\psi\colon G \to H$ on orbifold fundamental groups. The resulting orbifold $\overline{O}$ is a 'polygon': it is topologically a disc, whose boundary consists of segments which are either reflector curves or components of the orbifold boundary ∂O. If ρ' is not among these reflector curves then $\psi(\tau')$ is trivial. The canonical reflectorless cover of $\overline{O}$ therefore yields a quotient map to $\mathbb{Z}/2$ distinguishing τ from τ'.

So assume that ρ' survives in H, and is therefore one of the reflector curves of $\overline{O}$. Pass to a further quotient of H by abelianising the isotropy group of each corner reflector of $\overline{O}$ to obtain a copy of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, where the two incident reflector curves generate the two factors. We are left with a right-angled Coxeter group in which τ and τ' form part of a standard generating set. They thus have distinct images in first $\mathbb{Z}/2$-homology and hence are not conjugate in this quotient of G. This completes the proof. $\square$

Definition 6.2.13. A *hierarchical* (2-)orbifold will mean any 2-orbifold which is not on the following list:

- a sphere or projective plane with at most 3 cone points; or
- a disc or Möbius band, with ∂_{top} composed entirely of reflector curves and with at most one cone point and at most three corner reflectors.

Notice that in the above definition the reflectorless cover of any hierarchical orbifold is also hierarchical.

The reason for this definition is that all hierarchical orbifolds O admit a 'hierarchy' of the following type. If the orbifold has any genuine boundary curves or arcs, then cutting along arcs with both endpoints on a genuine boundary curve or arc (i.e. along an interval with trivial fundamental group) or along an arc with one endpoint on a genuine boundary curve or arc and the other endpoint on a reflector curve (i.e. along the quotient of an interval by a reflection, a 1-orbifold with fundamental group $\mathbb{Z}/2$) allows us to decompose the orbifold fundamental group into copies of $\mathbb{Z}$, $\mathbb{Z}/2 * \mathbb{Z}/2$, and finite p-groups glued along copies of $\mathbb{Z}$, $\mathbb{Z}/2$ or the trivial group. Note that in this case the reflectorless index 2 subgroup is correspondingly decomposed as a free product of copies $\mathbb{Z}$ and finite p-groups amalgamated over $\mathbb{Z}$ or the trivial group.

When the entirety of ∂_{top} is composed of reflector curves, and O is not on the above list, one may still obtain a hierarchy. We will not in fact use this hierarchy in the sequel, but it gives more consistency to the definition of 'hierarchical'. The first stage in the hierarchy is obtained as follows. If O is a disc, or Möbius band with reflector boundary and at least four corner reflectors, let l be an embedded 1-orbifold whose endpoints lie on the reflector curve such that at least two corner reflectors lie on either side of l. Note that the orbifold fundamental group of l is a copy of $\mathbb{Z}/2 * \mathbb{Z}/2$ along which G splits. If O is a cylinder with reflector boundary, choose an embedded 1-orbifold l with one endpoint on each reflector curve; again G splits over $\mathbb{Z}/2 * \mathbb{Z}/2 = \pi_1^{\rm orb} l$. Otherwise choose an essential simple closed curve l on O which does not pass through any cone points; such a curve exists for any orbifold other than those appearing in the above list.

Theorem 6.2.14. *Let $G = \pi_1^{orb} O$ be a residually p Fuchsian group, where O is a hyperbolic 2-orbifold that is orientable when $p \neq 2$. Suppose further that O is hierarchical. Then G is conjugacy p-separable.*

Proof. We note that each splitting of G given by the above hierarchy satisfies the conditions of Theorem 6.2.1. First consider the case when O has no reflector curves; this covers all cases when $p \neq 2$. When O has (genuine) boundary the result follows from Corollary 6.2.2 since then we may decompose G as a suitable free product of free groups and p-groups. Otherwise we have a splitting of G along a simple closed curve as an amalgamated free product or HNN extension of Fuchsian groups with (genuine) boundary, which are conjugacy p-separable. Passing to a regular cover of O which is a surface, the splittings along lifts of l are p-efficient by Proposition 6.1.8. It follows that the splitting of G is p-efficient. The action on the standard pro-p tree of the splitting is 1-acylindrical by Lemma 6.2.7. The remaining conditions 1, 2 and 3 in Theorem 6.2.1 hold by Corollary 6.2.5, Proposition 6.2.11, and Lemma 6.2.7 respectively. Hence we may apply Theorem 6.2.1 to conclude that G is conjugacy p-separable.

Now let $p = 2$ and suppose that O has reflector curves. Let O' be the canonical reflectorless degree 2 cover of O obtained by doubling O along its reflector curves and replacing any corner reflectors by cone points. Let $G' = \pi_1^{\rm orb} O'$. Note that O' is a hierarchical orbifold. Let $g_1, g_2 \in G$ be conjugate in the pro-2 completion $\widehat{G}_{(2)}$. If $g_1 \in G'$ then g_1 is conjugacy 2-distinguished in G' and hence in G by Lemma 6.2.15

below, so we are done. So suppose g_1 (hence g_2) is in $G \smallsetminus G'$. If g_1 has order 2 then since the only order 2 elements of $G \smallsetminus G'$ are in isotropy groups of reflector curves we are done by Lemma 6.2.12.

So suppose g_1 is of infinite order. Let $\gamma \in \widehat{G}_{(2)}$ be such that $g_1 = g_2^\gamma$. Conjugating g_2 by an element $\tau \in G'$ we may assume that γ lies in $\widehat{G'}_{(2)}$. Then g_1^2 is conjugate in $\widehat{G'}_{(2)}$ to g_2^2; since G' is conjugacy 2-separable, they are conjugate in G'. After a conjugation by an element of G' we may thus assume $g_1^2 = g_2^2$.

We claim that any infinite order element of a Fuchsian group has at most two square roots, differing by a reflection. For if $g = h^2$ is hyperbolic, then h is hyperbolic with the same fixed points on $\partial_\infty \mathbb{H}^2$ as g and hence with the same axis in $\mathbb{H}^2$. The translation length of h along this axis is half that of g, which determines h up to a reflection in the axis. If $g = h^2$ is parabolic then again g and h have the same fixed point at infinity and h is parabolic. Placing this fixed point at infinity in the upper half-space model of the hyperbolic plane, g and h are determined uniquely by their action (by isometries) on the x-axis, and a translation $x \mapsto x + a$ has precisely one square root $x \mapsto x + a/2$.

Hence either $g_1 = g_2$ as required or one of g_1 and g_2 is orientation preserving and the other is orientation reversing. In the latter case g_1 and g_2 have different images under the orientation homomorphism $G \to \mathbb{Z}/2$, and so cannot be conjugate in $\widehat{G}_{(2)}$. This concludes the proof. □

The extension of this to all Fuchsian groups does not follow immediately, since conjugacy separability is not a commensurability invariant (see [Gor86, CZ09, MM12]). In what follows we remind that reader that 'open subgroups $H \leq_p G$' are those subgroups of G with index a power of p such that H contains some normal subgroup of G with index a power of p. Note that G induces the full pro-p topology on such an H. Not all subgroups with index a power of p are necessarily open (for instance a symmetric group $S_{p-1} \leq_p S_p$ for $p \geq 5$).

Lemma 6.2.15 (cf Lemma 1 of [Ste70])**.** *Let $g \in G$, and suppose that $H \leq_p G$ is open in G and contains g. If g is conjugacy p-distinguished in H, then it is conjugacy p-distinguished in G.*

Proof. If $\{g_1, \dots, g_n\}$ is a complete set of right coset representatives of H in G, then

$$g^G = \bigcup_{i=1}^{n} (g^H)^{g_i}$$

where superscripts denote conjugation. By assumption g^H is closed in H, hence in G. Since g^G is a finite union of translates of g^H, the conjugacy class g^G is closed in G and g is conjugacy p-distinguished in G. $\square$

Proposition 6.2.16 (Theorem 3.9 of [Ste72])**.** *Let G be a group containing a free group or a surface group $F \lhd_p G$. Then elements of infinite order in G are conjugacy p-distinguished.*

Proof. The proof is identical with that of [Ste72, Theorem 3.9], noting that all finite index subgroups constructed there are open and have index a power of p in the present situation. $\square$

Lemma 6.2.17 (Lemma 3.8 of [Ste72])**.** *Let G be a group, $A \lhd_p G$. Suppose that A is a residually p abelian group. Then G is conjugacy p-separable.*

Proof. Again the proof in [Ste72] works with no modification. $\square$

Theorem 6.2.18. *Let O be a 2-orbifold, and suppose that $G = \pi_1^{orb} O$ is residually p and that O is orientable when $p \neq 2$. Then G is conjugacy p-separable.*

Proof. If O is not hyperbolic then G has an abelian subgroup $A \lhd_p G$, so that we are done by Lemma 6.2.17. By Theorem 6.2.14 we have reduced to the case of those non-hierarchical orbifolds appearing in the statement of Definition 6.2.13. Take $g \in G$. We must show that g is conjugacy p-distinguished. By Proposition 6.2.16, without loss of generality g is of finite order, say of order p^n. Since G is residually p, there are arbitrarily large p-group quotients $\phi \colon G \to P$ into which $\langle g \rangle$ injects. Choose $|P|$ sufficiently large that $H = \phi^{-1}(\langle \phi(g) \rangle)$ has rational Euler characteristic at most -3. Considering Definition 6.2.13 we see that all non-hierarchical 2-orbifolds have Euler characteristic strictly greater than -3. So H is the fundamental group of a hierarchical 2-orbifold. Then H is conjugacy p-separable by Theorem 6.2.14, so g is conjugacy p-distinguished in H. Note that $H \leq_p G$ is an open subgroup of G

containing g, hence g is conjugacy distinguished in G by Lemma 6.2.15. So G is conjugacy p-separable. $\square$

Given Theorem 6.2.18 the next two theorems follow from similar results in [Mar07] by simply checking that all finite-index subgroups constructed can be chosen to be normal of index a power of p.

Theorem 6.2.19 (Theorem 3.7 of [Mar07])**.** *Let G contain an orientable surface subgroup $\pi_1\Sigma \lhd_p G$. Then G is conjugacy p-separable.*

Lemma 6.2.20 (Lemma 4.2 of [Mar07])**.** *Let H be a group containing a normal p-power index orientable surface subgroup. Suppose G is a central extension of H by a finite p-group. Then G contains a normal orientable surface subgroup of index a power of p and hence is conjugacy p-separable.*

Theorem 6.2.21. *Let G be the fundamental group of a Seifert fibre space which has hyperbolic base orbifold. Assume that G is residually p. Then G is conjugacy p-separable.*

Remark. The analogous theorem for conjugacy separability is [Mar07, Theorem 5.2].

Proof. Suppose first that $p \neq 2$ and let g and g' be non-conjugate elements of $G = \pi_1 M$. Let h denote the homotopy class of a regular fibre of M and let O be the quotient orbifold of M, so that we have a central extension

$$1 \to \langle h \rangle \to G \to \pi_1^{\mathrm{orb}} O \to 1$$

If the images of g and g' in $\pi_1^{\mathrm{orb}} O$ are not conjugate, we are done by Theorem 6.2.18. So suppose g and g' are conjugate in $\pi_1^{\mathrm{orb}} O$. After a conjugacy we may assume that $g' = gh^n$ for some n. Choose some k such that $p^k > |n|$ and consider the quotient $\phi\colon G \to G' = G/\langle h^{p^k} \rangle$. Note that centralisers in Fuchsian groups are cyclic, so that the pre-image of the centraliser of g in $\pi_1^{\mathrm{orb}} O$ is a copy of $\mathbb{Z}^2$. Hence if $x \in G'$ conjugates $\phi(g)$ to $\phi(gh^m)$ for some m, then in fact $x \in \langle g, h \rangle$. So x commutes with $\phi(g)$ in G' giving a contradiction. Hence $\phi(g')$ is not conjugate to $\phi(g)$ in G'. By Lemma 6.2.20 the group G' is conjugacy p-separable and we are done.

Now let $p = 2$; the difference here is that O may be non-orientable. Let G^+ be the index 2 subgroup of G consisting of elements which centralise h. If $g \in G^+$ then g is conjugacy p-distinguished in G^+ and hence in G by Lemma 6.2.15. So suppose $g \in G \smallsetminus G^+$ and let $g' \in G$ be a non-conjugate of g. Again it suffices to deal with the case $g' = gh^n$. Now, since $g^{-1}hg = h^{-1}$, g is conjugate to gh^{2k} for all k; so n is odd. Consider the quotient $\phi\colon G \to G' = G/\langle h^2 \rangle$, which is conjugacy 2-separable. Suppose $x \in G'$ conjugates $\phi(g)$ to $\phi(g')$. Again the centraliser of the image of g in $\pi_1^{\text{orb}} O$ is cyclic, and the preimage of this group is a copy of $\mathbb{Z} \times \mathbb{Z}/2$ containing x. Hence $\phi(g)$ and $\phi(g')$ are not conjugate and we are done. $\square$

The restriction to hyperbolic base orbifolds in the above theorem was necessary to exclude problems with the geometry Nil, as the following example shows. Note that the three remaining Seifert fibred geometries ($\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, and $\mathbb{E}^3$) have no such issues as all these groups are finite or virtually abelian and are easily dealt with.

Example 6.2.22. We claim that the Heisenberg group $G = \mathcal{H}_3(\mathbb{Z})$ is not conjugacy p-separable for any prime p. Suppose $p \neq 2$, the $p = 2$ case being similar. We have a presentation

$$G = \langle x, y, h \,|\, [x, y] = h \text{ central} \rangle$$

By direct calculation, x^2 is not conjugate to x^2h. For any n we have

$$y^{-n}x^2y^n = x^2h^{2n}$$

In any p-group quotient $\phi\colon G \to P$, we have $\phi(x^2h) = \phi(x^2h^{2n})$ for some n, so that the image of x^2h is always conjugate to the image of x^2, proving the claim. Note that the congruence quotients exhibit that $\mathcal{H}_3(\mathbb{Z})$ is indeed residually p. See [Iva04] for a characterisation of conjugacy p-separable nilpotent groups.

Theorem 6.2.23. *Let G be the fundamental group of a p-efficient graph manifold in which all Seifert fibre spaces have hyperbolic base orbifold. Then G is conjugacy p-separable.*

Proof. The vertex groups are conjugacy p-separable by the previous result. By Proposition 6.2.10, the action on the standard pro-p tree of this splitting is 2-acylindrical.

Condition 1 of Theorem 6.2.1 holds by Corollary 6.2.6. Condition 2 holds by Proposition 6.2.11 since an element of a vertex group is conjugate into the boundary if and only if its image in the Fuchsian quotient is conjugate into the boundary. Condition 3 holds by Lemma 6.2.8. Hence Theorem 6.2.1 applies and G is conjugacy p-separable. $\square$

By [AF13, Proposition 5.2] any graph manifold has a finite-sheeted cover of the above type, so Theorem J follows immediately.

Chapter 7

Miscellany

7.1 RAAGs and special groups

This section is taken from the paper [KW16], which was joint work with Robert Kropholler. Right-angled Artin groups (RAAGs) have been the subject of much recent interest, especially because of their rich subgroup structure; in particular every special group embeds in a RAAG [HW08]. Furthermore RAAGs are linear and have excellent residual properties. Here we will show that RAAGs, and the closely related right-angled Coxeter groups (RACGs), are in fact completely determined by their finite quotient groups. The proofs will rely principally on the cohomological rigidity result of Koberda [Kob12, Theorem 6.4].

First let us recall some definitions.

Definition 7.1.1. Given a (finite simplicial) graph X, the *right-angled Artin group* $A(X)$ is the group with generating set $V(X)$ with the relation that vertices v and w commute imposed whenever v and w span an edge of X. The *right-angled Coxeter group* $C(X)$ is the quotient of $A(X)$ with the additional constraint that each generator has order 2.

The isomorphism type of a right-angled Artin group $A(X)$ uniquely determines the graph X up to isomorphism. This fact was first established by Droms [Dro87a]. We use a stronger cohomological criterion proved by Koberda [Kob12], who builds on earlier work of Subalka [Sab09], Droms [Dro87b], Gubeladze [Gub98] and Charney and Davis [CD95].

Theorem (Koberda [Kob12], Theorem 6.4). *Let X and Y be finite graphs. Then $X \cong Y$ if and only if there is an isomorphism of cohomology groups*

$$H^*(A(X);\mathbb{Q}) \cong H^*(A(Y);\mathbb{Q})$$

in dimensions one and two, which respects the cup product.

The proof relied solely on the following fact: each vertex $v \in X$ is dual to a cohomology class $f_v \in H^1(A(X);\mathbb{Q})$ for which the map

$$f_v \smile \bullet\colon H^1(A(X);\mathbb{Q}) \to H^2(A(X);\mathbb{Q})$$

has rank precisely the degree of the vertex v; moreover a vertex $w \in X$ is adjacent to v precisely if $f_v \smile f_w$ is non-zero.

Now the class $f_v \smile f_w$ is dual to an embedded 2-torus in the Salvetti complex of $A(X)$, hence gives a primitive element of $H^2(A(X);\mathbb{Z})$. It follows that changing the coefficient field $\mathbb{Q}$ to a finite field $\mathbb{Z}/p$ (for p a prime) changes neither the rank of the above map, nor the adjacency condition following it. Hence Koberda's cohomological rigidity result also holds with coefficient field $\mathbb{Z}/p$.

It remains to show that the pro-p completion of our right-angled Artin group detects the cohomology in dimensions one and two, and the cup product. As discussed in Section 2.2 it is frequently the case for groups arising in low-dimensional topology that the cohomology of a group is determined by its profinite completion. We always have substantial control over the cohomology in dimensions one and two. For the reader's convenience we recall from Section 2.2 the following result.

Proposition 2.2.4. *Let G be a discrete group and let p be a prime.*

- *$H^1(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^1(G;\mathbb{Z}/p)$ is an isomorphism;*
- *$H^2(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^2(G;\mathbb{Z}/p)$ is injective; and*
- *if H^{1+1} denotes that part of second cohomology generated by cup products of elements of H^1, then $H^{1+1}(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^{1+1}(G;\mathbb{Z}/p)$ is an isomorphism;*

where all the maps are the natural ones induced by $G \to \widehat{G}_{(p)}$.

By consideration of the standard classifying space for a RAAG [Cha07, Section 2.6], one sees that the dimension two cohomology is in fact generated by cup products. Thus in dimensions one and two, the algebra $H^*(A(X);\mathbb{Z}/p)$ is determined by the pro-p completion $\widehat{A(X)}_{(p)}$. Hence we have proved the following theorem.

Theorem 7.1.2. *Let X and Y be finite graphs and let p be a prime. Then $\widehat{A(X)}_{(p)} \cong \widehat{A(Y)}_{(p)}$ if and only if $X \cong Y$.*

In fact much more is true about the cohomology of the pro-p completion of $A(X)$.

Theorem 7.1.3 (Lorensen [Lor08, Lor10])**.** *The map from a right-angled Artin group to its pro-p completion (or profinite completion) induces an isomorphism of mod-p cohomology for any prime p.*

We can extend Theorem 7.1.2 to right-angled Coxeter groups by noting that there are natural isomorphisms

$$H^1(C(X);\mathbb{Z}/2) \cong H^1(A(X);\mathbb{Z}/2)$$

and

$$H^2(C(X);\mathbb{Z}/2) \cong H^2(A(X);\mathbb{Z}/2) \oplus (\mathbb{Z}/2)^{|V(X)|}$$

The second summand here corresponds to the relations $v^2 = 1$ and may be described intrinsically as follows. Modulo 2, we have the relations $(a+b)^2 = a^2 + b^2$ so that the image of the squaring map $a \to a \smile a$ is a subgroup Σ of $H^2(C(X))$, the image of the diagonal subgroup of $(H^1(C(X)))^2$. This subgroup Σ is precisely the second summand above. The quotient map

$$f\colon H^2(C(X);\mathbb{Z}/2) \to H^2(A(X);\mathbb{Z}/2)$$

with kernel Σ agrees with the natural map induced on cohomology by the natural map $A(X) \to C(X)$. From the description of Σ given above we see that this quotient map f is unique (i.e. does not depend on the presentation of $C(X)$ as a particular right-angled Coxeter group).

Thus the structure of the algebra $H^*(A(X))$ in dimensions one and two is determined by the behaviour of $H^*(C(X); \mathbb{Z}/2)$ in those dimensions, with the cup product map being given by the canonical map

$$(H^1(A(X)))^2 \xrightarrow{\cong} (H^1(C(X)))^2 \xrightarrow{\smile} H^2(C(X)) \to H^2(C(X))/\Sigma \cong H^2(A(X))$$

described above. Proposition 2.2.4 shows this algebra to be an invariant of the pro-2 completion. This completes the proof of the following theorem.

Theorem 7.1.4. *Let X and Y be finite graphs. Then $\widehat{C(X)}_{(2)} \cong \widehat{C(Y)}_{(2)}$ if and only if $X \cong Y$.*

Proposition 2.2.4 was sufficient to prove the Theorem. In fact right-angled Coxeter groups are 2-good, so that we have an isomorphism on cohomology in all dimensions. This follows from extension properties of 2-goodness applied to Proposition 9 of [LOS12] or from the work on graph products in [Sch14].

We may now prove that all virtually compact special groups are good. This was proved for hyperbolic groups by Schreve [Sch14] and later by Minasyan and Zalesskii [MZ16], both using virtual retraction properties. We give a proof using cube complex hierarchies. Hierarchies have also been used to prove goodness in other contexts, in particular in [GJZZ08].

The key feature of compact special groups is that they spilt along stabilisers of hyperplanes, giving a hierarchy terminating in the trivial group. These hierarchies interact well with separability properties, due to the following theorem. The hypothesis 'combinatorially quasiconvex' includes those subgroups arising in hierarchies.

Theorem (Haglund-Wise [HW08], Corollary 7.9)**.** *Combinatorially quasiconvex subgroups of virtually compact special groups are separable.*

Goodness is preserved under taking amalgamated free products and HNN extensions, given suitable conditions.

Theorem 7.1.5 (Proposition 3.6 of [GJZZ08])**.** *An efficient amalgamated free product or HNN extension of good groups is good.*

Theorem 7.1.6. *Virtually compact special groups are good.*

Proof. As noted in Section 2.2.1 we are free to pass to an arbitrary finite index subgroup of G and prove goodness there. We define a measure of complexity for a compact special group H. Set $n(H)$ to be the minimal dimension of a CAT(0) cube complex X on which H acts with special quotient. After subdividing, a hyperplane in this complex is an embedded 2-sided cubical subcomplex and H splits as an HNN extension or amalgamated free product over the stabiliser of this hyperplane. Iterating this process yields a rooted tree of groups in which each vertex has either two or three descendants (depending on whether the vertex splits as an HNN extension or amalgamated free product). Because H is compact special, this tree is finite and each branch terminates in the trivial group. Let $m(H)$ be the length of a longest branch over all such trees with minimal diameter—that is, the length of a shortest hierarchy for H. Now define the *quasiconvex hierarchy complexity* $\mu(G)$ of a special group G to be the pair of integers $(n(G), m(G))$. Order the pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$ lexicographically.

Now assume that all compact special groups H with $\mu(H) < \mu(G)$ are good. We have a splitting of G either as $A *_C B$ or $A *_C$. Now C is the stabiliser of a hyperplane. This hyperplane is a CAT(0) cube complex of dimension smaller than $n(G)$ whose quotient by C is special. Hence $n(C) < n(G)$, so $\mu(C) < \mu(G)$ and so C is good. Furthermore A and B have shorter hierarchies than G, so whether or not $n(A) = n(G)$, the complexities $\mu(A) < \mu(G)$ and $\mu(B) < \mu(G)$ do strictly decrease. Thus A, B and C are good. Furthermore combinatorially quasiconvex subgroups of G are separable. All finite index subgroups of A, B and C are combinatorially quasiconvex so the splitting is efficient and we may apply Theorem 7.1.5 to conclude that G is good. Note that the base case for the induction is simply the trivial group. $\square$

Recalling that Haglund and Wise [HW10] proved that all hyperbolic Coxeter groups are virtually compact special, we have:

Corollary 7.1.7. *Hyperbolic Coxeter groups are good.*

For right-angled Artin groups, Theorem 7.1.3 guaranteed that in fact the mod-p cohomology is determined by the pro-p completion. This property, which is sometimes called *p-goodness*, is rather rarer than straightforward goodness. In particular proofs will often require strong separability constraints in which only p-group quotients are available. These constraints are difficult to obtain in general.

We move now to a result of a rather different flavour. Often, properties of the underlying graph of a right-angled Artin or Coxeter group are expressible as group theoretic properties. As an example of such a property carrying over to the profinite world, we prove the following theorem.

Theorem 7.1.8. *Let X be a graph. Then $\widehat{A(X)}$ or $\widehat{C(X)}$ splits as a non-trivial profinite free product $\Delta_1 \amalg \Delta_2$ if and only if X is disconnected.*

Proof. If X is disconnected the result follows directly from the abstract case. So suppose that X is connected and that $\Gamma = \widehat{A(X)}$ splits as a profinite free product $\Delta_1 \amalg \Delta_2$. The case when X is a point is easy, so suppose that X is not a point.

The splitting of Γ as a free profinite product induces an action of Γ on a profinite tree T, where vertex stabilisers are precisely the conjugates of the Δ_i (Theorem 3.3.7). All edge stabilisers are trivial, so that no element of Γ can fix more than one point of T. By Proposition 3.1.20, any abelian group acting on T either fixes a point or is a subgroup of $\widehat{\mathbb{Z}}$. Each of the standard generators of $A(X)$ is contained in a copy of $\mathbb{Z}^2$ as there is some edge adjacent to the corresponding vertex. These copies of $\mathbb{Z}^2$ are retracts of the whole RAAG which therefore induces the full profinite topology on them. Hence the closure of these copies of $\mathbb{Z}^2$ are copies of $\widehat{\mathbb{Z}}^2$ in the profinite completion, which must therefore fix a vertex of T. Hence every generator of Γ fixes some (unique) vertex of T and so is contained in a (unique) conjugate of Δ_1 or Δ_2.

Note that for every edge $e = [v, w]$ of X, the subgroup $\overline{\langle v, w\rangle}$ is a rank 2 free abelian group so that v and w fix the same vertex of T. By connectedness of X, it follows that all of $\widehat{A(X)}$ fixes a vertex of T and so $\Delta_1 = 1$ or $\Delta_2 = 1$.

The case of a right-angled Coxeter group is similar. Indeed it is easier, as we may use the fact that all the standard generators have finite order and therefore fix a vertex of T by Theorem 3.1.17. □

7.2 Pro-p prime decompositions

In this section we show that studying Betti numbers and L^2-Betti numbers is enough to allow us to detect the prime decomposition of a closed 3-manifold with residually p fundamental group via the pro-p completion of its fundamental group. The corresponding result for profinite prime decompositions has now been proven for all 3-manifolds ([WZ17b, Theorem A] for closed manifolds, followed by [Wil18c, Theorem 6.22] for the bounded case) using techniques of cohomology and profinite trees.

Let $\mathcal{C}$ be a variety of finite groups—that is, a family of finite groups closed under taking finite direct products, subgroups, and quotients. We will also assume that whenever a finite cyclic group $\mathbb{Z}/p$ is in $\mathcal{C}$ then $\mathbb{Z}/p^n$ is also in $\mathcal{C}$ for all n, so that the pro-$\mathcal{C}$ completion $\widehat{\mathbb{Z}}_{\mathcal{C}}$ is a product $\prod_{p\in\pi}\mathbb{Z}_p$ for some set of primes π. This is to avoid certain unfortunate varieties of groups such as $\mathcal{C} = \{(\mathbb{Z}/2)^n\}_{n\in\mathbb{N}}$ for which $\widehat{\mathbb{Z}}_{\mathcal{C}} = \mathbb{Z}/2$. Under these assumptions $\mathbb{Z}$ is distinguished by its pro-$\mathcal{C}$ completion from all other finitely generated residually $\mathcal{C}$ groups.

For a finitely generated profinite group Γ set

$$b_1(\Gamma) = \sup_{p \text{ prime}} \max\left\{l \in \mathbb{N} \text{ such that } \Gamma \twoheadrightarrow (\mathbb{Z}/p^k)^l \text{ for all } k\right\}$$

Note that this supremum always exists because Γ is finitely generated. Furthermore if Γ is the pro-$\mathcal{C}$ completion of a discrete group $\pi_1 X$ then

$$b_1(\Gamma) = b_1(X) = b_1(\pi_1 X)$$

In fact $b_1(\Gamma)$ is the highest integer n such that the abelianisation $H_1(\Gamma;\mathbb{Z})$ of Γ contains $\mathbb{Z}_p^n$ for some p. The above definition was chosen to demonstrate the link with the usual notion of first Betti number. For finitely generated residually finite discrete groups, we have the following theorem.

Theorem (Lück Approximation Theorem [Lüc94])**.** *Let G be a finitely generated residually finite group and let $\{G_i\}$ be a nested sequence of finite index normal subgroups of G intersecting in the identity. Then the limit*

$$\lim_{i\to\infty} \frac{b_1(G_i)}{[G : G_i]}$$

exists and equals $b_1^{(2)}(G)$ independently of the sequence $\{G_i\}$.

Definition 7.2.1. Given a profinite group Γ we say that $b_1^{(2)}(\Gamma)$ *exists* if

$$\lim_{i\to\infty} \frac{b_1(\Lambda_i)}{[\Lambda : \Lambda_i]}$$

exists for every nested sequence $\{\Lambda_i\}$ of open normal subgroups of Γ intersecting in the identity. Note that if this limit exists for every such sequence, its value is independent of the choice of sequence; we define this value to be $b_1^{(2)}(\Gamma)$. Note that if $\Gamma = \widehat{G}_{\mathcal{C}}$ for a residually $\mathcal{C}$ group G then by the Lück Approximation Theorem $b_1^{(2)}(\Gamma)$ exists and equals $b_1^{(2)}(G)$.

Proposition 7.2.2. *Suppose $\Gamma = \coprod_{j=1}^{n} \Gamma_i$ is a free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups Γ_j and that $\{\Lambda_i\}$ is a nested sequence of open normal subgroups of Γ intersecting in the identity. Then*

$$\liminf_{i\to\infty} \frac{b_1(\Lambda_i)}{[\Gamma : \Lambda_i]} \geq (n-1) - \sum_{j=1}^{n} \frac{1}{|\Gamma_j|}$$

following the usual convention that $\frac{1}{|\Gamma_j|} = 0$ when Γ_j is infinite. Moreover if $b_1^{(2)}(\Gamma_j)$ exists for each j, then $b_1^{(2)}(\Gamma)$ exists and equals

$$b_1^{(2)}(\Gamma) = \sum_{j=1}^{n} \left(b_1^{(2)}(\Gamma_j) - \frac{1}{|\Gamma_j|} \right) + n - 1$$

Proof. Let $t_{i,j} = |\Lambda_i \backslash \Gamma / \Gamma_j|$. Consider the action of Γ_j on the right of $\Lambda_i \backslash \Gamma$. The number of orbits under this action is $t_{i,j}$. All of these orbits are the same size because the action of $\Lambda_i \backslash \Gamma$ on itself on the left permutes them. Hence $t_{i,j}$ is equal to $[\Gamma : \Lambda_i]$ divided by the size of each orbit. The stabiliser of any coset in $\Lambda_i \backslash \Gamma$ is $(\Gamma_j \cap \Lambda_i)$ so the group $(\Gamma_j \cap \Lambda_i) \backslash \Gamma_j$ acts freely on $\Lambda_i \backslash \Gamma$. Hence the number of orbits is precisely $[\Gamma_j : \Gamma_j \cap \Lambda_i]$. We conclude that

$$t_{i,j} = \frac{[\Gamma : \Lambda_i]}{[\Gamma_j : \Gamma_j \cap \Lambda_i]}$$

Hence, noting that b_1 is additive under pro-$\mathcal{C}$ free products, we find

$$\frac{b_1(\Lambda_i)}{[\Gamma : \Lambda_i]} = \sum_{j=1}^{n} \sum_{\tau \in \Lambda_i \backslash \Gamma / \Gamma_j} \frac{b_1(\Lambda_i \cap g_{j,\tau} \Gamma_j g_{j,\tau}^{-1})}{[\Gamma : \Lambda_i]} + \frac{1}{[\Gamma : \Lambda_i]} + (n-1) - \sum_{j=1}^{n} \frac{t_{i,j}}{[\Gamma : \Lambda_i]}$$

Note that since each Λ_i was normal, the values $b_1(\Lambda_i \cap g_{j,\tau} \Gamma_j g_{j,\tau}^{-1})$ depend only i and j. Each such term appears $t_{i,j}$ times. Hence applying the above formula for $t_{i,j}$ we

find
$$\frac{b_1(\Lambda_i)}{[\Gamma:\Lambda_i]} = \sum_{j=1}^{n} \frac{b_1(\Lambda_i \cap \Gamma_j)}{[\Gamma_j:\Gamma_j \cap \Lambda_i]} + \frac{1}{[\Gamma:\Lambda_i]} + (n-1) - \sum_{j=1}^{n} \frac{1}{[\Gamma_j:\Gamma_j \cap \Lambda_i]}$$
If each $b_1^{(2)}(\Gamma_j)$ exists, taking the limit as $i \to \infty$ gives the result. If not, note that the right-hand side is at least as large as the sum of its last three terms and apply $\liminf_{i\to\infty}$. □

Corollary 7.2.3. *Let G be a finitely generated residually $\mathcal{C}$ group such that $b_1^{(2)}(G)$ vanishes. Then the pro-$\mathcal{C}$ completion $\widehat{G}_{\mathcal{C}}$ is not isomorphic to any non-trivial pro-$\mathcal{C}$ free product unless $G \cong \mathbb{Z}/2 * \mathbb{Z}/2$.*

Proof. Let $\Gamma = \widehat{G}_{\mathcal{C}}$. If $\Gamma \cong \Gamma_1 \amalg \Gamma_2$ then from above we have
$$0 = b_1^{(2)}(G) = b_1^{(2)}(\Gamma) \geq 1 - \frac{1}{|\Gamma_1|} - \frac{1}{|\Gamma_2|}$$
whence either $\Gamma_1 \cong \mathbb{Z}/2 \cong \Gamma_2$ or some Γ_i is trivial. It only remains to show that the only residually-$\mathcal{C}$ group Γ with pro-$\mathcal{C}$ completion isomorphic to the pro-$\mathcal{C}$ completion of $\mathbb{Z}/2*\mathbb{Z}/2$ is $\mathbb{Z}/2*\mathbb{Z}/2$ itself. Note that G has an index 2 subgroup H whose pro-$\mathcal{C}$ completion is $\widehat{\mathbb{Z}}_{\mathcal{C}}$. Since H is finitely generated and residually-$\mathcal{C}$ this forces $H \cong \mathbb{Z}$. Since $\mathbb{Z}/2*\mathbb{Z}/2$ is distinguished from the other finitely generated groups with an index 2 subgroup isomorphic to $\mathbb{Z}$ by being non-abelian it follows that $G \cong \mathbb{Z}/2 * \mathbb{Z}/2$. □

Recall from Section 3.3.1 the following theorem.

Theorem (Kurosh Subgroup Theorem for pro-p groups; Theorem 9.6.2 of [Rib17])**.** *Suppose $\Gamma = \coprod_{x\in X} \Gamma_x$ is a free pro-p product of pro-p groups Γ_j. Let $\Delta \leq_c \Gamma$, indexed by a profinite space X. Then for each $x \in X$ there is a family $\{g_{x,\tau}\}$ of representatives of the double cosets $\tau \in \Delta\backslash\Gamma/\Gamma_x$ such that the family $\{\Delta \cap g_{x,\tau}\Gamma_x g_{x,\tau}^{-1}\}$ of subgroups of Γ is continuously indexed by*
$$\Delta\backslash(\Gamma \times X)/\!\sim\; = \bigcup_{x\in X} \Delta\backslash\Gamma/\Gamma_x = \{\Delta g_{x,\tau}\Gamma_x\}$$
and H is the free pro-p product
$$\Delta = \coprod_{\Delta\backslash(\Gamma\times X)/\sim} (\Delta \cap g_{x,\tau}\Gamma_x g_{x,\tau}^{-1}) \amalg F$$
where F is a free pro-p subgroup of Γ.

Hence any closed subgroup of a free pro-p product is either contained entirely in some conjugate of a free factor, splits as a free pro-p product, or is isomorphic to $\mathbb{Z}_p$. The following result may now be deduced from the above statements along with the calculations of Lott and Lück [LL95] that the first L^2-Betti number of a closed aspherical 3-manifold is zero.

Proposition 7.2.4. *Let M and N be closed connected orientable 3-manifolds with prime decompositions $M = \#_{i=1}^{m} M_i$ and $N = \#_{i=1}^{n} N_i$ respectively. Suppose that $\pi_1 M$ and $\pi_1 N$ are both residually p and have isomorphic pro-p completions. Then after possibly reordering the factors, $n = m$ and $\widehat{\pi_1(M_i)}_{(p)} \cong \widehat{\pi_1(N_i)}_{(p)}$ for all i.*

Proof. The fundamental groups $\pi_1 M$ and $\pi_1 N$ are given as free products of some family of aspherical 3-manifold groups, some finite p-groups and a free group. Finite p-groups do not split as non-trivial free pro-p products since the latter are always infinite. For an aspherical summand M_i, the pro-p completion of $\pi_1(M_i)$ does not split as a free pro-p product by Corollary 7.2.3 since their first L^2-Betti number is zero—in the case $\pi_1 M_i = \mathbb{Z}/2 * \mathbb{Z}/2$ then $M_i = \mathbb{RP}^3 \# \mathbb{RP}^3$ is not aspherical. Hence by the pro-p Kurosh subgroup theorem it only remains to determine the rank of the free group factor, which may now be deduced from the mod-p homology. □

Every 3-manifold group is virtually residually p for all but finitely many primes p by a theorem of Aschebrenner and Friedl [AF13]. Note that the finite p-groups which arise as fundamental groups of closed 3-manifolds are very restricted—they are cyclic for $p \neq 2$. For $p = 2$ there are also generalised quaternion groups

$$Q_{4n} = \left\langle x, y \,\middle|\, x^2 = (xy)^2 = y^n \right\rangle$$

where $n \geq 2$ is a power of 2. See [AFW15, Section 1.7] for a complete list of finite 3-manifold groups.

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