Profinite Groups, Exercise Sheet 4 — Lent 2021

*[Key questions are marked with an obelus †. Expansion questions are marked with a star *.]*

1. Let G be a group.

(i) Show that each basis element $g \in G$ is a unit in the ring $\mathbb{Z}G$.

(ii) Show that if g is a finite order element of G then $g-1$ is a zero-divisor in $\mathbb{Z}G$.

Remark. It is an open question (one of the Kaplansky conjectures) whether these are the only units in $\mathbb{Z}G$, and whether $\mathbb{Z}G$ can have any zero divisors if G is torsion-free.

2. (Invariants of a module). Let G be a group and let M be a G-module. Define the *invariants* of M to be the set

$$M^G = \{m \in M : g \cdot m = m \forall g \in G\}$$

(i) Prove that M^G is a submodule of M.

(ii) Let $\alpha\colon M_1 \to M_2$ be a morphism of G-modules. Show that α restricts to a map $\alpha\colon M_1^G \to M_2^G$ (i.e. that $\alpha(M_1^G) \subseteq M_2^G$).

(iii) Consider a short exact sequence of G-modules

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

Show that the sequence

$$0 \longrightarrow M_1^G \xrightarrow{\alpha} M_2^G \xrightarrow{\beta} M_3^G$$

is exact.

[Note the absence of the '$\to 0$' at the end: there is no condition to check at M_3^G.]

(iv) Find an example to show that $M_2^G \to M_3^G$ need not be surjective.

3. Let $E = M \rtimes G$.

(i) Let $s\colon G \to E$ be a group homomorphism such that $G \to E \to G$ is the identity—i.e. a splitting of the extension E. Define $\psi_s\colon G \to M$ by

$$s(g) = (\psi_s(g), g) \in M \rtimes G.$$

Show that $\psi_s \in Z^1(G, M)$.

(ii) Let s and s' be group homomorphisms as above. Show that ψ_s and $\psi_{s'}$ differ by a 1-coboundary if and only if there exists $m \in M$ such that

$$(m, 1)s(g)(m, 1)^{-1} = s'(g)$$

for all $g \in G$. We call this relation M-*conjugacy.*

(iii) Given a 1-cocycle $\phi \in Z^1(G, M)$, construct a splitting $s\colon G \to E$ such that $\phi = \psi_s$.

(iv) Deduce that M-conjugacy classes of splittings are in bijection with elements of $H^1(G, M)$.

†4. (Functorial behaviour of extensions). Let G be a group and let M be a G-module. Let E be an extension of G by M, representing a cohomology class $\zeta \in H^2(G, M)$.

(i) Let $\alpha\colon M \to M'$ be a G-linear map. Let E act on M' via the quotient $E \to G$, and form the semidirect product $M' \rtimes E$.

Prove that $N = \{(\alpha(m), m^{-1}) \mid m \in M\}$ is a normal subgroup of $M' \rtimes E$.

[Note: in E we use multiplicative notation, and in M and M' we use additive notation. This has the unfortunate effect that when $m \in M$ is thought of as an element of E, its inverse is m^{-1}. The map α then has the property $\alpha(m^{-1}) = -\alpha(m)$.]

Define the group E' to be the quotient of $M' \times E$ by the above normal subgroup. Show that E' is an extension of G by M', that there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\alpha"] & E \arrow[r] \arrow[d] & G \arrow[r] & 1 \\
0 \arrow[r] & M' \arrow[r] & E' \arrow[ur] & &
\end{tikzcd}$$

and that E' represents the image $\alpha_*(\zeta)$ of ζ under the map

$$\alpha_*\colon H^2(G, M) \to H^2(G, M').$$

(ii) Let $f\colon G' \to G$ be a group homomorphism. Let G' act on M in the usual way, by the formula $g' \cdot m = f(g') \cdot m$. Find an extension E' of G' by M which represents the cohomology class $f^*(\zeta) \in H^2(G', M)$.

5. Let G be a cyclic group of order n. Let t be a generator of G.

†(i) Show that the sequence

$$\cdots \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}$ by G-modules, where α and β are the maps

$$\alpha(x) = x(t-1), \quad \beta(x) = x(1 + t + \cdots + t^{n-1})$$

and ϵ is the map given by sending $g \mapsto 1$ for all $g \in G$.

†(ii) Show that

$$H^k(G, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \text{ is odd} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k \geq 2 \text{ is even} \end{cases}$$

where $\mathbb{Z}$ has the trivial G-action.

(iii) Suppose $n = pq$ for integers p and q. Let $M = \mathbb{Z}^p$ and let G act on M via cyclic permutation of the basis elements. Find $H^k(G, M)$.

(iv) Suppose $n = 2$. Let $M = \mathbb{Z}$ where G acts via the map $1 \mapsto -1$. Find $H^k(G, M)$.

†(v) Construct, as in Theorem 5.2.1, a chain map f_n in dimensions at most 2 from the bar resolution to the above resolution:

$$\begin{tikzcd}
\mathbb{Z}G\{G^{(2)}\} \arrow[r, "d_2"] \arrow[d, "f_2"] & \mathbb{Z}G\{G^{(1)}\} \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G\{G^{(0)}\} \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z}G \arrow[r, "\beta"] & \mathbb{Z}G \arrow[r, "\alpha"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Hence find a 2-cocycle $\phi \colon G \times G \to \mathbb{Z}$ representing a generator of $H^2(G, \mathbb{Z}) \cong \mathbb{Z}$. (It is not necessary to check that it is a cocycle.)

[Hint: denote the elements of G by $\{t^k : 0 \leq k < n\}$. When constructing the chain map f_2 on generators $[t^k|t^l]$ you will need to distinguish between the cases $k + l < n$ and $k + l \geq n$.]

†(vi) Write down the group structure of the extension of G by $\mathbb{Z}$ corresponding to a generator of $H^2(G, \mathbb{Z})$. Prove that this extension is isomorphic to $\mathbb{Z}$.

(vii) By using Question 4, or otherwise, find explicit forms for all equivalence classes of central extensions of G by $\mathbb{Z}$. How many isomorphism types are there?

6. Let G be a finite group and let M be a G-module.

(i) Let $\phi \in Z^n(G, M)$, and define

$$\psi(g_2, \ldots, g_n) = \sum_{\gamma \in G} \gamma^{-1} \cdot \phi(\gamma, g_2, \ldots, g_n)$$

Prove that

$$d^{n-1}\psi = |G| \cdot \phi$$

†(ii) Assume that M is finitely generated as an abelian group. Show that $H^n(G, M)$ is a finite abelian group whose order divides some power of $|G|$.

†(iii) Assume that M is finite and suppose $|G|$ is coprime to $|M|$. Show that $H^n(G, M) = 0$ for all n.

(iv) Suppose a finite group E has an abelian normal subgroup M such that $|M|$ is coprime to $|E/M|$. Show that E has a subgroup isomorphic to E/M, and show that all such subgroups are conjugate.

7. (a) Let A and B be G-modules. Show that a G-module M is isomorphic to the direct sum $A \oplus B$ if and only if there exist maps

$$i_A \colon A \to M, \quad p_A \colon M \to A, \quad i_B \colon B \to M, \quad p_B \colon M \to B$$

such that $i_A p_A + i_B p_B = \mathrm{id}_M$, $p_A i_A = \mathrm{id}_A$, $p_B i_B = \mathrm{id}_B$, $p_A i_B = 0$ and $p_B i_A = 0$.

(b) Let M_1 and M_2 be G-modules. Prove that

$$H^n(G, M_1 \oplus M_2) \cong H^n(G, M_1) \oplus H^n(G, M_2)$$

for all n.

8. Let G be a group and let K be a subgroup of G. Let M be a K-module. Define the *coinduced module*[1] to be the abelian group

$$\mathrm{coind}_G^K(M) = \mathrm{Hom}_K(\mathbb{Z}G, M),$$

with G-action given by

$$(g \cdot f)(x) = f(xg)$$

where $f \in \mathrm{Hom}_K(\mathbb{Z}G, M)$ and $x \in \mathbb{Z}G$.

(i) Prove that the above formula is a valid action of G on $\mathrm{Hom}_K(\mathbb{Z}G, M)$.

(ii) Let F be a G-module. Show that the map

$$\Psi\colon \mathrm{Hom}_K(F, M) \to \mathrm{Hom}_G(F, \mathrm{coind}_G^K(M))$$

defined by

$$\Psi(h)(p)\colon x \mapsto h(xp)$$

for $h \in \mathrm{Hom}_K(F, M)$, $p \in F$, $x \in \mathbb{Z}G$ is a well-defined map of abelian groups. By constructing an inverse, or otherwise, show that it is an isomorphism.

(iii) Show that if F is a free G-module then F is also free as a K-module.

(iv) Prove *Shapiro's Lemma*:

$$H^n(K, M) = H^n(G, \mathrm{coind}_G^K(M))$$

(v) Deduce that $\mathrm{cd}(K) \leq \mathrm{cd}(G)$. Show that if a group has finite cohomological dimension then it is torsion-free (i.e. does not contain any elements of finite order except the identity).

†9. Let G be a pro-p group and let M be a finite p-primary G-module. Assume that M is simple.

(i) Show that M is killed by p—i.e. that $pm = 0$ for all $m \in M$.

(ii) Let z be an element of G such that z^p acts trivially on M. Show that $M^{\langle z \rangle}$ is non-trivial.

(iii) Let H be a group and let N be an H-module. Suppose that x is central in H. Show that $N^{\langle x \rangle}$ is an H-submodule of N.

(iv) Show that G acts trivially on M, and deduce that M is isomorphic to $\mathbb{F}_p$ with the trivial G-action.

***10.** Let G be a profinite group. A *discrete torsion G-module* is an abelian group M, equipped with the discrete topology, in which every element has finite order, equipped with a continuous map $G \times M \to M$ describing a G-action.

(i) Prove that a contnuous function from a compact space to a discrete space has finite image.

(ii) Prove that a discrete torsion module M is the union of its finite submodules.

[1] Also called the induced module in the literature. Because life is unfair sometimes.

(iii) Let (A_i) be a directed system of abelian groups. Show that any element of the directed limit

$$\varinjlim A_i = \bigoplus A_i / \langle a_i \sim \phi_{ij}(a_j) \rangle \rangle$$

is the image of an element of one of the A_i under the natural map. *[see Exercise Sheet 1 for the construction of the direct limit* $\varinjlim$*]*

(iv) Let $M = \bigcup M_i$ be a discrete torsion module, expressed as a union of finite submodules. Show that there is a natural isomorphism

$$C^n(G, M) \cong \varinjlim C^n(G, M_i).$$

Deduce that

$$H^n(G, M) \cong \varinjlim H^n(G, M_i).$$

(v) Let $i \in I$ be an inductive system. Let

$$0 \longrightarrow A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \longrightarrow 0$$

be a short exact sequence of abelian groups for every $i \in I$, such that for all $i \preceq j$ we have a diagram of transition maps

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "\alpha_i"] \arrow[d, "\phi^A_{ji}"] & B_i \arrow[r, "\beta_i"] \arrow[d, "\phi^B_{ji}"] & C_i \arrow[r] \arrow[d, "\phi^C_{ji}"] & 0 \\
0 \arrow[r] & A_j \arrow[r, "\alpha_j"] & B_j \arrow[r, "\beta_j"] & C_j \arrow[r] & 0
\end{tikzcd}$$

making each of $(A_i), (B_i), (C_i)$ into a directed system. Prove that we have an exact sequence

$$0 \longrightarrow \varinjlim A_i \xrightarrow{\alpha} \varinjlim B_i \xrightarrow{\beta} \varinjlim C_i \longrightarrow 0$$

These limiting properties mean, in effect, that properties of cohomology with coefficients in discrete torsion modules can be derived from the equivalent properties for finite modules. This makes cohomology theory well-behaved if we allow discrete torsion modules.

(vi) Let A be a compact abelian group and let M be a discrete torsion abelian group. Show that the group of continuous homomorphisms $\mathrm{Hom}(A, M)$ is a torsion abelian group.

This sort of proposition ultimately allows one to define the correct replacement for coinduced modules in the cohomology theory of profinite groups, filling one of the gaps in the technology used to establish the Course Convention properly.