Profinite Groups, Exercise Sheet 2 — Lent 2021

[Key questions are marked with an obelus **†**. *Expansion questions are marked with a star *.]*

1. Finite quotients of $\mathbf{SL}_N(\mathbb{Z})$. The obvious way to get a finite quotient of a matrix group $\mathrm{SL}_N(\mathbb{Z})$ is to reduce all the coefficients modulo some integer $m > 0$. The image is a matrix with coefficients in the ring $\mathbb{Z}/m\mathbb{Z}$ which will of course have determinant 1 modulo m. That is, there is a natural function $f_m\colon \mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\mathbb{Z}/m\mathbb{Z})$.

(i) Prove that f_m is a group homomorphism.

(ii) Let $a, b \in \mathbb{Z}$ be such that $ab \equiv 1$ modulo m. Show that the matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$$

lies in the image of f_m.

(iii) Let $(a_1, \ldots, a_N)^T$ be a column vector in $\mathbb{Z}^N$. Show that left-multiplication by matrices in $\mathrm{SL}_N(\mathbb{Z})$ can be used to reduce this column vector to the form $(b, 0, \ldots, 0)^T$ where $b = \mathrm{hcf}(a_1, \ldots, a_N)$.
Deduce that if $(\bar{a}_1, \ldots, \bar{a}_N)^T$ is a column vector in $(\mathbb{Z}/m\mathbb{Z})^N$ then left-multiplication by elements in $\mathrm{im}(f_m)$ can be used to reduce it to the form $(b, 0, \ldots, 0)^T$.

(iv) Deduce that $\mathrm{im}(f_m) = \mathrm{SL}_N(\mathbb{Z}/m\mathbb{Z})$. Deduce also that the natural map $\mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\widehat{\mathbb{Z}})$ has dense image.

2. $\widehat{\mathbf{SL}_2(\mathbb{Z})}$ and $\mathbf{SL}_2(\widehat{\mathbb{Z}})$. As you have just seen, the obvious quotients of the matrix group $\mathrm{SL}_2(\mathbb{Z})$ are the finite matrix groups $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. One might guess that these are effectively all the finite quotients ('effectively all' meaning 'a cofinal subsystem of the set of quotients'). We will see in this exercise that this fails dramatically, and the two profinite groups $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ and $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ are radically different.

(i) Show that the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

generate $\mathrm{SL}_2(\mathbb{Z})$. Deduce also that $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

[Hint: first show that the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

associated to elementary row/column operations live in the subgroup generated by A and B. Then imitate Question 1.]

*(ii) Let F be the free group generated by two elements a and b, and let K be the kernel of the map $F \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ sending

$$a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Using a suitable covering space of a wedge of two circles, compute a finite generating set S for K. *[It may be helpful to recall that* $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ *is isomorphic to the symmetric group* S_3*.]*

*(iii) Let $f\colon F \to \mathrm{SL}_2(\mathbb{Z})$ be the map given by $a \mapsto A$, $b \mapsto B$ (which is surjective by (i)). Noting that $f(K)$ equals the kernel $\Gamma_2 = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}))$, so that $f(S)$ generates this kernel, show that Γ_2 is generated by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

It is a fact that the subgroup generated by the final two matrices is in fact *free*, so Γ_2 is in fact isomorphic to $\mathbb{Z}/2\mathbb{Z} \times F$. In particular, Γ_2 admits a surjective homomorphism to $\mathbb{Z}$, and therefore to $\mathbb{Z}/n\mathbb{Z}$ for all n.

(iv) Now consider $G_2 = \ker(\mathrm{SL}_2(\mathbb{Z}_2) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}))$. Show that the three matrices from part (iii), considered as elements of $\mathrm{SL}_2(\mathbb{Z}_2)$, generate G_2 topologically. Show that

$$\begin{pmatrix} 1 & 16 \\ 0 & 1 \end{pmatrix}$$

must lie in the kernel of any map from G_2 to an abelian group.

[Hint: recall that 3 *has a multiplicative inverse in* $\mathbb{Z}_2$*, so that* $\mathrm{SL}_2(\mathbb{Z}_2)$ *contains a matrix*

$$\begin{pmatrix} 3 & 0 \\ 0 & 3^{-1} \end{pmatrix}$$

which may be useful.]

Conclude that there is no surjective continuous map $G_2 \to \mathbb{Z}/16\mathbb{Z}$.

(v) Show that if the natural map $\widehat{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{SL}_2(\widehat{\mathbb{Z}}) = \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$ is an isomorphism then $\widehat{\Gamma}_2 \cong G_2 \times \prod_{p \neq 2} \mathrm{SL}_2(\mathbb{Z}_p)$.

(vi) By considering the image of a suitable generating set show that if $p \neq 2$ then any homomorphism $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z}) \to \mathbb{Z}/16\mathbb{Z}$ has trivial image. Deduce that any continuous homomorphism $\mathrm{SL}_2(\mathbb{Z}_p) \to \mathbb{Z}/16\mathbb{Z}$ has trivial image.

(vii) Conclude that $\widehat{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{SL}_2(\widehat{\mathbb{Z}}) = \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$ is not an isomorphism.

Remark. It is a remarkable theorem of Bass–Lazard–Serre that exactly the opposite is true for matrices of size greater than 2: that is, $\widehat{\mathrm{SL}_N(\mathbb{Z})} \cong \mathrm{SL}_N(\widehat{\mathbb{Z}})$ for $N \geq 3$. Or, more plainly, any finite quotient $\mathrm{SL}_N(\mathbb{Z}) \to F$ factors as a map $\mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\mathbb{Z}/m\mathbb{Z}) \to F$ for some m.

†3. Prove that a profinite group $G = \varprojlim G_j$ has a topological generating set with d elements if and only if every $p_j(G)$ has a generating set with d elements.

†4. Let X be a finite set, let F be the abstract free group with basis X and let $\widehat{F}$ be its profinite completion. Show that $\widehat{F}$ is a *free profinite group with basis* X in the following sense: for any profinite group G and any function $i\colon X \to G$, there is a unique continuous homomorphism $\hat{\imath}\colon \widehat{F} \to G$ extending i.

5. Higman's Group.

(i) Let p be an odd prime. By considering the group $(\mathbb{Z}/p\mathbb{Z})^\times$, show that $p \mid 2^{p-1} - 1$.

(ii) Let $p \in \mathbb{N}$ and let $a, b, n \in \mathbb{N}$. Show that if $p \mid 2^n - 1$ then $p \mid 2^{an} - 1$. Show that if $p \mid 2^n - 1$ and $p \mid 2^{n+b} - 1$ then $p \mid 2^b - 1$. Deduce that there exists $r > 0$ such that

$$\{n \text{ such that } p \mid 2^n - 1\} = r\mathbb{Z}$$

(iii) Let $n > 1$. Show that the smallest prime factor of n is smaller than the smallest prime factor of $2^n - 1$.

(iv) Let G be a finite group. Let $g_1, \ldots, g_4 \in G$ satisfy the relations

$$\begin{matrix} g_1^{-1} g_2 g_1 = g_2^2 & g_2^{-1} g_3 g_2 = g_3^2 \\ g_3^{-1} g_4 g_3 = g_4^2 & g_4^{-1} g_1 g_4 = g_1^2 \end{matrix}$$

By considering the orders of $g_1, \ldots, g_4$, show that $g_1, \ldots, g_4$ are all equal to the identity of G. Deduce that the group

$$H = \left\langle x_1, \ldots, x_4 \;\middle|\; \begin{matrix} x_1^{-1} x_2 x_1 = x_2^2, & x_2^{-1} x_3 x_2 = x_3^2, \\ x_3^{-1} x_4 x_3 = x_4^2, & x_4^{-1} x_1 x_4 = x_1^2 \end{matrix} \right\rangle$$

has no finite quotients other than the trivial group.

6. Prove that a direct product of two residually finite groups is residually finite.

7. Let Γ be a finitely generated abstract group.

(i) A *characteristic subgroup* of Γ is a subgroup Δ such that $f(\Delta) = \Delta$ for every automorphism $f\colon \Gamma \to \Gamma$. Show that any finite index subgroup of Γ contains a finite index characteristic subgroup.

(ii) Let Γ be a finitely generated and residually finite abstract group. Show that $\mathrm{Aut}(\Gamma)$ is residually finite.

(iii) Let Γ and Λ be residually finite abstract groups, with Γ finitely generated. Let $\phi\colon \Lambda \to \mathrm{Aut}(\Gamma)$ be an injective homomorphism. Show that the semidirect product $\Gamma \rtimes_\phi \Lambda$ is residually finite.

8. Another characteristic subgroup one could define in the spirit of the Frattini subgroup would be

$$\Psi(G) = \bigcap \{N \mid N \text{ a maximal proper } \textit{normal} \text{ subgroup of } G\}$$

Show that $\Phi(G) \subseteq \Psi(G)$, and give an example to show that equality need not hold.

†9. Frattini subgroups of p-groups.

(i) Let G be a p-group and let M be a maximal proper subgroup of G. Show that M is normal and has index p.

(ii) Show that $\Phi(\mathbb{F}_p^d) = \{0\}$ for $d \geq 1$.

(iii) Show that $\Phi(G) = [G, G]G^p$.

†10. Decide, for each of the following sets S, whether they (topologically) generate:

(a) The free pro-3 group $\widehat{F}_{(3)}$ on two generators a and b;

(b) The free pro-5 group $\widehat{F}_{(5)}$ on two generators a and b;

(c) The free discrete group F on two generators a and b;

(i) $S = \{aba^{-1}b^3a^2,\ b^2a^2b^{-1}\}$

(ii) $S = \{ba^{-1}b^{-3},\ b^3ab^2ab^{-2}\}$

(iii) $S = \{bab^{-3},\ b^3ab^2ab^{-2}\}$

(iv) $S = \{babab^{-1}a^{-1},\ b^2aba\}$