# Profinite Groups, Exercise Sheet 1 Lent 2021

*[Key questions are marked with an obelus* **†**. *Expansion questions are marked with a star *.]*

**1.** (Pullbacks) Pullbacks are the dual construction to push-outs. Consider the category $\mathsf{J}$ with three objects $0, 1, 2$ and two non-identity arrows $1 \to 0$ and $2 \to 0$. A diagram of shape $\mathsf{J}$ in $\mathsf{Grps}$ consists of groups $G_0, G_1, G_2$ and homomorphisms $\phi_{10} \colon G_1 \to G_0$ and $\phi_{20} \colon G_2 \to G_0$. Give an explicit description of the limit of such a diagram.

[*Hint: Pushouts are quotients of coproducts. What sort of construction should be dual to that?*]

**2.** Let $\mathsf{J}$ be the category with two objects 1 and 2 and two morphisms $1 \to 2$. Consider the diagram of shape $\mathsf{J}$ in $\mathsf{Grps}$ shown below.

$$G_1 \overset{\phi}{\underset{1}{\rightrightarrows}} G_2$$

where $\phi$ is a group homomorphism and 1 denotes the trivial homomorphism. What is the limit of this diagram? What is the colimit?

**3.** Let $(J, \preceq)$ be a totally ordered set. What is the product of two objects $i$ and $j$ in the poset category $\mathsf{J}$? What is the colimit?

If $J$ is now merely a *partially* ordered set, how would you describe the product of two objects of $\mathsf{J}$? Does it always have to exist?

**4.** Let $J = \mathbb{N}$, with the partial ordering $n \preceq m$ if and only if $n \geq m$. Consider the diagram of shape $\mathsf{J}$ in $\mathsf{Sets}$ shown below, and assume that all the transition maps $\phi_{nm}$ are *inclusions*. What is the inverse limit of the $X_n$?

$$\cdots \longrightarrow X_3 \xrightarrow[\subseteq]{\phi_{32}} X_2 \xrightarrow[\subseteq]{\phi_{21}} X_1 \xrightarrow[\subseteq]{\phi_{10}} X_0$$

***5.** (Direct limits) Work in the category $\mathsf{AbGrp}$ of abelian groups for this question.

(i) Let $\mathsf{J}$ be a category consisting of a set $J$ of objects, with no morphisms other than the necessary identities. What is the colimit of a diagram $\mathbf{X} \colon \mathsf{J} \to \mathsf{AbGrp}$?

(ii) Now let $\mathsf{J}$ be a poset category. Suppose that $\mathsf{J}$ is a *directed system* in the sense that for any $i, j \in \mathrm{Obj}(\mathsf{J})$ there exists some $k \in \mathrm{Obj}(\mathsf{J})$ such that $i \preceq k$ and $j \preceq k$. Find an explicit form, akin to Proposition 1.2.8, for the colimit of a diagram $\mathbf{X} \colon \mathsf{J} \to \mathsf{AbGrp}$. Such a diagram is called a *directed system of abelian groups*, and its colimit is called the *direct limit*[1] $\varinjlim X_j$.

(iii) Let $\mathsf{J}$ be the poset category whose objects are the positive natural numbers with the ordering $n \preceq m$ if $n | m$. Let $X_n = \mathbb{Z}/n\mathbb{Z}$ and let $\phi_{nm} \colon X_n \to X_m$ be the inclusion map given by $1 \mapsto m/n$ for $n | m$. Show that $\varinjlim X_n \cong \mathbb{Q}/\mathbb{Z}$.

[1] Don't blame me, I didn't choose the name...

**†6.** Let $G_j$ ($j \in J$) be a family of finite groups, each with the discrete topology. Let $G = \prod G_j$ be the product and let $p_j \colon G \to G_j$ be the projection maps. Recall that the subbasic open sets on the product $\prod G_j$ are by definition $U_{j,x} = p_j^{-1}(x)$ for $j \in J$, $x \in G_j$.

(i) Let $m \colon G \times G \to G$ be the multiplication map and let $i \colon G \to G$ be the inversion map. Prove that $i$ and $m$ are continuous. For any $g \in G$, prove that the left-multiplication map $\mu_g \colon G \to G$, $h \to gh$ is continuous.

It follows that if $(G_j)_{j \in J}$ is an inverse system of finite groups, the restrictions of these maps to the inverse limit $\widehat{G} = \varprojlim G_j$ are also continuous, so that $\widehat{G}$ is a topological group.

(ii) Prove that $\widehat{G}$ is also the limit of the $G_j$ in the *category of topological groups*: that is, if $H$ is a topological group admitting *continuous* homomorphisms to all $G_j$ compatible with the transition maps, then the canonical homomorphism $H \to \widehat{G}$ is also continuous.

(iii) Show that if $f \colon \Gamma \to \Delta$ is a group homomorphism, then there is a natural continuous homomorphism $\hat{f} \colon \widehat{\Gamma} \to \widehat{\Delta}$ between the profinite completions, such that $\iota \circ f = \hat{f} \circ \iota$.

[*Some of the concepts assumed here may perhaps not have made it into the Metric Spaces and Topology course. There is an supplementary document on Moodle.*]

**†7.** Let $(X_j)_{j \in J}$ be an inverse system of finite groups such that all transition maps are surjective. Prove that all the projections $p_k \colon \varprojlim X_j \to X_k$ are surjective. Prove that a subset $Z \subseteq \varprojlim X_j$ is dense if and only if $p_j(Z) = X_j$ for all $j \in J$.

**†8.** Let $G$ be a compact topological group. Prove that a subgroup of $G$ is open if and only if it has finite index and is closed.

**9.** Prove that $\{0, 2\}^{\mathbb{N}}$ is homeomorphic to a Cantor set.

Let $(X_j)_{j \in J}$ be an inverse system of finite sets. Assume $J$ is countably infinite. Construct a countable set $I$ and a continuous injective function $\varprojlim X_j \to \{0, 1\}^I$. Deduce that $\varprojlim X_j$ is homeomorphic to a closed subset of a Cantor set.

**†10.** Let $(J, \preceq)$ be an inverse system, where $J$ is countable. Show that there exists a linearly ordered cofinal subsystem $J'$ of $J$.

Let $(G_j)_{j \in J}$ be an inverse system of finite groups where $J$ is countably infinite and linearly ordered. Assume that the transition maps are surjective. Prove that $\varprojlim G_j$ is uncountable or finite.

**†11.** Show that the ring $\mathbb{Z}_p$ has no zero-divisors.

***12. The $p$-adic rationals $\mathbb{Q}_p$.** Recall that the field of fractions of the integral domain $\mathbb{Z}_p$ is defined to be the set

$$\mathbb{Q}_p = \big\{(a, b) \mid a, b \in \mathbb{Z}_p, b \neq 0\big\} / \sim$$

with the equivalence relation

$$(a,b) \sim (c,d) \quad \Leftrightarrow \quad ad = bc$$

Of course the equivalence class of $(a,b)$ is denoted by $\frac{a}{b}$. Addition and multiplication are defined by the usual formulae

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

The ring $\mathbb{Z}_p$ is considered to be contained in $\mathbb{Q}_p$ in the usual way $a = a/1$.

(i) Show that any non-zero element $\alpha$ of $\mathbb{Q}_p$ may be written uniquely as $\alpha = p^k\alpha'$ for some $\alpha' \in \mathbb{Z}_p^\times$ and some $k \in \mathbb{Z}$ (where for $k < 0$, $p^k$ of course means $1/p^{-k}$).

Define the *p-adic norm* on $\mathbb{Q}_p$ by

$$|\alpha|_p = p^{-k} \quad \text{where } \alpha = p^k\alpha' \text{ for } \alpha' \in \mathbb{Z}_p^\times$$

and by $|0|_p = 0$, and define a function $d\colon \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}$ by $d(\alpha,\beta) = |\alpha - \beta|_p$. Note that $d$ is the usual metric on $\mathbb{Z}_p$.

(ii) Show that $d$ is a metric on $\mathbb{Q}_p$. Show that the sets

$$\{\alpha \in \mathbb{Q}_p \text{ such that } |\alpha|_p = p^{-k}\}, \quad \{\alpha \in \mathbb{Q}_p \text{ such that } |\alpha|_p \le p^{-k}\}$$

are both open and closed, for any $k$.

(iii) Show that $\mathbb{Q}_p$ is complete but not compact. *[You may use the fact that $\mathbb{Z}_p$ is complete].*

(iv) Show that the addition and multiplication maps $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}_p$ and the inversion map $\mathbb{Q}_p \setminus \{0\} \to \mathbb{Q}_p$ are continuous with this metric.

**13. Procyclic groups.** A *procyclic group* is an inverse limit of finite cyclic groups. You have already seen the procyclic groups $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$. In this question you will classify all of the procyclic groups.

(i) Show that a closed subgroup of a procyclic group is procyclic.

(ii) Exhibit a procyclic group $G$ and a subgroup $H$ which is isomorphic to $\mathbb{Z}^2$ (and is of course not a closed subgroup).

(iii) Let $G = \varprojlim_{j\in J} G_j$ be the inverse limit of a surjective inverse system of finite cyclic groups whose orders are powers of a fixed prime $p$. If $i \preceq j$ show that for any generator $x_j$ of $G_j$ every preimage $x_i \in \phi_{ij}^{-1}(x_j)$ is a generator of $G_i$. Show that there is a choice of generator $x_i \in G_i$ for all $i \in I$ such that $\phi_{ij}(x_i) = x_j$ for every $i \preceq j$. Deduce that there is a surjective continuous homomorphism $\mathbb{Z}_p \to G$.

(iv) Prove that the only infinite index closed subgroup of $\mathbb{Z}_p$ is the trivial subgroup. Show that any finite index open subgroup of $\mathbb{Z}_p$ is equal to $\ker(\mathbb{Z}_p \to \mathbb{Z}/p^e\mathbb{Z})$ for some $e \ge 0$.

(v) Use the Chinese Remainder Theorem to show that any procyclic group $G$ has the form $\prod_{p \text{ prime}} G_p$ where $G_p$ is a continuous quotient of $\mathbb{Z}_p$. Deduce a classification of all procyclic groups.