UNIVERSITY OF CAMBRIDGE

MAT3, MAMA

MATHEMATICAL TRIPOS Part III

Specimen Paper

PAPER 151

PROFINITE GROUPS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let (X_i) be a system of finite sets with transition maps $\phi_{ij}\colon X_i \to X_j$ for $i \preceq j$, indexed over a poset $(I, \preceq)$.

(i) Define what it means for a poset $(I, \preceq)$ to be an *inverse system*.

(ii) Give *two* definitons of the *limit* $\varprojlim X_i$: a definition by a universal property, and an explicit definition. [You need not prove that these definitions are equivalent.]

(iii) Show that if I is an inverse system then $\varprojlim X_i$ is non-empty.

(b) Let $(J, \preceq)$ be a poset with six elements $1, 2, 3, a, b, c$, related by:

$$c \preceq 1 \qquad c \preceq 2 \qquad b \preceq 1 \qquad b \preceq 3$$
$$a \preceq 2 \qquad a \preceq 3$$

and with no other non-trivial relations.

(i) Is J an inverse system? [Include your reasons in your answer.]

(ii) Find a system of finite sets $(X_j)_{j\in J}$ indexed over J, such that all transition maps $\phi_{ij}\colon X_i \to X_j$ for $i \preceq j$ are surjective, but such that the limit $\varprojlim X_i$ is the empty set.

(c) Consider the abstract group

$$\Gamma = \langle a, b \mid ba^2b^{-1} = a^3 \rangle \,.$$

(i) Let Q be a finite group and let $f\colon \Gamma \to Q$ be a surjective homomorphism. Show that the order of $f(a)$ is coprime to 6.

(ii) If n is an integer coprime to 6, construct a finite group Q_n and a surjective homomorphism $f_n\colon \Gamma \to Q$ such that $f(a)$ has order n.

(iii) Let $\iota\colon \Gamma \to \widehat{\Gamma}$ be the canonical map from Γ to its profinite completion. Prove that

$$\overline{\langle \iota(a) \rangle} \cong \prod_{p\neq 2,3} \mathbb{Z}_p$$

where p ranges over the prime numbers not equal to 2 or 3.

[You may use any theorems from lectures provided you state them clearly.]

2 (a) Let $G = \varprojlim G_j$ be a profinite group, where $(G_j)_{j \in J}$ is an inverse ststem of finite groups.

(i) Define what is meant by a *topological generating set* for G.

(ii) Let S be a subset of G. State a criterion, in terms of the G_j, for S to be a topological generating set for G.

(iii) Give an example of a profinite group G, a finitely generated abstract subgroup $\Gamma \leqslant G$, and a subset S of Γ such that S is not an abstract generating set for Γ but is a topological generating set for G. [You need not give a detailed proof that your example works.]

(b)

(i) State and prove Gaschutz's Lemma for Finite Groups.

(ii) Deduce Gaschutz's Lemma for Profinite Groups.

[You may assume any standard results concerning inverse systems.]

(c)

(i) Let F be a finitely generated free group, let Q be a finite group and let $f\colon F \to Q$ be a surjective homomorphism. Let $\hat{f}\colon \widehat{F} \to Q$ be the extension of f to $\widehat{F}$. Show that for any automorphism $\phi \in \mathrm{Aut}(Q)$ there exists a continuous automorphism $\hat{\phi} \in \mathrm{Aut}(\widehat{F})$ such that $\hat{f}\hat{\phi} = \phi\hat{f}$.

[You may freely use standard properties of profinite groups and of $\widehat{F}$ *provided you state them clearly.]*

(ii) Give an example of F, Q, and f such that there is no automorphism $\tilde{\phi} \in \mathrm{Aut}(F)$ such that $f\tilde{\phi} = \phi f$.

3 Let G be a group. Define a topology $\mathcal{T}$ on G whose basic open sets are those of the form

$$U_{Q,f,q} = f^{-1}(q)$$

where Q is a finite group, $f\colon G \to Q$ is a group homomorphism, and $q \in Q$.

(a)

(i) Define what it means for $(G, \mathcal{T})$ to be a *topological group.*

(ii) Define what it means for G to be *residually finite.* Show that G is residually finite if and only if $(G, \mathcal{T})$ is a Hausdorff topological space.

(b) A subset X of G is called *separable in* G if for every $g \in G \smallsetminus X$ there exists a finite group Q and a group homomorphism $f\colon G \to Q$ such that $f(g) \notin f(X)$.

(i) Show that X is separable in G if and only if X is closed in $(G, \mathcal{T})$.

(ii) Show that $X = G \smallsetminus \{1\}$ is separable if and only if G is finite.

(iii) Let H be a subgroup of G and suppose there exists a homomorphism $\rho\colon G \to H$ such that $\rho(h) = h$ for all $h \in H$. Show that the map

$$\sigma\colon G \to G, \quad \sigma(g) = \rho(g)g^{-1}$$

is continuous. [You may assume that $(G, \mathcal{T})$ is a topological group.]

If G is residually finite, prove that H is separable in G.

(c) Let F be a free group on two generators a and b.

(i) Show that the subgroups $\langle a \rangle$ and $\langle ab \rangle$ are separable in F.

(ii) Show that the subgroup $\langle a^n \rangle$ is separable in F for any $n > 0$.

(iii) Let $w = a^{-1}b^2a^2b^{-1}a \in F$. Using the method of Stallings folding, or otherwise, produce an explicit homomorphism $f\colon F \to Q$ to a finite group Q such that $f(b) \notin f(\langle w \rangle)$.

4 (a) Let $(I, \preceq)$ be an inverse system and let $(G_i)_{i \in I}$ be an inverse system of finite groups indexed over I. Let $G = \varprojlim G_i$ and let $p_i \colon G \to G_i$ be the natural projection map.

(i) Give a basis of open sets for the topology on G.

(ii) Let $X \subseteq G$, $H \subseteq G$ be subsets. Show that $H \subseteq \overline{X}$ if and only if $p_i(H) \subseteq p_i(X)$ for all $i \in I$.

(b) For a commutative ring R with unity, define

$$T(R) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(R) \right\} .$$

(i) Show that there exists $\kappa \in \mathbb{Z}_3$ such that $2\kappa = 1$. [Do not assume Gaschutz's Lemma.]

(ii) Show that $T(\mathbb{Z})$ is not dense in $T(\mathbb{Z}_3)$.

(iii) Show that $T(\mathbb{Z}_3) \subseteq \overline{\mathrm{SL}_2(\mathbb{Z})}$.

[You may assume without proof that $\mathrm{SL}_2(\mathbb{Z}_3) = \varprojlim \mathrm{SL}_2(\mathbb{Z}/3^n\mathbb{Z})$*.]*

(c) State and prove Hensel's Lemma for Square Roots.

(d) Find an integer matrix $A \in \mathrm{GL}_2(\mathbb{Z}_3)$ such that

$$A^3 \equiv \begin{pmatrix} 82 & 9 \\ 9 & 1 \end{pmatrix} \mod 81 .$$

[It is sufficient to express A in terms of explicit integer matrices; you are not required to simplify such an expression.]

5 (a) Let G be a group and let M be a (left) G-module.

(i) Consider the homomorphism of abelian groups

$$\Phi\colon \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^r, M) \to M^r$$

$$\Phi(f) = \begin{pmatrix} f(e_1) & f(e_2) & \dots & f(e_r) \end{pmatrix}^{\mathrm{Tr}}$$

where e_i is the element of $(\mathbb{Z}G)^r$ with a 1 in the i^{th} place and zero elsewhere.

Show that Φ is an isomorphism.

(ii) Write elements of $(\mathbb{Z}G)^r$ as row vectors and elements of M^r as column vectors. Suppose that $\psi\colon (\mathbb{Z}G)^3 \to (\mathbb{Z}G)^2$ is a morphism of G-modules given by right-multiplication by a matrix $A = (a_{ij})$:

$$\psi(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

where $a_{ij} \in \mathbb{Z}G$. Let

$$\psi^*\colon \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^2, M) \to \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^3, M)$$

be the dual map.

For $f \in \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^2, M)$, let $\Phi(f) = \begin{pmatrix} f_1 & f_2 \end{pmatrix}^{\mathrm{Tr}}$. Show that

$$\Phi(\psi^*(f)) = A \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = A \cdot \Phi(f)$$

(b) Let G be the group defined by the presentation

$$G = \langle a, b \,|\, a^6 = 1, b^4 = 1, a^3 b^2 = 1 \rangle.$$

(i) Define what it means for a sequence of G-modules

$$C_2 \xrightarrow{\psi} C_1 \xrightarrow{\phi} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

to be a *chain complex*. Define what it means for the sequence to be *exact*.

(ii) Let $\psi\colon (\mathbb{Z}G)^3 \to (\mathbb{Z}G)^2$ be the map defined by

$$\begin{aligned} \psi(x, y, z) &= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 + a + a^2 + a^3 + a^4 + a^5 & 0 \\ 0 & 1 + b + b^2 + b^3 \\ 1 + a + a^2 & a^3(1 + b) \end{pmatrix} \\ &=: \begin{pmatrix} x & y & z \end{pmatrix} C, \end{aligned}$$

let $\phi\colon (\mathbb{Z}G)^2 \to (\mathbb{Z}G)$ be given by

$$\begin{aligned} \phi(u, v) &= \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} a - 1 \\ b - 1 \end{pmatrix} \\ &=: \begin{pmatrix} u & v \end{pmatrix} D \end{aligned}$$

and let $\epsilon \colon \mathbb{Z}G \to \mathbb{Z}$ be the augmentation map.

Show that the sequence

$$(ZG)^3 \xrightarrow{\psi} (\mathbb{Z}G)^2 \xrightarrow{\phi} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{1}$$

is a chain complex.

(c) Assume now that there is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules whose final four terms are the sequence (1).

(i) Compute $H^1(G, \mathbb{Z})$ and $H^1(G, \mathbb{Z}/12\mathbb{Z})$, where each coefficient module has trivial G-action.

(ii) Write down, without proof, a long exact sequence corresponding to the exact sequence of G-modules

$$0 \to \mathbb{Z} \xrightarrow{12} \mathbb{Z} \longrightarrow \mathbb{Z}/12\mathbb{Z} \to 0.$$

Deduce that $H^2(G, \mathbb{Z}) \neq 0$.

[You do not need to describe the maps in the long exact sequence.]

END OF PAPER