1 Let $\widehat{\mathbb{Z}}$ denote the profinite integers and let $\mathbb{Z}_p$ denote the p-adic integers for p a prime number. Throughout this question you may freely use the Chinese Remainder Theorem.

(a)(i) Prove that the ring $\mathbb{Z}_p$ has no zero-divisors.

(ii) Let $n \in \mathbb{Z}$. Show that n is not a zero-divisor in $\widehat{\mathbb{Z}}$.

(iii) Let $n \in \mathbb{Z}$ and $\alpha \in \widehat{\mathbb{Z}}$. Show that if $\alpha n \in \mathbb{Z}$ then $\alpha \in \mathbb{Z}$.

(b)(i) Let $(G_i)_{i \in I}$ be an inverse system of finite groups, let $G = \varprojlim G_i$ and let $p_i \colon G \to G_i$ be the projection map. Let H be a subgroup of G. Show that H is closed if and only if $H = \varprojlim p_i(H)$.

(ii) Is $\mathbb{Z}$ a closed subgroup of $\widehat{\mathbb{Z}}$?

(iii) Show that $\widehat{\mathbb{Z}}/\mathbb{Z}$, equipped with the quotient topology, is a torsion-free compact group. Prove that it is not a profinite group.

(c) Show that $\alpha \in \widehat{\mathbb{Z}}$ topologically generates $\widehat{\mathbb{Z}}$ if and only if α does not equal $p\beta$ for any prime number p and any $\beta \in \widehat{\mathbb{Z}}$.