Properly forking formulas in Urysohn spaces

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Abstract

In this informal note, we demonstrate the existence of forking and nondividing formulas in continuous theory of the complete Urysohn sphere, as well as the discrete theories of the integral Urysohn spaces of diameter n (where $n \geq 3$). Whether or not such formulas existed was asked in thesis work of the author, as well as joint work with Terry. We also show an interesting phenomenon in that, for $n = 3$, forking and dividing over $\emptyset$ are the same for formulas in the integral Urysohn sphere of diameter n, while this is not the case for (at least) $n \geq 8$.

1 Introduction

A broad question one can ask about a complete first-order theory T is whether or not forking is the same as dividing. This question can be focused in a few specific ways including:

1. Are forking and dividing the same for any formula over any set of parameters?
2. Are forking and dividing the same for any complete type over any set of parameters?
3. Given a fixed set of parameters C, are forking and dividing over C the same for any formula?
4. Given a fixed set of parameters C, are forking and dividing over C the same for any complete type?

In a simple theory, we have the strongest answer, which says that forking and dividing are the same for any formula over any set of parameters. This is generalized to NTP_2 theories as follows.

Theorem 1.1 (Chernikov-Kaplan [2]). *Suppose T is NTP_2 and $C \subset \mathbb{M}$ is a set of parameters. The following are equivalent.*

(i) For any formula $\varphi(x, y)$ and $b \in \mathbb{M}$, $\varphi(x, b)$ forks over C if and only if $\varphi(x, b)$ divides over C.

(ii) C is an extension base for non-forking*, i.e., if $p \in S_n(C)$ then p does not fork over C.*

Recall that, in any theory, every set of parameters C is an extension base for *non-dividing*. So $(i) \Rightarrow (ii)$ is true without the assumption of NTP_2. There are many classes of NTP_2 theories in which all sets are extension bases for non-forking (e.g. simple theories and o-minimal theories). On the other hand, the canonical example of a theory in which some sets fail to be extension bases for non-forking is the cyclic order on the rationals, which is NTP_2 (in fact NIP).

In [3] I showed that, for any fixed $n \geq 3$, if T_n is the theory of the generic K_n-free graph then all sets of parameters are extension bases for non-forking, but there are forking and non-dividing formulas. Therefore, it is unclear whether there are generalizations of Chernikov and Kaplan's result to other classes of theories defined by the usual combinatorial dividing lines. In particular, T_n has SOP_3, so perhaps the following question still has a positive answer.

Question 1.2. Does the analog of Theorem 1.1 hold for NSOP_3 theories? Or even just NTP_1?

On the other hand, my work in [3] also showed that, in T_n, we still have a bit more good behavior beyond just the fact that any set is an extension base for non-forking. In particular, we have a positive answer to one of the other variations of the main question given above.

Theorem 1.3 (Conant [3])**.** *If $n \geq 3$ then, in T_n, forking and dividing are the same for any complete type over any set of parameters.*

Altogether, T_n becomes the first example of a complete theory in which forking and dividing are the same for complete types, but not for formulas.

This also motivates the following question, which asks for a weaker version of Chernikov and Kaplan's result. As far as I know, this question is still open.

Question 1.4. Suppose T is a complete theory in which all sets are extension bases for non-forking. Are forking and dividing the same for complete types (over any set of parameters)?

Even if this question has a negative answer, one can still relativize it to certain classes (e.g. NSOP or NSOP_n for some $n \geq 3$).

2 A properly forking formula in the continuous theory of the Urysohn sphere

In joint work with Caroline Terry [5], we show that in the theory of the complete Urysohn sphere in continuous logic, there is similar behavior as in the generic K_n-free graphs.

Theorem 2.1 (Conant-Terry [5])**.** *If T is the theory of the complete Urysohn sphere in continuous logic, then forking and dividing are the same for any complete type over any set of parameters.*

We then ask the obvious question:

Question 2.2. [5] Are forking and dividing the same for formulas in the theory of the complete Urysohn sphere (in continuous logic)?

My suspicion was that the Urysohn sphere should behave like the generic K_n-free graphs, and the answer to this question should be no. In this note, I will show this is indeed the case. I will also note some other interesting examples and behavior found in the discrete "generalized" Urysohn spaces I studied in my thesis [4]. At the moment, the only purpose of writing this down seems to be to satisfy my own curiosity. However, I still believe there is interesting content to the main question (Question 1.4), and its variants.

Let T be the theory of the complete Urysohn sphere considered as a continuous first-order structure in the "empty" continuous language containing only the distance function. Let $\mathbb{U}$ denote a sufficiently saturated monster model of T. For the rest of the note, we use letters $a, b, c, \ldots$ only for *singletons*. In this section, $+$ denotes truncated addition on $[0, 1]$.

Definition 2.3. Fix $C \subset \mathbb{U}$ and $b_1, b_2 \in \mathbb{U}$. Define

$$d_{\max}(b_1, b_2/C) = \inf_{c \in C}(d(b_1, c) + d(c, b_2))$$
$$d_{\min}(b_1, b_2/C) = \max\left\{\tfrac{1}{3}d(b_1, b_2), \sup_{c \in C}|d(b_1, c) - d(c, b_2)|\right\}.$$

By convention, $\sup \emptyset = 0$ and $\inf \emptyset = 1$.

Lemma 2.4 (Conant-Terry [5]). *Fix $C \subset \mathbb{U}$, $b_1, b_2 \in \mathbb{U}$, and $\gamma \in [0, 1]$. The following are equivalent.*

(*i*) *There is a C-indiscernible sequence $(b_1^i, b_2^i)_{i<\omega}$ such that $(b_1^0, b_2^0) = (b_1, b_2)$ and $d(b_1^0, b_2^1) = \gamma$.*

(*ii*) $d_{\min}(b_1, b_2/C) \leq \gamma \leq d_{\max}(b_1, b_2/C)$.

Given $r \in [0, 1]$, define the formula $d_r(x, y) = |d(x, y) - r|$. In particular, if $a, b \in \mathbb{U}$ then $d_r(a, b) = 0$ if and and only if $d(a, b) = r$.

Define the formula $\varphi(x, y_1, y_2) = \max(d_1(x, y_1), d_{\frac{1}{2}}(x, y_2))$.

Proposition 2.5. *If $b_1, b_2 \in \mathbb{U}$ and $d(b_1, b_2) = 1$ then $\varphi(x, b_1, b_2)$ divides over $\emptyset$.*

Proof. We have $d_{\min}(b_1, b_2) = \frac{1}{3}$ and $d_{\max}(b_1, b_2/\emptyset) = 1$. By Lemma 2.4, there is an indiscernible sequence $(b_1^i, b_2^i)_{i<\omega}$ such that $(b_1^0, b_2^0) = (b_1, b_2)$ and $d(b_1^0, b_2^1) = \frac{1}{3}$. To show $\varphi(x, b_1, b_2)$ divides over $\emptyset$, we show $\{\varphi(x, b_1^0, b_2^0) = 0, \varphi(x, b_1^1, b_2^1) = 0\}$ is inconsistent. If a realizes this type, then

$$1 = d(a, b_1^0) \leq d(a, b_2^1) + d(b_1^0, b_2^1) = \tfrac{1}{2} + \tfrac{1}{3},$$

which is a contradiction. □

Now define the formula $\psi(x, y_1, y_2, y_3, y_4) = \min_{i \neq j} \varphi(x, y_i, y_j)$.

Proposition 2.6. *Fix a tuple $\bar{b} = (b_1, b_2, b_3, b_4) \in \mathbb{U}$ such that $d(b_i, b_j) = 1$ for all $i \neq j$. Then $\psi(x, \bar{b})$ forks over $\emptyset$, but does not divide over $\emptyset$.*

Proof. By Proposition 2.5, $\psi(x, \bar{b})$ forks over $\emptyset$. To show $\psi(x, \bar{b})$ does not divide over $\emptyset$, we fix an indiscernible sequence $\bar{b}^n)_{n<\omega}$, with $\bar{b}^0 = \bar{b}$, and show $\{\psi(x, \bar{b}^n) = 0 : n < \omega\}$ is consistent.

First, we claim that there are $1 \leq i < j \leq 4$ such that $d(b_i^0, b_j^1) \geq \frac{1}{2}$ and $d(b_j^0, b_i^1) \geq \frac{1}{2}$. Suppose this fails. Without loss of generality, assume $d(b_1^0, b_2^1) < \frac{1}{2}$. Considering the triangle (b_1^0, b_2^1, b_3^0), it follows that $d(b_2^1, b_3^0) > \frac{1}{2}$ and so, by assumption $d(b_2^0, b_3^1) < \frac{1}{2}$. By a similar argument, $d(b_3^0, b_4^1) < \frac{1}{2}$. Considering the triangle (b_1^0, b_4^1, b_3^0), it follows that $d(b_1^0, b_4^1) > \frac{1}{2}$ and so, by assumption, $d(b_1^1, b_4^0) < \frac{1}{2}$. Considering the triangle (b_1^0, b_3^1, b_2^0), we must have $d(b_1^0, b_3^1) > \frac{1}{2}$ and so, by assumption, $d(b_1^1, b_3^0) < \frac{1}{2}$. This contradicts the triangle (b_1^1, b_3^0, b_4^0).

Fix $1 \leq i < j \leq 4$ as above. By indiscernibility $d(b_i^m, b_j^n) \geq \frac{1}{2}$ for all $m, n < \omega$. By the triangle inequality, there is a point $a \in \mathbb{U}$ such that $d(a, b_i^n) = 1$ and $d(a, b_j^n) = \frac{1}{2}$ for all $n < \omega$. Then a realizes $\{\varphi(x, b_i^n, b_j^n) = 0 : n < \omega\}$ and therefore realizes $\{\psi(x, \bar{b}^n) = 0 : n < \omega\}$. □

3 Properly forking formulas in generalized Urysohn spaces

At the end of [5], we discuss certain discrete analogs of the Urysohn sphere, which are well suited for the study of relational structures in classical first-order logic. Specifically, given an integer $n > 0$, one can define the *integral Urysohn space with diameter n*, i.e. the Fraïssé limit of the class of finite metric spaces with distances in $\{0, 1, \ldots, n\}$. This structure is also called the *free n^{th} root* *of the complete graph* by Casanovas and Wagner [1]. For example, if $n = 1$ this structure is just an infinite complete graph, and if $n = 2$ it is the same as the random graph.

The work in [5] can be adapted to these discrete Urysohn spaces. In particular, given $n > 0$, let T_n be the theory of the integral Urysohn space with diameter n, in a finite relational language $\mathcal{L}_n = \{d_0(x, y), \dots, d_n(x, y)\}$, where $d_t(x, y)$ is interpreted as distance t. Then T_n has quantifier elimination and, adapting the results of [5], it follows that forking and dividing are the same for complete types in T_n (over any set of parameters). This is shown explicitly and in much greater generality in my thesis [5] (see final discussion below).

Again, the natural question is whether forking and dividing are the same for formulas. From the observations above, we see that T_n is simple with $n \in \{1, 2\}$, and so the question has a positive answer for these cases because of general simplicity theory. However, T_n is not simple if $n \geq 3$ (see [1]). In fact, for $n \geq 3$, T_n has SOP$_n$ and NSOP$_{n+1}$ (see [5] or [4]). Once again, my suspicion was that forking and dividing should not be the same for formulas in T_n, when $n \geq 3$.

The most obvious approach to verify this would be to generalize the example in the previous section. However, somewhat surprisingly, this doesn't work. In fact, as I will show below, forking and dividing *over* $\emptyset$ are the same for formulas in T_3. One can still construct an example of a properly forking formula, but parameters are necessary. On the other hand, if $n = 6$ or $n \geq 8$, then the example from the previous section can be adapted to T_n after replacing 1, $\frac{1}{2}$, and $\frac{1}{3}$ by, respectively, n, $\lceil \frac{n}{2} \rceil$, and $\lceil \frac{n}{3} \rceil$ (one also needs the analog of Lemma 2.4, which I will discuss in the next paragraph).

Given $n > 0$, let $\mathbb{U}_n$ be a sufficiently saturated model of T_n. I will show that, if $n \geq 3$, then there is a suitable parameter set $C \subset \mathbb{U}_n$ and a formula that properly forks over C. The only tool we need is the analog of Lemma 2.4, which is obtained using the natural adaptations of $d_{\max}$ and $d_{\min}$ to $\mathbb{U}_n$. These are defined using addition truncated at n and, for $b_1, b_2 \in \mathbb{U}_n$, replacing $\frac{1}{3} d(b_1, b_2)$ with $\lceil \frac{d(b_1,b_2)}{3} \rceil$. We also use the convention $\sup \emptyset = 0$ and $\inf \emptyset = n$.

Fix $n \geq 3$, and work in T_n. Let $\varphi(x, y_1, y_2)$ be the formula $d_1(x, y_1) \wedge d_3(x, y_2)$.

Proposition 3.1. *Suppose $b_1, b_2 \in \mathbb{U}_n$ and $C \subset \mathbb{U}_n$ are such that*

(*i*) $d(b_1, b_2) = 3$,

(*ii*) *there are distinct $c_1, c_2 \in C$ with $d(b_i, c_i) = 1$ for $i \in \{1, 2\}$,*

(*iii*) $d(c, c') = 2$ *for any distinct $c, c' \in C$,*

(*iv*) $d(b_i, c) = 2$ *for any $i \in \{1, 2\}$ and $c \in C$ with $c \neq c_i$.*

Then $\varphi(x, b_1, b_2)$ divides over C.

Proof. First, define a sequence $(b_1^l, b_2^l)_{l<\omega}$ such that $\bar{b}^l \equiv_C \bar{b}^l$ for all $l < \omega$ and

- $d(b_i^l, b_2^m) = 1$ for all $l < m < \omega$,
- $d(b_i^l, b_j^m) = 2$ for all $l < m < \omega$ and $(i, j) \neq (1, 2)$.

If this sequence satisfies the triangle inequality, then it is an indiscernible sequence in $\mathbb{U}_n$. Since all distances in the sequence are among $\{1, 2, 3\}$, the only possible violation of the triangle inequality would occur from a triangle with distances $(1, 1, 3)$. Suppose $\{u, v, w\}$ is such a triangle. Then there are $l < m$ such that $b_1^l, b_2^m \in \{u, v, w\}$; say $\{u, v\} = \{b_1^l, b_2^m\}$. If $d(b_1^l, w) = 1$ then $w = b_2^k$ for some $k > l$, contradicting $d(w, b_2^m) = 3$. If $d(b_2^m, w) = 1$ then $w = b_1^k$ for some $k < m$, contradicting $d(w, b_1^l) = 3$.

Finally, we show $\varphi(x, b_1^0, b_2^0) \wedge \varphi(x, b_1^1, b_2^1)$ is inconsistent. Indeed, if a realizes this formula the $\{a, b_1^0, b_2^1\}$ is a triangle with distances $(1, 1, 3)$. □

Proposition 3.2. *Fix $\bar{b} = (b_1, b_2, b_3, b_4) \in \mathbb{U}_n$, with $d(b_i, b_j) = 3$ for all $i \neq j$. Let $C = \{c_1, c_2, c_3, c_4\} \subset \mathbb{U}_n$ be such that $d(b_i, c_i) = 1$ for all $i \leq 4$, $d(c_i, c_j) = 2$ for all $i \neq j$, and $d(b_i, c_j) = 2$ for all $i \neq j$. Define the formula*

$$\psi(x, \bar{b}) = \bigvee_{i \neq j} \varphi(x, b_i, b_j).$$

Then $\varphi(x, \bar{b})$ forks over C but does not divide over C.

Proof. For any $i \neq j$, (b_i, b_j) and C satisfy the conditions of Proposition 3.1, and so $\varphi(x, b_i, b_j)$ divides over C. Therefore $\psi(x, \bar{b})$ forks over C.

Fix a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$, with $\bar{b}^0 = \bar{b}$. By similar deductions as in the proof of Proposition 2.6, we may fix $1 \leq i < j \leq 4$ such that $d(b_i^0, b_j^1) \geq 2$ and $d(b_j^0, b_i^1) \geq 2$. Considering the triangles $\{b_i^0, b_j^1, c_i\}$ and $\{b_j^0, b_i^1, c_j\}$, it follows that $d(b_i^0, b_j^1), d(b_j^0, b_i^1) \in \{2, 3\}$. We claim that there is some $a \in \mathbb{U}_n$ such that $d(a, b_i^l) = 1$ and $d(a, b_j^l) = 3$ for all $l < \omega$. For if not then, again, there must be a triangle with distances $(1,1,3)$. This triangle must be of the form $\{a, b_i^l, b_i^m\}$ for some $l < m < \omega$, with $d(a, b_i^l) = d(a, b_i^m) = 1$ and $d(b_i^l, b_i^m) = 3$. But this contradicts

$$d(b_i^l, b_i^m) \leq d(b_i^l, c_i) + d(b_i^m, c_i) = d(b_i, c_i) + d(b_i, c_i) = 2.$$

We have found a realization of $\{\varphi(x, b_i^l, b_j^l) : l < \omega\}$, which therefore realizes $\{\psi(x, \bar{b}^l) : l < \omega\}$. This shows that $\psi(x, \bar{b}^l)$ does not divide over C. □

Finally, we prove the curious result mentioned earlier that, at least in the case $n = 3$, the nonempty parameter C is necessary.

Proposition 3.3. *Suppose $\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_k(\bar{x}, \bar{y})$ are formulas (possibly with suppressed parameters), and $\bar{b} \in \mathbb{U}_3$ is such that $\varphi_t(\bar{x}, \bar{b})$ divides over $\emptyset$ for all $1 \leq t \leq k$. Let $\psi(\bar{x}, \bar{y}) = \bigvee_{t=1}^k \varphi_t(\bar{x}, \bar{y})$. Then $\psi(\bar{x}, \bar{b})$ divides over $\emptyset$.*

Proof. Fix $1 \leq t \leq k$. We first show that $\varphi_t(\bar{x}, \bar{b}) \vdash \bigvee_{i,j} d(x_i, b_j) \leq 1$. Suppose not, and fix a tuple $\bar{a} \in \mathbb{U}_3$ such that $\mathbb{U}_3 \models \varphi_t(\bar{a}, \bar{b})$ and $d(a_i, b_j) \geq 2$ for all i, j. Let $(\bar{b}^l)_{l<\omega}$ be a C-indiscernible sequence, with $\bar{b}^l = \bar{b}$. We will find a tuple $\bar{a}^*$ such that $\bar{a}^* \bar{b}^l \equiv_C \bar{a}\bar{b}$ for all $l < \omega$, which contradicts the assumption that $\varphi_t(\bar{x}, \bar{b})$ divides over C. Suppose no such $\bar{a}^*$ exists. Then the type

$$\{d(x_i, b_j^l) = d(a_i, b_j) : l < \omega,\ i \leq |\bar{a}|,\ j \leq |\bar{b}|\} \cup \{d(x_i, c) = d(a_i, c) : c \in C,\ i \leq |\bar{a}|\}$$

is inconsistent and therefore implies some violation of the triangle inequality. Since $d(a_i, b_j) \geq 2$ for all i, j, it follows that the violation of the triangle inequality involves at most one b_j^l in $\bigcup_{l<\omega} \bar{b}^l$. By indiscernibility, this implies a violation of the triangle inequality among the points in $\bar{a}\bar{b}C$, which is a contradiction.

Altogether, we have shown $\bigvee_{t=1}^k \varphi_t(\bar{x}, \bar{b}) \vdash \bigvee_{i,j} d(x_i, b_j) \leq 1$. Therefore, it suffices to show $\bigvee_{i,j} d(x_i, b_j) \leq 1$ divides over $\emptyset$.

Let $(\bar{b}^l)_{l<\omega}$ be a sequence such that $\bar{b}^l \equiv \bar{b}$ and $d(b_i^l, b_j^m) = 3$ for all $l < m < \omega$ and $i, j \leq |\bar{b}|$. Note that this is an indiscernible sequence. Suppose, toward a contradiction, that there is a tuple $\bar{a} \in \mathbb{U}_3$ such that $\mathbb{U}_3 \models \bigvee_{i,j} d(a_i, b_j^l) \leq 1$ for all $l < \omega$. For each $l < \omega$, there are $i_l, j_l \leq |\bar{b}|$ such that $d(a_{i_l}, b_{j_l}^l) \leq 1$. By pigeonhole, there are $l < m < \omega$ such that $i := i_l = i_m$. Then the triangle $\{a_i, b_{j_l}^l, b_{j_m}^m\}$ has distances $(1,1,3)$, which is a contradiction. Altogether, $\{\bigvee_{i,j} d(x_i, b_j^l) \leq 1 : l < \omega\}$ is inconsistent, and so $\bigvee_{i,j} d(x_i, b_j) \leq 1$ divides over $\emptyset$. □

In my thesis, I studied the model theory of generalized Urysohn spaces in much larger generality. Specifically, I developed a general framework for studying the Urysohn space over an arbitrary countable linearly ordered commutative monoid with least element 0 (or *distance* monoid for short). Such a monoid, denoted $\mathcal{R} = (R, \oplus, \leq, 0)$, serves as the set of distances for the "generalized Urysohn space over $\mathcal{R}$", denoted $\mathcal{U}_{\mathcal{R}}$. For example, if $n > 0$ and $\mathcal{R} = (\{0, 1, \ldots, n\}, +_n, \leq, 0)$, where $+_n$ is addition truncated at n, then $\mathcal{U}_{\mathcal{R}}$ is the integral Urysohn space of diameter n described above. Similarly, $(\mathbb{Q}^{\geq 0}, +, \leq, 0)$ yields the *rational Urysohn space* and $(\mathbb{Q} \cap [0, 1], +_1, \leq, 0)$ yields the *rational Urysohn sphere*. One can also obtain *ultrametric Urysohn spaces* by using the binary operation max instead of usual addition.

It turns out that, while quantifier elimination can fail if the distance monoid $\mathcal{R}$ is exotic enough, it seems to hold in most naturally occurring cases, including all of those mentioned above. Using this, it is not hard to adapt the example from the previous section to construct a properly forking (over $\emptyset$) formula in the theories of the rational Urysohn space and rational Urysohn sphere. This does require, however, a certain analysis of saturated models as generalized metric spaces with distances in a canonical monoid extension $\mathcal{R}^*$ of $\mathcal{R}$. (In fact, the theory of $\mathcal{U}_{\mathcal{R}}$ has quantifier elimination if and only if the addition operation in $\mathcal{R}^*$ is separately continuous.)

In my thesis, I characterized precisely when the theory of $\mathcal{U}_{\mathcal{R}}$ is simple (assuming quantifier elimination). It turns out that simplicity is equivalent to NSOP_3 in this class of structures, and can be characterized using a fairly straightforward algebraic property of the monoid $\mathcal{R}$. As in the cases above, I suspect that if the theory of $\mathcal{U}_{\mathcal{R}}$ is not simple (i.e. is SOP_3) then forking and dividing will not be the same for formulas. On the other hand, forking and dividing is always the same for complete types, using essentially the same argument as in [5].

At the moment, I don't have a general argument for this claim. The example with the smallest distance set, for which I have not yet constructed a properly forking formula, is the Urysohn space with distances $\{0, 1, 2, 5, 6, 7\}$. In this example, both of the methods carried out above don't seem to immediately work.

References

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