

ASSOCIATIVITY OF THE MORLEY PRODUCT OF INVARIANT MEASURES IN NIP THEORIES

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ABSTRACT. In light of a gap found by Krupiński, we give a new proof of associativity for the Morley (or “nonforking”) product of invariant measures in NIP theories.

Let T be a complete first-order \mathcal{L} -theory, and fix a sufficiently saturated monster model \mathcal{U} . Given a tuple of variables x , we let $\mathfrak{M}_x(\mathcal{U})$ denote the space of global Keisler measures on $\text{Def}_x(\mathcal{U})$. Recall that $\mu \in \mathfrak{M}_x(\mathcal{U})$ corresponds to the space of regular Borel probability measures on the Stone space $S_x(\mathcal{U})$ of global types in x .

A measure $\mu \in \mathfrak{M}_x(\mathcal{U})$ is **invariant** if there is a small model $M \prec \mathcal{U}$ such that, for any \mathcal{L} -formula $\phi(x, y)$ and $b, b' \in \mathcal{U}^y$, $b \equiv_M b'$ implies $\mu(\phi(x, b)) = \mu(\phi(x, b'))$. If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is M -invariant, and $\phi(x, y)$ is an \mathcal{L}_M -formula, then we have a well-defined function $F_\mu^\phi: S_y(M) \rightarrow [0, 1]$ such that $F_\mu^\phi(q) = \mu(\phi(x, b))$ for some/any $b \models q$. Then μ is **Borel-definable** (over M) if F_μ^ϕ is a Borel function for any $\phi(x, y)$. If each of these maps is *continuous*, then μ is called **definable** over M .

Now fix measures $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$, and assume μ is Borel-definable (over some small model). The **Morley product** $\mu \otimes \nu$ (originally defined by Hrushovski and Pillay in [2]) is constructed as follows. Given an $\mathcal{L}_\mathcal{U}$ -formula $\phi(x, y)$, let $M \prec \mathcal{U}$ be a small model that contains the parameters in $\phi(x, y)$, and is such that μ is Borel-definable over M . Then

$$(\mu \otimes \nu)(\phi(x, y)) = \int_{S_y(M)} F_\mu^\phi d\nu.$$

One can verify that this does not depend on the choice of M , and yields a well-defined Keisler measure in $\mathfrak{M}_{xy}(\mathcal{U})$. Moreover, if ν is M -invariant then so is $\mu \otimes \nu$.

Now assume T is NIP. In this case, any M -invariant Keisler measure is automatically Borel-definable over M (see [2, Corollary 4.9] or [5, Proposition 7.19]), and so one can iterate the Morley product. This naturally raises the question of associativity. In [5, Chapter 7], a proof of associativity is sketched, but a gap in the proof was recently found by Krupiński. The purpose of this note is to provide a new proof. In Section 1, we recall a few fundamental results on smooth measures that are needed for the proof. In Section 2, we review the proof of associativity for types (which motivates the proof for measures) and then prove the main result. In Section 3, we sketch a second proof of associativity, and then briefly discuss the gap in the proposed proof from [5].

1. PRELIMINARIES ON SMOOTH MEASURES

Given a tuple $\bar{a} \in (\mathcal{U}^x)^n$, define $\text{Av}_{\bar{a}} \in \mathfrak{M}_x(\mathcal{U})$ such that

$$\text{Av}_{\bar{a}}(\phi(x)) = \frac{1}{n} |\{1 \leq i \leq n : \mathcal{U} \models \phi(a_i)\}|$$

for any $\mathcal{L}_{\mathcal{U}}$ -formula $\phi(x)$. Given $r, s \in \mathbb{R}$ and $\epsilon > 0$, we write $r \approx_{\epsilon} s$ if $|r - s| < \epsilon$.

Definition 1.1. Fix $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $M \prec \mathcal{U}$.

- (1) μ is **smooth** over M if μ is the unique measure in $\mathfrak{M}_x(\mathcal{U})$ extending $\mu|_M$.
- (2) μ is **finitely approximated** in M if for any \mathcal{L} -formula $\phi(x, y)$ and $\epsilon > 0$, there is a tuple $(a_1, \dots, a_n) \in (M^x)^n$ such that $\text{Av}_{\bar{a}}(\phi(x, b)) \approx_{\epsilon} \mu(\phi(x, b))$ for any $b \in \mathcal{U}^y$. In this case, we call \bar{a} a $(\phi(x, y), \epsilon)$ -**approximation** for μ .

Given measures $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$, we say that $\lambda \in \mathfrak{M}_{xy}(\mathcal{U})$ is a **separated amalgam** of μ and ν if $\lambda(\phi(x) \wedge \psi(y)) = \mu(\phi(x))\nu(\psi(y))$ for any $\mathcal{L}_{\mathcal{U}}$ -formulas $\phi(x)$ and $\psi(y)$.

Proposition 1.2. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$ is smooth over $M \prec \mathcal{U}$.*

- (a) *Let $\phi(x, y)$ be an \mathcal{L} -formula and fix $\epsilon > 0$. Then there are \mathcal{L}_M -formulas $\theta_1^-(x), \dots, \theta_n^-(x), \theta_1^+(x), \dots, \theta_n^+(x), \psi_1(y), \dots, \psi_n(y)$, for some $n \geq 1$, such that:

 - (i) the formulas $\psi_1(y), \dots, \psi_n(y)$ partition \mathcal{U}^y ,
 - (ii) for all $i \leq n$, if $\mathcal{U} \models \psi_i(b)$ then $\theta_i^-(x) \subseteq \phi(x, b) \subseteq \theta_i^+(x)$, and
 - (iii) for all $i \leq n$, $\mu(\theta_i^+(x)) - \mu(\theta_i^-(x)) < \epsilon$.
 Moreover, this implies μ is definable over M .*
- (b) *If $\nu \in \mathfrak{M}_y(\mathcal{U})$ is Borel-definable, then $\mu \otimes \nu$ is the unique separated amalgam of μ and ν in $\mathfrak{M}_{xy}(\mathcal{U})$, and so $\mu \otimes \nu = \nu \otimes \mu$.*
- (c) *μ is finitely approximated in M .*

Proof. See Lemma 2.3 and Corollary 2.5 of [3] for parts (a) and (b). As noted in [3], the symmetry claim in part (b) follows since $\mu \otimes \nu$ and $\nu \otimes \mu$ are both separated amalgams of μ and ν . See [5, Proposition 7.10] for part (c). \square

Part (c) of Proposition 1.2 is also evident from the proof of [3, Corollary 2.6] (see also [3, Corollary 2.8]). The next result is [6, Corollary 3.17], which is stated without proof, and so we take the opportunity here to provide details.

Corollary 1.3. *If $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $\nu \in \mathfrak{M}_y(\mathcal{U})$ are smooth over $M \prec \mathcal{U}$, then $\mu \otimes \nu$ is smooth over $M \prec \mathcal{U}$.*

Proof. Suppose $\lambda \in \mathfrak{M}_{xy}(\mathcal{U})$ is such that $\lambda|_M = (\mu \otimes \nu)|_M$. We want to show that $\lambda = \mu \otimes \nu$. By Proposition 1.2(b), it suffices to show that λ is a separated amalgam of μ and ν . So fix $\mathcal{L}_{\mathcal{U}}$ -formulas $\phi(x)$ and $\psi(y)$. Fix $\epsilon > 0$. By Proposition 1.2(a), there are \mathcal{L}_M -formulas $\theta^-(x), \theta^+(x), \chi^-(y), \chi^+(y)$ such that:

- (i) $\theta^-(x) \subseteq \phi(x) \subseteq \theta^+(x)$ and $\chi^-(y) \subseteq \psi(y) \subseteq \chi^+(y)$;
- (ii) $\mu(\theta^+(x)) - \mu(\theta^-(x)) < \epsilon$ and $\nu(\chi^+(y)) - \nu(\chi^-(y)) < \epsilon$.

(For example, write $\phi(x)$ as $\phi_0(x, b)$ for some \mathcal{L} -formula $\phi_0(x, z)$ and $b \in \mathcal{U}^z$, and obtain $\theta_i^-(x), \theta_i^+(x), \psi_i(z)$ by applying Proposition 1.2(a) to $\phi_0(x, z)$ and ϵ . Then choose i such that $\psi_i(b)$ holds, and let $\theta^-(x) = \theta_i^-(x)$ and $\theta^+(x) = \theta_i^+(x)$.)

Note that $\theta^-(x) \wedge \chi^-(y) \subseteq \phi(x) \wedge \psi(y) \subseteq \theta^+(x) \wedge \chi^+(y)$. Since $\lambda|_M = (\mu \otimes \nu)|_M$, we have

$$\begin{aligned} \lambda(\theta^-(x) \wedge \chi^-(y)) &= (\mu \otimes \nu)(\theta^-(x) \wedge \chi^-(y)) =: r, \text{ and} \\ \lambda(\theta^+(x) \wedge \chi^+(y)) &= (\mu \otimes \nu)(\theta^+(x) \wedge \chi^+(y)) =: s. \end{aligned}$$

So $\lambda(\phi(x) \wedge \psi(y)), (\mu \otimes \nu)(\phi(x) \wedge \psi(y)) \in [r, s]$. Moreover,

$$\begin{aligned} s - r &= \mu(\theta^+(x))\nu(\chi^+(y)) - \mu(\theta^-(x))\nu(\chi^-(y)) \\ &= \mu(\theta^+(x))\left(\nu(\chi^+(y)) - \nu(\chi^-(y))\right) + \nu(\chi^-(y))\left(\mu(\theta^+(x)) - \mu(\theta^-(x))\right) < 2\epsilon. \end{aligned}$$

Therefore $\lambda(\phi(x) \wedge \psi(y)) \approx_{2\epsilon} (\mu \otimes \nu)(\phi(x) \wedge \psi(y))$. Since $\epsilon > 0$ was arbitrary, we have the desired result. \square

Finally, we recall a main result about NIP theories, namely, the existence of smooth extensions (see [4, Theorem 3.26] or [5, Proposition 7.9]).

Theorem 1.4 (Keisler). *Assume T is NIP. Given $\mu \in \mathfrak{M}_x(\mathcal{U})$ and $M \prec \mathcal{U}$, there is $\nu \in \mathfrak{M}_x(\mathcal{U})$ such that $\mu|_M = \nu|_M$ and ν is smooth over some $N \succ M$.*

2. ASSOCIATIVITY

Before starting the proof, we briefly recall the argument for associativity of the Morley product for invariant types (which holds in any theory).

Fix global types $p \in S_x(\mathcal{U})$ and $q \in S_y(\mathcal{U})$, and assume p is invariant over some small model. Then an $\mathcal{L}_{\mathcal{U}}$ -formula $\phi(x, y)$ is in $p \otimes q$ if and only if $\phi(x, b) \in p$ for some/any $b \models q|_M$, where $M \prec \mathcal{U}$ contains any parameters in $\phi(x, y)$ and p is M -invariant. Now assume q is invariant, and fix a third type $r \in S_z(\mathcal{U})$. Let $\phi(x, y, z)$ be an $\mathcal{L}_{\mathcal{U}}$ -formula, and choose $M \prec \mathcal{U}$ such that $\phi(x, y, z)$ is over M and p and q are M -invariant. Let $c \models r|_M$ and $b \models q|_N$, where $N \prec \mathcal{U}$ contains Mc . Then it is straightforward to show that $(b, c) \models (q \otimes r)|_M$, and thus $\phi(x, y, z) \in p \otimes (q \otimes r)$ if and only if $\phi(x, b, c) \in p$. On the other hand, since $\phi(x, y, c)$ is over N ,

$$\phi(x, y, z) \in (p \otimes q) \otimes r \iff \phi(x, y, c) \in p \otimes q \iff \phi(x, b, c) \in p.$$

The proof of associativity for measures follows the same rough strategy, although the individual steps become more intricate. In the above argument, $\text{tp}(b/\mathcal{U})$ and $\text{tp}(c/\mathcal{U})$ are *isolated* global types extending $q|_N$ and $r|_M$, respectively. For the case of measures, we will replace these by *smooth extensions* (recall that a global type is smooth if and only if it is isolated). Thus we will need to make an NIP assumption to know that such extensions exist (via Theorem 1.4). Also, in place of realizations of isolated global types, we will use ϵ -approximations of smooth measures (via Proposition 1.2(c)). Finally, when adapted to measures, several trivial maneuvers in the argument for types require the results in Section 1. For example, the proof above implicitly uses the obvious facts that isolated types commute with any invariant type, and that the product of two isolated types is isolated.

Next, we record a few easy observations.

Remark 2.1.

(a) If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is Borel definable over $M \prec \mathcal{U}$, and $\nu, \nu' \in \mathfrak{M}_y(\mathcal{U})$ are such that $\nu|_M = \nu'|_M$ then, for any \mathcal{L}_M -formula $\phi(x, y)$,

$$(\mu \otimes \nu)(\phi(x, y)) = (\mu \otimes \nu')(\phi(x, y)).$$

(b) If $\mu \in \mathfrak{M}_x(\mathcal{U})$ is finitely approximated in $M \prec \mathcal{U}$, $\phi(x, y)$ is an \mathcal{L}_M -formula, and $\bar{a} \in (M^x)^n$ is a $(\phi(x; y), \epsilon)$ -approximation for μ then, for any $\nu \in \mathfrak{M}_y(\mathcal{U})$,

$$(\mu \otimes \nu)(\phi(x, y)) \approx_{\epsilon} (\text{Av}_{\bar{a}} \otimes \nu)(\phi(x, y)) = \frac{1}{n} \sum_{i=1}^n \nu(\phi(a_i, y)).$$

We now prove the main result.

Theorem 2.2. *Assume T is NIP, and suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$. If μ and ν invariant, then $\mu \otimes (\nu \otimes \lambda) = (\mu \otimes \nu) \otimes \lambda$.*

Proof. Fix a $\mathcal{L}_{\mathcal{U}}$ -formula $\phi(x, y, z)$. We want to show

$$(\mu \otimes (\nu \otimes \lambda))(\phi(x, y, z)) = ((\mu \otimes \nu) \otimes \lambda)(\phi(x, y, z)).$$

Let $M \prec \mathcal{U}$ be a model such that $\phi(x, y, z)$ is over M , and μ and ν are M -invariant. By Theorem 1.4, there is $N \succ M$ and $\hat{\lambda} \in \mathfrak{M}_z(\mathcal{U})$ such that $\lambda|_M = \hat{\lambda}|_M$ and $\hat{\lambda}$ is smooth over N . Similarly, there is $\hat{\nu} \in \mathfrak{M}_y(\mathcal{U})$ such that $\nu|_N = \hat{\nu}|_N$ and $\hat{\nu}$ is smooth over some small model containing N . Note that $\hat{\nu} \otimes \hat{\lambda}$ is smooth by Corollary 1.3.

Claim: $(\hat{\nu} \otimes \hat{\lambda})|_M = (\nu \otimes \lambda)|_M$.

Proof: Let $\psi(y, z)$ be an \mathcal{L}_M -formula. Then

$$\begin{aligned} (\nu \otimes \lambda)(\psi(y, z)) &= (\nu \otimes \hat{\lambda})(\psi(y, z)) = (\hat{\lambda} \otimes \nu)(\psi(y, z)) \\ &= (\hat{\lambda} \otimes \hat{\nu})(\psi(y, z)) = (\hat{\nu} \otimes \hat{\lambda})(\psi(y, z)), \end{aligned}$$

where the first and third equalities use Remark 2.1(a), while the second and fourth use Proposition 1.2(b). \dashv _{claim}

Now fix some $\epsilon > 0$. Let $\phi_1(z; x, y)$, $\phi_2(y; x, z)$, and $\phi_3(y, z; x)$ denote various partitions of the variables in $\phi(x, y, z)$ into object and parameter variables. By Proposition 1.2(c), we may let $\bar{c} \in (N^z)^n$ be a $(\phi_1(z; x, y), \epsilon)$ -approximation for $\hat{\lambda}$, and let $\bar{b} \in (\mathcal{U}^y)^m$ be a $(\phi_2(y; x, z), \epsilon)$ -approximation for $\hat{\nu}$. A straightforward calculation then shows that $(b_i, c_j)_{i \leq m, j \leq n}$ is a $(\phi_3(y, z; x), 2\epsilon)$ -approximation for $\hat{\nu} \otimes \hat{\lambda}$. Therefore we have

$$\begin{aligned} (\mu \otimes (\nu \otimes \lambda))(\phi(x, y, z)) &= (\mu \otimes (\hat{\nu} \otimes \hat{\lambda}))(\phi(x, y, z)) \\ &= ((\hat{\nu} \otimes \hat{\lambda}) \otimes \mu)(\phi(x, y, z)) \approx_{2\epsilon} \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \mu(\phi(x, b_i, c_j)), \end{aligned}$$

where the first equality uses Remark 2.1(a) (and the Claim), the second equality uses Proposition 1.2(b) (and smoothness of $\hat{\nu} \otimes \hat{\lambda}$), and the final approximation uses Remark 2.1(b). On the other hand,

$$\begin{aligned} ((\mu \otimes \nu) \otimes \lambda)(\phi(x, y, z)) &= ((\mu \otimes \nu) \otimes \hat{\lambda})(\phi(x, y, z)) = (\hat{\lambda} \otimes (\mu \otimes \nu))(\phi(x, y, z)) \\ &\approx_{\epsilon} \frac{1}{n} \sum_{j=1}^n (\mu \otimes \nu)(\phi(x, y, c_j)) = \frac{1}{n} \sum_{j=1}^n (\mu \otimes \hat{\nu})(\phi(x, y, c_j)) \\ &= \frac{1}{n} \sum_{j=1}^n (\hat{\nu} \otimes \mu)(\phi(x, y, c_j)) \approx_{\epsilon} \frac{1}{mn} \sum_{j=1}^n \sum_{i=1}^m \mu(\phi(x, b_i, c_j)), \end{aligned}$$

where the first and third equalities use Remark 2.1(a), the second and fourth equalities use Proposition 1.2(b), and the approximations use Remark 2.1(b).

Altogether, $(\mu \otimes (\nu \otimes \lambda))(\phi(x, y, z)) \approx_{4\epsilon} ((\mu \otimes \nu) \otimes \lambda)(\phi(x, y, z))$. Since $\epsilon > 0$ was arbitrary, we have $(\mu \otimes (\nu \otimes \lambda))(\phi(x, y, z)) = ((\mu \otimes \nu) \otimes \lambda)(\phi(x, y, z))$. \square

3. FINAL REMARKS

3.1. Alternate proof via associativity for smooth measures. A quicker summary of the proof of associativity for invariant types in arbitrary theories is as follows. Fix $p \in S_x(\mathcal{U})$, $q \in S_y(\mathcal{U})$, and $r \in S_z(\mathcal{U})$, and assume that p and q are invariant. Let $M \prec \mathcal{U}$ be an arbitrary model such that p and q are M -invariant, and choose $c \models r|_M$, $b \models q|_{Mc}$, and $a \models p|_{Mbc}$. Then one easily shows that $((p \otimes q) \otimes r)|_M = (p \otimes (q \otimes r))|_M = \text{tp}(a, b, c/M)$ (see also [5, Fact 2.20]). This argument essentially reduces associativity to the case of isolated types. We can make an analogous reduction in the case of measures, and thus provide a proof of Theorem 2.2 that avoids explicit use of the fact that smooth measures are finitely approximated (Proposition 1.2(c)). We only need one general fact about associativity in the presence of a smooth measure.

Proposition 3.1. *Suppose $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$ are such that μ is smooth, and ν and $\mu \otimes \nu$ are Borel-definable. Then $(\mu \otimes \nu) \otimes \lambda = \mu \otimes (\nu \otimes \lambda)$.*

Proof. Let $\omega = (\mu \otimes \nu) \otimes \lambda$. Since μ is smooth, it suffices by Proposition 1.2(b) to show that ω is a separated amalgam of μ and $\nu \otimes \lambda$. So fix $\mathcal{L}_{\mathcal{U}}$ -formulas $\phi(x)$ and $\psi(y, z)$, and let $\theta(x, y, z) := \phi(x) \wedge \psi(y, z)$. Fix $M \prec \mathcal{U}$ such that $\theta(x, y, z)$ is over M , and μ, ν , and $\mu \otimes \nu$ are Borel-definable over M . Then

$$\omega(\theta(x, y, z)) = \int_{S_z(M)} F_{\mu \otimes \nu}^{\theta} d\lambda = \mu(\phi(x)) \int_{S_z(M)} F_{\nu}^{\psi} d\lambda = \mu(\phi(x))(\nu \otimes \lambda)(\psi(y, z)),$$

as desired. \square

Now assume T is NIP, and fix measures $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$, with μ and ν invariant. Fix an $\mathcal{L}_{\mathcal{U}}$ -formula $\phi(x, y, z)$. Let $M \prec \mathcal{U}$ be a small model, such that $\phi(x, y, z)$ is over M , and μ and ν are invariant over M . By Theorem 1.4, there are models $N_1 \succ N_0 \succ M$ and measures $\hat{\mu} \in \mathfrak{M}_x(\mathcal{U})$, $\hat{\nu} \in \mathfrak{M}_y(\mathcal{U})$, $\hat{\lambda} \in \mathfrak{M}_z(\mathcal{U})$ such that $\hat{\lambda}|_M = \lambda|_M$, $\hat{\nu}|_{N_0} = \nu|_{N_0}$, $\hat{\mu}|_{N_1} = \mu|_{N_1}$, $\hat{\lambda}$ is smooth over N_0 , $\hat{\nu}$ is smooth over N_1 , and $\hat{\mu}$ is smooth over some model containing N_1 . By Corollary 1.3, $\hat{\nu} \otimes \hat{\lambda}$ is smooth over N_1 . Now, using Remark 2.1(a) and Proposition 1.2(b) (as in the Claim in the proof of Theorem 2.2), we have

$$(\hat{\mu} \otimes \hat{\nu})|_{N_0} = (\mu \otimes \nu)|_{N_0} \quad \text{and} \quad (\hat{\nu} \otimes \hat{\lambda})|_M = (\nu \otimes \lambda)|_M.$$

So using Remark 2.1(a), Proposition 1.2(b), and Proposition 3.1, we have

$$\begin{aligned} ((\mu \otimes \nu) \otimes \lambda)(\phi(x, y, z)) &= ((\mu \otimes \nu) \otimes \hat{\lambda})(\phi(x, y, z)) \\ &= (\hat{\lambda} \otimes (\mu \otimes \nu))(\phi(x, y, z)) = (\hat{\lambda} \otimes (\hat{\mu} \otimes \hat{\nu}))(\phi(x, y, z)) \\ &= ((\hat{\mu} \otimes \hat{\nu}) \otimes \hat{\lambda})(\phi(x, y, z)) = (\hat{\mu} \otimes (\hat{\nu} \otimes \hat{\lambda}))(\phi(x, y, z)) \\ &= ((\hat{\nu} \otimes \hat{\lambda}) \otimes \hat{\mu})(\phi(x, y, z)) = ((\hat{\nu} \otimes \hat{\lambda}) \otimes \mu)(\phi(x, y, z)) \\ &= (\mu \otimes (\hat{\nu} \otimes \hat{\lambda}))(\phi(x, y, z)) = (\mu \otimes (\nu \otimes \lambda))(\phi(x, y, z)). \end{aligned}$$

Note that in the previous argument, we only needed the version of Proposition 3.1 in which all three measures are smooth. Since smooth measures are definable (by Proposition 1.2(a)), one could instead use associativity for definable measures, which is proved in [1, Proposition 2.6].

3.2. Product measures of Borel sets. We finish with a brief discussion of the “gap” in the proof of associativity given in [5] (as identified by Krupiński). Assume T is NIP and fix measures $\mu \in \mathfrak{M}_x(\mathcal{U})$, $\nu \in \mathfrak{M}_y(\mathcal{U})$, and $\lambda \in \mathfrak{M}_z(\mathcal{U})$, all invariant over $M \prec \mathcal{U}$. Toward proving $\mu \otimes (\nu \otimes \lambda) = (\mu \otimes \nu) \otimes \lambda$, the argument in [5] uses [5, Proposition 7.11] to reduce to the case that μ is an invariant type $p \in S_x(\mathcal{U})$. Now fix a formula $\phi(x, y, z)$. Since p is Borel-definable over M , the set

$$B(y, z) = \{s \in S_{yz}(M) : \phi(x, b, c) \in p \text{ for some/any } (b, c) \models s\}$$

is Borel. For simplicity and to highlight the issue, we assume that $B(y, z)$ is a closed subset of $S_{yz}(M)$. In particular, we may write $B(x, y) = \bigcap_{i \in I} [\psi_i(y, z)]$ where each $\psi_i(y, z) \in \mathcal{L}_{yz}(M)$. Define $F_\nu^B : S_z(M) \rightarrow [0, 1]$ such that $F_\nu^B(r) = \nu(B(y, c))$ for some/any $c \models r$ (where $B(y, c) = \bigcap_{i \in I} [\psi_i(y, c)]$). Then $(p \otimes (\nu \otimes \lambda))(\phi(x, y, z)) = (\nu \otimes \lambda)(B)$, while $((p \otimes \nu) \otimes \lambda)(\phi(x, y, z)) = \int_{S_z(M)} F_\nu^B d\lambda$. Altogether, the proof of associativity boils down to the following equality

$$(\nu \otimes \lambda)(B) = \int_{S_z(M)} F_\nu^B d\lambda.$$

Note that if B is *clopen* (i.e., a formula), then this is precisely the definition of the Morley product. However, for the case that B is closed, the above equality naively requires one to interchange an integral with an infimum over some family of functions (via regularity of $\nu \otimes \lambda$ as a measure on $S_{yz}(\mathcal{U})$). In the setting of abstract integration, this is not always possible. On the other hand, Theorem 2.2 implies (*a posteriori*) that the above equality does indeed hold when T is NIP.

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