# The dual complex of genus one mapping spaces

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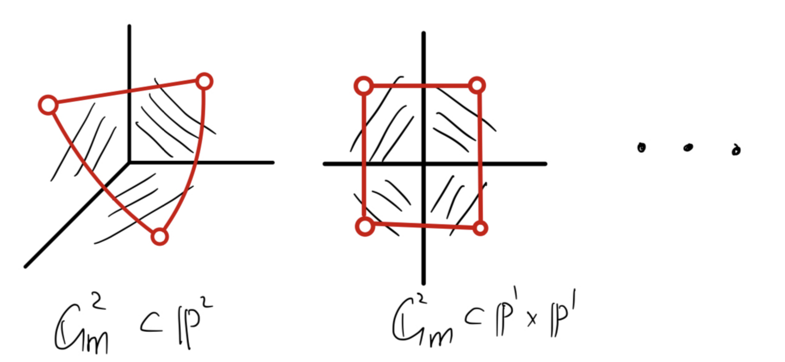
## Dual complex of a smooth variety

Let $X$ be a smooth variety or DM stack, and let $X \subset \overline{X}$ be some *normal crossings* compactification.

**Definition/Theorem**. Let $\{D_i\}_{i \in I}$ be the components of $\overline{X} \setminus X$. The **boundary complex** of $X$ is the generalised cell complex with cells labeled $\{D_J | J \subset I\}$ glued along inclusions of strata. The homotopy type of boundary complex is ***independent*** of the choice of the normal crossings compactification.

**Example**. The boundary complex of $\mathbb{G}_m^2$ is homotopic to $S^1$.



**Key example**. The boundary complex of $\mathcal{M}_{g,n}$ agrees with the link of $\mathrm{M}_{g,n}^{\mathrm{trop}}$, the moduli of *tropical curves of genus $g$ and $n$ marked points* ([2], [1]) via the Deligne-Mumford compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$.

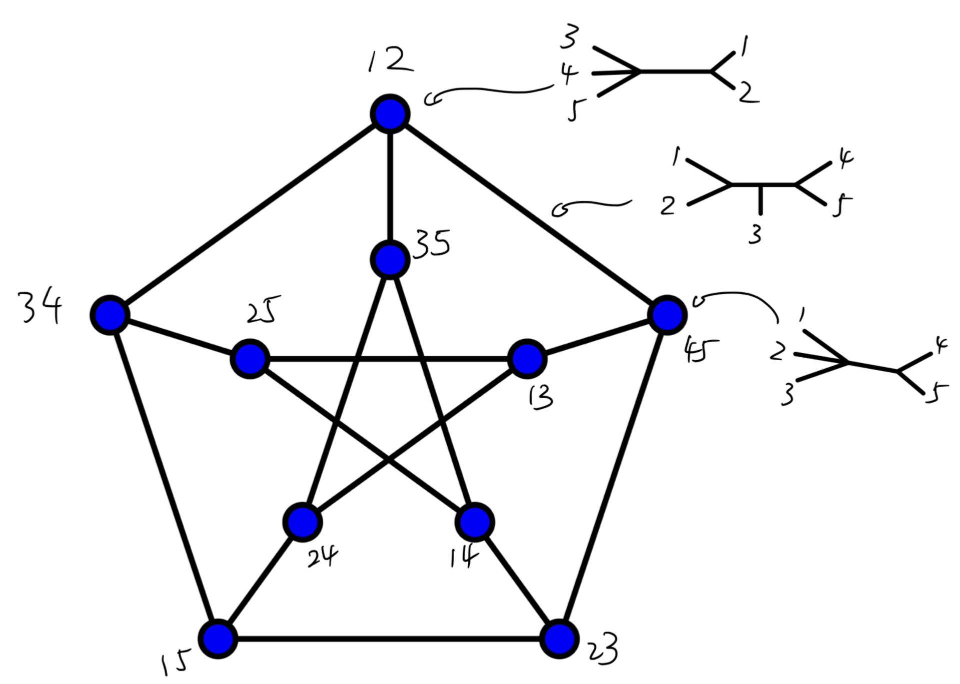


Figure 1. The boundary complex of $\mathcal{M}_{0,5}$, reproducing [3, §4.3]

Our work concern **mapping spaces** from curves to the projective space. Let $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$ be maps from smooth genus $g$, $n$-marked curves to $\mathbb{P}^r$ of degree $d$ up to isomorphisms.

## Mapping spaces and stable maps

**Question**. What is the topology of the dual complex of $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$?

**Theorem 1**. For $d > 0$, the dual complex of $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ is contractible.

In genus zero, the *stable map* compactification $\mathcal{M}_{0,n}(\mathbb{P}^r, d) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is of normal crossings. The boundary strata correspond to trees + degree labelings. The dual complex from such data admits an explicit deformation retract to a point:

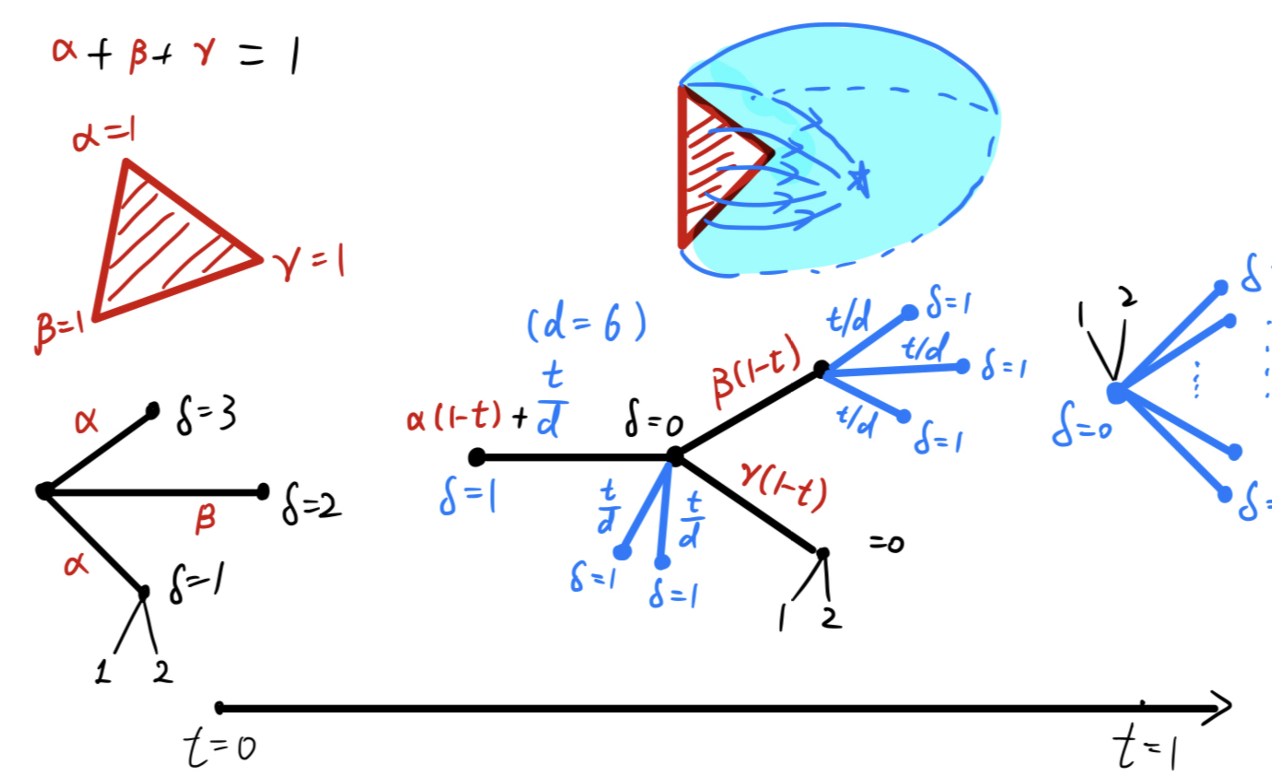


Figure 2. From vertices of degree $> 1$, sprout off degree one vertices. Shrink all existing edges and grow the new edges linearly.

In fact, this works for dual graph + degrees in any genus:

**Theorem 2**. The *virtual* complex for $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ - cone complex with cones indexed by stable map dual graphs - is contractible.

**Problem!** The stable map space $\mathcal{M}_{1,n}(\mathbb{P}^r, d) \subset \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ has strata of dimension *greater* than the interior. The combinatorics of the graphs might not recover the genuine dual complex.

## Vakil–Zinger type desingularisation

From now on, let $d > 1$. The closure of $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$ in $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ was explicitly desingularised by Vakil–Zinger [5]. A further blow-up, denoted $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ admits a *modular* interpretation by Ranganathan–Santos-Parker–Wise [4]:

| | |
|---|---|
| Combinatorics | (*Extra data*) A generic vector in the direct sum of tangent spaces at nodes 'equidistant' from the genus one component. |
| Geometry | (*Closed condition*) The stable map has to *collapse* the choice of vector at the *contraction radius*. |

**Combinatorics of the mapping space**. The mapping space is stratified by 'radially aligned' stable map dual graphs $(\mathbf{G}, \rho)$ with new specialisation relations:

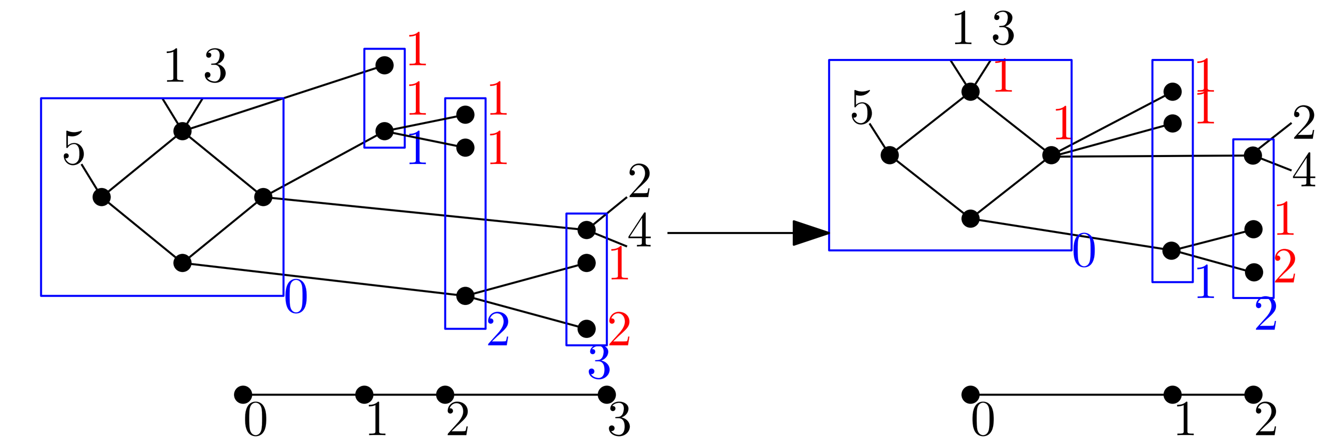


Figure 3. Merging blocks 0 and 1. The two edges connecting the blocks need to be contracted *simultaneously*.

**Geometry of the strata**. For each $(\mathbf{G}, \rho)$, the stratum $\widetilde{\mathcal{M}}(\mathbf{G}, \rho)$ of maps with the aligned dual graph is *connected*. There is an explicit criterion for when it is non-empty, ie., realisable.

## Main results

**Theorem 3**. The dual complex of the *realisable* radially aligned stable map dual graphs agrees with the (geometric) dual complex of $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$.

This is a consequence of strata connectedness and that $\mathcal{M}_{1,n}(\mathbb{P}^r, d) \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ has normal crossings boundary.

**Main Theorem**. The dual complex of $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$ is homeomorphic to a subspace of the virtual dual complex and preserved under the deformation retraction. Hence, it is contractible.

### Future directions

Dual complexes of other moduli spaces including $\mathcal{M}_{2,n}(\mathbb{P}^r)$, multi-scale differentials, and logarithmic maps.

## References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