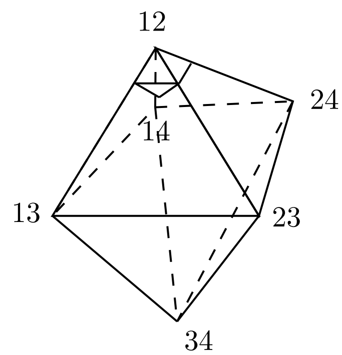
Defining the weighted Dressian

We fix a pair (r,n) for rank-r matroids on $[n]$. Let $\mathbf{w} \in \mathbb{Q}^n$ satisfy that $0 < w_i \leq 1$ and $\sum_{i=1}^n w_i > r$.

Definition 0.1. The $\mathbf{w}$-cut hypersimplex (in $\Delta(r,n)$) is $\Delta_w(r,n) := \{\mathbf{x} \in \Delta(r,n) | \forall i \in [n] : x_i \leq w_i\}$.

Example 0.2. The figure below shows $\Delta_{\mathbf{w}}$ with $(r,n) = (2,4)$, and $\mathbf{w} = (1,1,\epsilon,\epsilon)$, with $0 < \epsilon \ll 1$. This is a dilated copy of $\Delta(2,4)$ along the vertex $\mathbf{e}_{24}$.



Definition 0.3. A regular subdivision of $\Delta_{\mathbf{w}}$ is a *partial matroid tiling* if there exists a face-fitting collection of matroid polytopes $\mathcal{P} = \{P\}$ such that the subdivision is given by $\{P \cap \Delta_{\mathbf{w}} | P \in \mathcal{P}\}$.

The $\mathbf{w}$-weighted Dressian (modulo lineality) $\overline{\mathrm{Dr}}_{\mathbf{w}} \subset \mathrm{SecFan}(\Delta_{\mathbf{w}})$ consists of (classes of) height functions that induce partial matroid tilings of $\Delta_{\mathbf{w}}$. It can be presented as the image of a map between fans and hence carries some fan structure.

Example 0.4. For the weight $\mathbf{1} = (1,1,\ldots,1)$, $\Delta_{\mathbf{1}} = \Delta(r,n)$ recovers the Dressian $\overline{\mathrm{Dr}}(r,n)$.

Example 0.5. For the Losev-Manin weight (first r entries as 1, and the rest as ϵ), since $\Delta_{\mathbf{w}}$ is a 'small closed neighbourhood' around the vertex $\mathbf{e}_{[r]}$ any matroid polytope that contribute to the tiling needs to have $\mathbf{e}_{[r]}$ as a vertex. As a consequence, the partial tiling is determined by its restriction on the vertex figure of $\mathbf{e}_{[r]}$ (in $\Delta_{\mathbf{w}}$).

Further, a result Rincón [Rin13, Prop. 3.2] states that all regular subdivisions of the vertex figure comes from some partial matroid tiling. Therefore, the weighted Dressian agrees with the secondary fan of the vertex figure.

Remark 0.6. The definition does not record which matroid polytope contributes to the partial tiling. For instance, with $(r,n) = (2,4), \mathbf{w} = (1,1,\epsilon,\epsilon)$, both the upper pyramid and the full hypersimplex restrict to the trivial subdivision of $\Delta_{\mathbf{w}}$.

Motivation

The motivation comes from a few moduli spaces in algebraic geometry. Let's start by recalling the following result of Kapranov,

Theorem 0.7. [Kap92, §4.1] The boundary complex of the compactification $M_{0,n} \subset \overline{M}_{0,n}$ can be naturally identified with the Dressian $\overline{\mathrm{Dr}}(2,n)$ (modulo its lineality space).

Remark 0.8. See [MS15, §4.3] for more perspectives and references.

Namely, the tropical linear space associated to the Dressians are the tropical curves showing up in $M_{0,n}^{\text{trop}}$.

There are two ways to generalise the pair $(\overline{M}_{0,n}, \overline{\text{Dr}}(2,n))$. The first one is varying the weights of the marked points:

Definition 0.9.
- $\mathbf{w} \in \mathbb{Q}^n$ is an admissible weight vector if for each $i \in [n]$, $0 < w_i \leq 1$ and $\sum_{i=1}^n w_i > 2$.
- $M_{0,\mathbf{w}}$ (resp. $\overline{M}_{0,\mathbf{w}}$) parametrises n ordered points $p_1, \dots, p_n$ on $\mathbb{P}^1$ (resp. genus zero nodal curves) modulo isomorphism, such that p_i, p_j are distinct when $w_i + w_j > 1$, and on each component C, $\sum_{i\in[n]:p_i\in C} w_i + \#$ of nodes on $C > 2$.

Theorem 0.10. [Has03]

- $M_{0,\mathbf{w}} \subset \overline{M}_{0,\mathbf{w}}$ is a compactification.
- There is an explicit wall-and-chamber decomposition of the set of admissible $\{\mathbf{w}\}$ given by affine hyperplanes $\{\sum_{i\in I} w_i = m\}_{I\subset[n], m\in\mathbb{N}}$, such that $\overline{M}_{0,\mathbf{w}}$ is identical in each chamber interior, and there are two types of natural morphisms:
  - *Specialisation*: let $\text{Ch}(\mathbf{w}_1)$ denote the chamber interior that $\mathbf{w}_1$ belongs to, then for all $\mathbf{w}_2 \in \overline{\text{Ch}(\mathbf{w}_1)}$, there is $\overline{M}_{0,\mathbf{w}_1} \to \overline{M}_{0,\mathbf{w}_2}$.
  - *Reduction*: for $\mathbf{w}_1 \geq \mathbf{w}_2$ (coordinate-wise), there is $\overline{M}_{0,\mathbf{w}_1} \to \overline{M}_{0,\mathbf{w}_2}$.

Remark 0.11. When $\mathbf{w} = (1,1,\epsilon,\dots,\epsilon)$, $M_{0,\mathbf{w}} \subset \overline{M}_{0,\mathbf{w}}$ recovers the Losev-Manin space, which is the toric variety associated to the permutahedral fan.

Definition 0.12. $M_{0,\mathbf{w}}^{\text{trop}}$ is the subfan of $M_{0,n}^{\text{trop}}$ consisting of cones corresponding to $\mathbf{w}$-stable rational tropical curves: for all vertices v, the weighted number of legs and the valency sum to > 2.

The fan $M_{0,\mathbf{w}}^{\text{trop}}$ (or its underlying cone complex) coincides with the boundary complex of the compactification $M_{0,\mathbf{w}} \subset \overline{M}_{0,\mathbf{w}}$ and has been studied in the previous works of Ulirsch [Uli15] and Cavalieri - Hampe - Markwig - Ranganathan [CHMR16].

The second way is to see a collection of points on $\mathbb{P}^1$ as a hyperplane arrangement. The analogue of $M_{0,n}$ is now $M^\circ(r,n) := n$ ordered hyperplanes in $\mathbb{P}^{r-1}$ having underlying matroid $U_{r,n}$, up to change of coordinates.

Theorem 0.13. [HKT06] A compactification $M^\circ(r,n) \subset M(r,n)$ of MMP (KSBA) stable hyperplane arrangements exist, and the degenerate arrangements in the moduli space are precisely the ones induced by matroid subdivisions of the hypersimplex.

[also [Kap92]] The case for $r = 2$ recovers the compactification $M_{0,n} \subset \overline{M}_{0,n}$.

Remark 0.14. The compact moduli spaces $M(r,n)$ are typically reducible and not of pure dimension.

Therefore, the higher rank Dressians $\overline{\text{Dr}}(r,n)$ can still be seen as an 'atlas' for $M^\circ(r,n) \subset M(r,n)$ in that they parametrise the combinatorics of the degneration. Still, the correspondence is much less explicit than the case of $M_{0,n} \subset \overline{M}_{0,n}$.

In algebraic geometry, the two generalisations can be mixed together as well: there are moduli spaces of weighted hyperplane arranegments parametrising pairs $(\mathbb{P}^{r-1}, \sum_{i=i}^n w_i H_i)$ and their degenerations.

Theorem 0.15. [Ale08] The space $M(r,n)$ fits into a class of compact moduli spaces $M_{\mathbf{w}}(r,n)$. Further, the statements about wall-and-chamber decompositions and natural maps still hold for $M_{\mathbf{w}}(r,n)$.

In view of the previous results of Kapranov and Hacking - Keel - Tevelev, the construction of the weighted Dressian is motivated by the following

Question. What is the analogue of the Dressian that acts as an atlas of $M_{\mathbf{w}}(r,n)$?

The appearance of the $\mathbf{w}$-cut hypersimplex $\Delta_{\mathbf{w}}$ comes from a hint given by Alexeev:

Theorem 0.16. [Ale08, Theorem 2.12] A hyperplane arranegment $(\mathbb{P}^{r-1}, H_1, \ldots, H_n)$ is $\mathbf{w}$-stable (resp. semistable) if and only if the interior of its matroid polytope (resp. the full matroid polytope) covers the $\mathbf{w}$-cut hypersimplex.

Results

Proposition 0.17. In the case of $r = 2$, the construction $\overline{\mathrm{Dr}}_{\mathbf{w}}$ carries a fan structure that is isomorphic to $M^{\mathrm{trop}}_{0,\mathbf{w}}$.

The construction is analogous to the identification $\overline{\mathrm{Dr}}(2,n) \cong M^{\mathrm{trop}}_{0,n}$. The technical detail is to check that the construction of the $\mathbf{w}$-stable tropical curve only depends on the subdivision of $\Delta_{\mathbf{w}}$ and not the choice of matroid polytopes that contribute to the tiling.

Proposition 0.18. The collection of fans $\{\overline{\mathrm{Dr}}_{\mathbf{w}}\}_{\mathbf{w}}$ satisfies the wall-crossing morphisms (of fans) as in the statement of Theorem 0.10.

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