$\chi^{S_n}(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$

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Strategy

Contribution from $\overline{\mathcal{M}}^{\mathrm{nrt}}_{1,n}(\mathbb{P}^r, d)^{\mathbb{C}^*}$ with $d > 0$:

$$\mathrm{ch}_{S_2 \wr \mathbb{S}}(\mathsf{Dih}_r) \circ_{S_2} \sum_{n \geq 0} \chi^{S_2 \times S_n}(\mathrm{Cat}_{2,n}).$$

$$\mathrm{ch}_{S_2 \wr \mathbb{S}}(\mathsf{Dih}_r) = \sum_{k \geq 2} \sum_{H \subset D_k} n_H \cdot \chi^{S_2 \wr S_k}(S_2 \wr S_k / H) \in \Lambda(S_2).$$

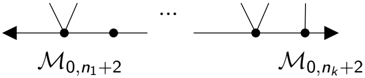
$$n_H := \#\{\text{decorated } k\text{-cycles with Aut} = H\}$$

Characteristics of Cat

$$\Lambda = \mathbb{Q}[[p_1, p_2, \dots]], \Lambda_2 \otimes \Lambda = (p_1^2 \otimes \Lambda) \oplus (p_2 \otimes \Lambda).$$

$$\mathsf{a}_0 := \sum_{n \geq 3} \chi^{S_n}(\mathcal{M}_{0,n}).$$

$\mathrm{Cat}_2 = \bigsqcup_n \mathrm{Cat}_{2,n}$ is stratified by



$$\sum_{n \geq 0} \chi^{S_2 \times S_n}(\mathrm{Cat}_{2,n}) = \frac{1}{2} p_1^2 \otimes \frac{1}{1 - \mathsf{a}_0''} + \frac{1}{2} p_2 \otimes \frac{1 + 2\dot{\mathsf{a}}_0}{1 - \psi_2(\mathsf{a}_0'')}.$$

Operations: $\mathsf{f}' = \partial \mathsf{f} / \partial p_1, \dot{\mathsf{f}} = \partial \mathsf{f} / \partial p_2, \psi_i(\mathsf{f}) = p_i \circ \mathsf{f}$.This gives us the second term in

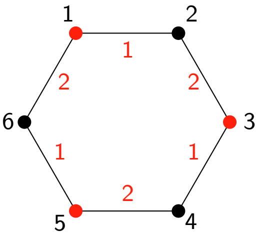
$$\mathrm{ch}_{S_2 \wr \mathbb{S}}(\mathsf{Dih}_r) \circ_{S_2} \sum_{n \geq 0} \chi^{S_2 \times S_n}(\mathrm{Cat}_{2,n}).$$

Characters $\chi^{S_2 \wr S_k}(S_2 \wr S_k / H)$

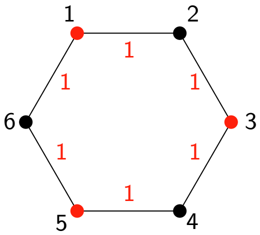
The characteristics live in $\Lambda(S_2) = \mathbb{Q}[[\mathfrak{p}_i, \mathfrak{q}_i \mid i > 0]]$.

$H \subset D_k = \langle \rho, \tau \rangle \subset S_2 \wr S_k$.

H	$\chi^{S_2 \wr S_k}(S_2 \wr S_k / H)$
$\langle \rho^j \rangle$ for $j \mid k$	$\frac{j}{k} \sum_{s \mid \frac{k}{j}} \varphi(s) \mathfrak{p}_s^{k/s}$
$\langle \rho^j, \rho\tau \rangle$ for $j \mid k$ and j even	$\underbrace{\frac{1}{2}\mathfrak{p}_2^{k/2-1}\mathfrak{q}_1^2}_{\text{flips}} + \frac{j}{2k} \sum_{s \mid \frac{k}{j}} \varphi(s) \mathfrak{p}_s^{k/s}$



$$\frac{1}{3}(\mathfrak{p}_1^6 + 2\mathfrak{p}_3^2)$$



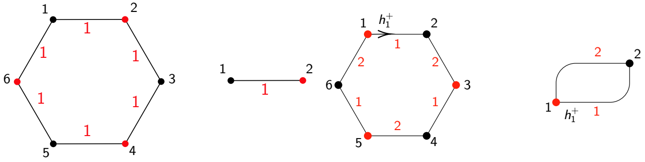
$$\frac{1}{2}\mathfrak{p}_2^2\mathfrak{q}_1^2 + \frac{1}{6}(\mathfrak{p}_1^6 + 2\mathfrak{p}_3^2)$$

Enumerating decorated cycles

Given $H \subset D_k$, we want to know the number n_H of decorated cycles with automorphism group H.

Related quantity: number $n_{\geq H}$ of decorated cycles with automorphism group *containing* H.

These are vertex coloring and degree assignment on 'quotient' cycles:



Enumerating decorated cycles

H	$n_{\geq H}$
$\langle \rho^j \rangle$ for $j \mid k$	$\underbrace{\binom{\frac{dj}{k}-1}{j-1}}_{\text{degree assignment}} \quad \underbrace{\varpi_j(r+1)}_{\text{chromatic polynomial on } j\text{-cycle}}$
$\langle \rho^{2\ell}, \rho\tau \rangle$ for $2\ell \mid k$	$\underbrace{\binom{\frac{d\ell}{k}-1}{\ell-1}}_{\text{degree assignment}} \quad \underbrace{\omega_k(r+1)}_{\text{chromatic polynomial on path with } k \text{ edges}}$

$$n_{\geq H} = \sum_{H \leq H'} n_H \Rightarrow n_H = \sum \mu(H, H') n_{\geq H'}$$

where $\mu(H, H') \in \pm 1$ is the Möbius function of the poset $\{H \subset D_k\}$.

Calculating the plethysm

Now we know both terms of

$$\mathrm{ch}_{S_2\wr\mathbb{S}}(\mathsf{Dih}_r)\circ_{S_2}\sum_{n\geq 0}\chi^{S_2\times S_n}(\mathrm{Cat}_{2,n}).$$

$\circ_{S_2}$ is a map $\Lambda(S_2)\times(\Lambda_2\otimes\Lambda_{>0})\to\Lambda$.
From first principles:

- $\mathfrak{p}_n\circ(\frac{1}{2}p_1^2\otimes f_1+\frac{1}{2}p_2\otimes f_2)=p_n\circ f_1$,
- $\mathfrak{q}_n\circ(\frac{1}{2}p_1^2\otimes f_1+\frac{1}{2}p_2\otimes f_2)=p_n\circ f_2$,
- $\mathfrak{p}_n^j\circ_{S_2} f=(\mathfrak{p}_n\circ_{S_2} f)^j$, $\mathfrak{q}_n^j\circ_{S_2} f=(\mathfrak{q}_n\circ_{S_2} f)^j$.

Example:

$$\mathfrak{p}_1\circ_{S_2}\sum_{n\geq 0}\chi^{S_2\times S_n}(\mathrm{Cat}_{2,n})=\frac{1}{1-\mathsf{a}_0''}$$

$$\mathfrak{q}_1\circ_{S_2}\sum_{n\geq 0}\chi^{S_2\times S_n}(\mathrm{Cat}_{2,n})=\frac{1+2\dot{\mathsf{a}}_0}{1-\psi_2(\mathsf{a}_0'')}.$$

Hence the main formula only involves ordinary symmetric functions Λ rather than S_2-wreath symmetric functions $\Lambda(S_2)$.

Main formula

$$\mathrm{b}^{\mathrm{nrt}}_{1,r} = \overbrace{(r+1)\left(\mathrm{a}_1 + \frac{\dot{\mathrm{a}}_0(\dot{\mathrm{a}}_0+1)+\frac{1}{4}\psi_2(\mathrm{a}_0'')}{1-\psi_2(\mathrm{a}_0'')} - \frac{1}{2}\sum_{n\geq 1}\frac{\varphi(n)}{n}\log(1-\psi_n(\mathrm{a}_0''))\right)}^{\overline{\mathcal{M}}^{\mathrm{nrt}}_{1,n}(\mathbb{P}^r,0)}$$

$$+\sum_{d\geq 2} q^d \sum_{k=2}^{d} \eta_{k,d}(r)\frac{(1+2\dot{\mathrm{a}}_0)^2}{(1-\psi_2(\mathrm{a}_0''))^{k/2+1}} + \sum_{j|k}\theta_{j,k,d}(r)\frac{1}{(1-\psi_j(\mathrm{a}_0''))^{k/j}}.$$

The factors $\eta_{k,d}(r)$ and $\theta_{j,k,d}(r)$ are polynomials in $\mathbb{Q}[r]$ given by the formulas

$$\eta_{k,d}(r) = \frac{1}{4}\sum_{j|k}\sum_{i|j}\sum_{\substack{\ell \\ 2\ell|i \text{ and } k|d\ell}} \mu\left(\frac{i}{2\ell}\right)\binom{\frac{d\ell}{k}-1}{\ell-1}(r^{\ell+1}+r^{\ell}),$$

$$\theta_{j,k,d}(r) = \frac{\varphi(j)}{2k}\sum_{i|\frac{k}{j}}\sum_{\substack{\ell \\ \ell|i \text{ and } k|d\ell}} \mu\left(\frac{i}{\ell}\right)\binom{\frac{d\ell}{k}-1}{\ell-1}(r^{\ell}+(-1)^{\ell}r).$$

Main formula

$$\mathrm{b}_{1,r} = \mathrm{b}^{\mathrm{nrt}}_{1,r} \circ \left(p_1 + \overbrace{\frac{\mathrm{b}'_{0,r}}{r+1}}^{\overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r,d)} \right).$$

n	$\chi^{S_n}\left(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,2)\right)$
0	$24\binom{r+1}{3}+17\binom{r+1}{2}$
1	$\left[108\binom{r+1}{3}+58\binom{r+1}{2}\right]s_1$
2	$\left[339\binom{r+1}{3}+171\binom{r+1}{2}+6\binom{r+1}{1}\right]s_2+\left[168\binom{r+1}{3}+65\binom{r+1}{2}+6\binom{r+1}{1}\right]s_{1,1}$
3	$\left[1176\binom{r+1}{3}+498\binom{r+1}{2}+6\binom{r+1}{1}\right]s_3+\left[996\binom{r+1}{3}+396\binom{r+1}{2}+12\binom{r+1}{1}\right]s_{2,1}$ $+\left[144\binom{r+1}{3}+40\binom{r+1}{2}+6\binom{r+1}{1}\right]s_{1,1,1}$

n	$\chi^{S_n}\left(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,3)\right)$
0	$216\binom{r+1}{4}+247\binom{r+1}{3}+55\binom{r+1}{2}$
1	$\left[1300\binom{r+1}{4}+1365\binom{r+1}{3}+260\binom{r+1}{2}\right]s_1$
2	$\left[5380\binom{r+1}{4}+5319\binom{r+1}{3}+945\binom{r+1}{2}\right]s_2+\left[3156\binom{r+1}{4}+2991\binom{r+1}{3}+503\binom{r+1}{2}\right]s_{1,1}$

Features

- The S_n-equivariant characteristics are of the form

$$\sum_\lambda \sum_j n_{\lambda,j} \binom{r+1}{j} s_\lambda$$

  for $n_{\lambda,j} \in \mathbb{Z}$.
  Fact: true for all $\chi^{S_n}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d))$ [Kannan - S].
- For $r \gg 0$, rational tails contribution dominates the S_n-characteristics:

n	$\chi^{S_n}\left(\overline{\mathcal{M}}^{\mathrm{nrt}}_{1,n}(\mathbb{P}^r,2)\right)$	$\mathrm{b}^{\mathrm{rt}}_{1,r}(n,2)$
0	0	$24\binom{r+1}{3} + 17\binom{r+1}{2}$
1	$2\binom{r+1}{2}s_1$	$\left[108\binom{r+1}{3} + 56\binom{r+1}{2}\right] s_1$
2	$3\binom{r+1}{2}s_2 + \binom{r+1}{2}s_{1,1}$	$\left[339\binom{r+1}{3}+168\binom{r+1}{2}+6\binom{r+1}{1}\right]s_2$ $+\left[168\binom{r+1}{3}+64\binom{r+1}{2}+6\binom{r+1}{1}\right]s_{1,1}$
3	$4\binom{r+1}{2}s_3 + 2\binom{r+1}{2}s_{2,1}$	$\left[1176\binom{r+1}{3}+494\binom{r+1}{2}+6\binom{r+1}{1}\right]s_3$ $+\left[996\binom{r+1}{3}+394\binom{r+1}{2}+12\binom{r+1}{1}\right]s_{2,1}$ $+\left[144\binom{r+1}{3}+40\binom{r+1}{2}+6\binom{r+1}{1}\right]s_{1,1,1}$

Future directions

- Other classes of spaces: Vakil–Zinger desingularization, quasimaps; targets: projective toric varieties and flag varieties.
- Higher genus:
  - a decorated k-cycle Γ has $\mathrm{Aut}(\Gamma) \subset D_k \subset S_2 \wr S_k \Rightarrow$ calculations in $\Lambda(S_2)$.
  - More generally, any decorated graph G has $\mathrm{Aut}(G) \subset \prod_{i \geq 0} S_i \wr S_{\nu(G)_i}$ ($\nu(G)_i :=$ # vertices with valency i) $\Rightarrow$ calculations in $\Lambda^{[2]} := \bigotimes_{i \geq 0} \Lambda(S_i)$. [Kannan - S. ongoing]