TOPOLOGY OF GENUS ONE MAPPING SPACES TO PROJECTIVE SPACE

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ABSTRACT. We derive a set of generators for the rational cohomology of the desingularised genus one mapping space $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ constructed by Vakil–Zinger and the pure weight cohomology of the stable maps space $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ and discuss a way to recover all relations. The results build on a detailed study of the stratifications of the moduli spaces coming from tropical geometry and the constraints coming from the weight filtration on the cohomology of strata, extending the techniques from the previous study on $\overline{\mathcal{M}}_{g,n}$. Our results imply that the even cohomology of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is tautological and controlled by genus zero and reduced genus one Gromov–Witten theory. We also completely describe the rational Picard group of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ and recover known results on the vanishing of odd cohomology.

CONTENTS

1. INTRODUCTION

Let $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$ be the moduli space of degree d maps from n-marked smooth elliptic curves. Modular compactifications of the space, and its higher genus analogues, are of interest to both enumerative geometry and understanding of degenerations of line bundles and linear systems on curves.

The stable maps compactification $\mathcal{M}_{1,n}(\mathbb{P}^r, d) \subset \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ parametrise maps from nodal genus one curves to the projective space and is hence stratified by the dual graphs of domain curves with degree decorations. The associated locally closed strata of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ are smooth, but in general they have dimension strictly greater than $\dim \mathcal{M}_{1,n}(\mathbb{P}^r, d)$, so that $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is reducible. The work of Vakil–Zinger [VZ08] constructs an iterative blow-up $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ such that $\mathcal{M}_{1,n}(\mathbb{P}^r, d) \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is a normal crossings compactification. Their construction is given a modular interpretation by Ranganathan – Santos-Parker – Wise [RSPW19] with input from logarithmic geometry and elliptic singularities.

This work studies the topology of $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$ and its modular compactifications on an equal footing to $\mathcal{M}_{1,n}, \overline{\mathcal{M}}_{1,n}$, and their universal Picard groups. We describe the rational cohomology $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d); \mathbb{Q})$

and the pure weight cohomology $\mathrm{gr}^W_\star H^\star(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d);\mathbb{Q})$ in terms of their natural stratifications and cohomology of genus zero stable maps.

We begin by briefly discussing the generators in our main result. See Section 4.2 for more details.

Definition 1.1. Let $\mathrm{Pic}^d_{1,n} \to \mathcal{M}_{1,n}$ be the relative degree d Picard group over $\mathcal{M}_{1,n}$. Let $\Theta \in H^2(\mathrm{Pic}^d_{1,n})$ be the class of a relative polarisation of $\mathrm{Pic}^d_{1,n} \to \mathcal{M}_{1,n}$.

A point in $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ parametrises a marked elliptic curve, a degree d line bundle, and a projectivised $(r+1)$-tuple of sections that have basepoint. For $d \geq 2$, the projectivised tuples of sections are parametrised by a vector bundle $\mathcal{E}_d^{\oplus r+1}$ over $\mathrm{Pic}^d_{1,n}$ such that the fibre over $(C,\mathbf{p},\mathcal{L})$ is given by $H^0(C,\mathcal{L})^{\oplus r+1}$. Then $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ admits an open embedding in the projectivisation $\mathcal{M}_{1,n}(\mathbb{P}^r,d) \subset \mathbb{P}_{\mathrm{Pic}^d_{1,n}}(\mathcal{E}_d^{\oplus r+1})$.

Definition 1.2. Let $H_{\hat{\mathcal{Q}}} \in H^2(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$ be the pullback of the hyperplane class on $\mathbb{P}_{\mathrm{Pic}^d_{1,n}}(\mathcal{E}_d^{\oplus r+1})$ given by the projective bundle formula. A similar discussion applies to maps from a nodal elliptic curve, in which $H_{\hat{\mathcal{Q}}}$ is still well-defined: see Section 4.3.

Finally, we consider the Artin stacks $\mathsf{M}_{0,1}$ and $\mathsf{M}_{0,2}$ of smooth rational curves with one and two marked points, respectively. They are identified with classifying stacks of automorphisms of $\mathbb{P}^1$ that preserves the set of marked points. In particular, there are isomorphisms $H^\star(\mathsf{M}_{0,i}) \cong \mathbb{Q}[\psi]$ for $i = 1,2$, where $\psi \in H^2(\mathsf{M}_{0,i})$ is the ψ-class of any marked point and is well-defined up to a sign.

Theorem A (Corollary 5.9)**.** $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ and $\mathrm{gr}^W_\star H^\star(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ are generated by $(\iota_{[\mathbf{G},\rho]})_*F$, where $(\mathbf{G},\rho)$ labels a boundary stratum closure $\iota_{[\mathbf{G},\rho]} : \overline{\mathcal{M}}_{(\mathbf{G},\rho)} \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, and F is a polynomial of the following classes:

- pullback from $W_\star H^\star(\mathcal{M}_{1,m})$ for $m \leq n+d$,
- $\Theta \in H^2(\mathrm{Pic}^{d'}_{1,m})$ and $H_{\hat{\mathcal{Q}}} \in H^2(\mathcal{M}_{1,n'}(\mathbb{P}^r,d'))$ defined above,
- pullback from $H^\star(\mathsf{M}_{0,1})$ and $H^\star(\mathsf{M}_{0,2})$, each generated by their ψ-classes,
- pullback from rational tails, which are isomorphic to $\overline{\mathcal{M}}_{0,n'}(\mathbb{P}^r,d')$,

and $(\iota_{(\mathbf{G},\rho)})_*$ is the Gysin pushforward on cohomology. For $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$, we identify the source of relations among the generators in Section 6.

The precise statement of our main result is more refined than as stated above and discussed in Section 5.2. The generators of $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ come in two flavours. On the one hand, all but the first class of generators in the list admit lifts to algebraic cycles which are tautological in the sense of Bae [Bae20]: the cohomology of genus zero stable maps are known to be tautological cycles from the work of Oprea [Opr06a]. Thus, the cohomology $H^\star((\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)))$ is generated by tautological classes and pullback of $W_\star H^\star(\mathcal{M}_{1,m})$ for $m \leq n+d$. Since the even cohomology of $\overline{\mathcal{M}}_{1,n}$ are all spanned by boundary classes [Pet14], the even cohomology of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ is tautological and is hence entirely captured by reduced genus one Gromov–Witten invariants, which are intersection numbers of the tautological cycles on $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$. This is a genus one analogue of the fact that the cohomology of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is captured by genus zero Gromov–Witten invariants.

In the other direction, Theorem A describes the Hodge structures in $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$. Following [CLP24], we denote $\mathsf{L} := H^2(\mathbb{P}^1)$ and $\mathsf{S}_{k+1} := W_k H^k(\mathcal{M}_{1,k})$. By Eichler–Shimura isomorphism, S_{k+1} correspond to $\mathrm{SL}_2(\mathbb{Z})$-cusp forms of weight $k+1$ and are possibly non-zero for odd $k \geq 11$. The generation result implies that the Hodge structures present in $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ are products of L and S_{k+1} for $k \geq n+d$. In particular, the odd weight Hodge structures in $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ have weight at least 11. Since $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ has pure weight cohomology, we recover the following result of Fontanari [Fon07].

Corollary B. For odd $k < 11$ and all n, $H^k(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ vanishes.

The presence of Hodge structures $\mathsf{L}^j \prod_k \mathsf{S}_{k+1}^{\ell_k}$ is tied to the graphical stratification of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ and plethysms of $\mathbb{S}$-modules arising from the fibre product descriptions of the strata. This allows more explicit bounds and estimates on the cohomological degrees and ranks that each Hodge structure contributes.

Example 1.3 (Section 5.3)**.** For $r \geq 11$, the lowest d such that $H^\star(\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r,d))$ has non-vanishing odd cohomology is $d = 11$. In this case, $H^{123}(\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r,11)) \neq 0$ is the first non-vanishing odd cohomological degree. The lowest odd cohomological degree k with $H^k(\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r,d))$ non-vanishing is realised by $k = 13$ when $d \geq 66$.

Another application of our results is the Picard group of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, which was previously unknown.

Corollary C (Proposition 6.1)**.** The second cohomology group $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ has a basis given by the pullbacks of $\Theta, H_{\hat{\mathcal{Q}}}$, and boundary divisors. Further, the cycle class map $A^1_{\mathbb{Q}}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)) \to H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ is an isomorphism.

The proof that no relations hold on $H^2(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$ illustrates our general approach to relations. The relations come from non-pure weight part of the cohomology groups of the interior that are images under coboundary maps of the generators: see Section 6 for more details. Our calculation of these classes leads to the following qualitative expectations:

- apart from pullbacks of Getzler relation [Get97, Pet14], few relations hold in each fixed cohomological degree that is small compared to r,
- relations begin to appear in cohomological degrees growing linearly with r.

The dependence on r suggests that these relations come from the geometry of the linear systems or the presence of the target $\mathbb{P}^r$, which is indeed the case: the geometric descriptions of strata typically involve basepoint free loci of linear systems or polynomials, and they are locally modeled by the complements of high codimension subspace arrangements in affine spaces.

Even in the case of genus zero stable maps, relations among tautological cycles, especially their interactions with the graphical stratifications, seem to be understudied. A more detailed study on relations among the decorated strata classes in both genus zero and genus one mapping spaces will be carried out in [Son25].

Context. We note that in contrast to $\mathcal{M}_{g,n}, \overline{\mathcal{M}}_{g,n}$ and their (compactified) Picard groups, little is known about the topology of mapping spaces of curves to projective space. The presence of the target $\mathbb{P}^r$ raises the following challenges:

- the moduli spaces now parametrise the complete linear system associated to the universal Picard group, which in general have complications studied in Brill–Noether theory,
- the degree decorated dual graphs have a much higher combinatorial complexity compared to $\overline{\mathcal{M}}_{g,n}$ and compactified Jacobians.

In high genus, both factors contribute to the reducibility and singularity of the stable maps moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$. Even in genus zero, the existing results on the Chow ring and cohomology of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$

[Pan99, Opr06a, MM06] are already significantly more subtle than the appealing description in the case of $\overline{\mathcal{M}}_{0,n}$ due to Keel [Kee92]. The present work takes a first step in the analogous investigation on higher genus mapping spaces.

1.1. **Overview.** Our calculation of $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ proceeds in two steps:

- using descriptions of graphical strata in $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ [RSPW19, KS24a] as fibre products, we obtain partial computations on the weight filtrations of the locally closed strata;
- the strata calculations fit together in a spectral sequence that converges to $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ or $\mathrm{gr}^W_\star H^\star \overline{\mathcal{M}}^{\mathrm{main}}_{1,n}(\mathbb{P}^r, d)$.

As such, the work combines the recursive combinatorial structure of stable maps, the tropical geometry of radially aligned curves, explicit parametrisation of maps from rational curves, and weight filtrations in the stratification and Leray spectral sequences. We now give a more detailed discussion of these ingredients.

1.1.1. *Graphical stratification.* It is well-known that moduli spaces of stable maps, in particular $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, are stratified by prestable dual graphs with suitable decorations. Extending this, the desingularisation construction in [VZ08] receives a modular interpretation in [RSPW19] as stable maps from centrally aligned curves satisfying the factorisation property. Thus, analogous to the ordinary stable maps case, they are stratified by aligned dual graphs with decorations.

The decorated dual graph gives the corresponding stratum a iterated fibration structure that we describe in Section 3 involving open mapping spaces $\mathcal{M}_{1,n'}(\mathbb{P}^r, d'), \mathcal{M}_{0,n'}(\mathbb{P}^r, d')$ or closed subscheme in products $\prod_{j=1}^k \mathcal{M}_{0,n_j}(\mathbb{P}^r, d_j)$ cut out by the conditions on the derivatives. As the combinatorics of the boundary strata is intricate and grows exponentially with d, we instead take partial closures of certain strata and work with a coarser stratification that is indexed only by a pair of partitions of d and $[n]$. The strata we consider are hence fibre products involving genus zero *stable* maps, rather than locally closed strata therein: see Definition 3.4 for more details. Working with the coarser stratification provides results on genus one mapping spaces *assuming* the knowledge of $H^\star(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$, which has been approached in [Opr06a] and [MM06, MM07, MM08].

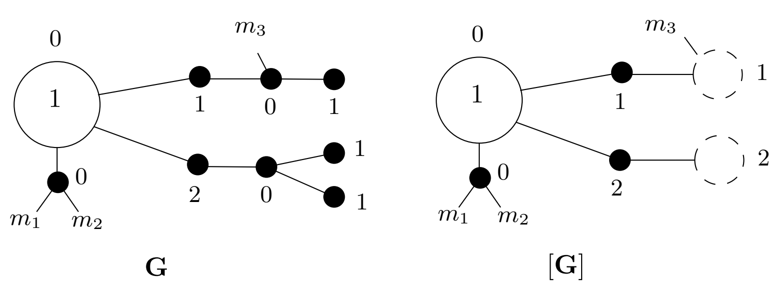

FIGURE 1. **G** is a stable maps dual graph, and [**G**] denotes its *contraction equivalence* class, which identifies dual graphs past the *contraction radius*.

A more conceptual motivation for stratifying with full stable maps spaces comes from the theory of operads. Let $\overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r, d)$ be any fibre of the Zariski locally trivial fibration $\mathrm{ev}_{n+1} : \overline{\mathcal{M}}_{0,n+1}(\mathbb{P}^r, d) \to \mathbb{P}^r$. It is an S_n-space stratified by degree decorated tree with a distinguished leg representing the frozen point and the formal sum $\sum_{n,d} \overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r, d) q^d$ carries the structure of a degree-graded operad. Informally, the operation of attaching rational tails to genus one maps endows the cohomology groups $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ and $\mathrm{gr}^W_\star H^\star(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ with the structure of algebras over this operad, and our computation with the coarser stratification reflects this structure.

Regarding the full stratification, the *boundary complex* - generalised cell complex consisting of boundary strata and their face relations - has been proven to be contractible in both $\mathcal{M}_{1,n}(\mathbb{P}^r,d) \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ for $d > 1$ and $\mathcal{M}_{0,n}(\mathbb{P}^r,d) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ for $d > 0$ [KS24a]. The computation of S_n-equivariant cohomology and Euler characteristics involves a detailed investigation of the combinatorics: see [GP06] for genus zero stable maps and [KS24b] for more recent development on the genus one mapping spaces. In the latter work, separating the contribution of rational tails also plays a crucial role.

1.1.2. *Weight filtration on the strata.* The fibration structure of each stratum gives rise to a Leray spectral sequence that computes the cohomology of the stratum, which then fit in a spectral sequence associated to the filtration[1] of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ given by strata closures. Carrying out the full calculation as stated is technically difficult because, for instance, it requires the knowledge of $H^\star(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$ and its various fibre products. The situation is simplified by considering the weight filtration on the strata, which we now introduce.

The rational (co)homology of algebraic varieties and stacks carry canonical mixed Hodge structures, and the weight filtration $W_\bullet H^\star(-)$ is the unique increasing filtration such that $\mathrm{gr}^W_i H^j(-) \otimes \mathbb{C}$ carries a Hodge structure of weight i. In view of the Hodge decomposition on smooth projective varieties, the weight filtration measures how far a variety is from being smooth and projective. Mixed Hodge structures are also present on cohomology with compact support, ordinary homology, and Borel–Moore homology, and they are compatible with standard dualities and isomorphisms. Importantly, the cycle class map $A_k(-) \to H^{\mathrm{BM}}_{2k}(-)$ always lands in $F^k\mathrm{gr}^W_{2k} H^{\mathrm{BM}}_{2k}(-)$. Dualising back to cohomology groups, the *pure weight cohomology* $\mathrm{gr}^W_\star H^\star(-)$ can thus be seen as the cohomological shadow of intersection theory on our moduli spaces.

In our setting, the locally closed strata of genus one mapping spaces carry non-pure weight filtrations due to their non-compactness. On the other hand, $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ is pure because $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ and we have restricted to calculating the pure weight subquotient $\mathrm{gr}^W_\star H^\star(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$. As our goal is to calculate the pure weight cohomology, the compatibility of the spectral sequence with weights implies that *only the pure weight part of each stratum can contribute to* $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$, since there is no non-trivial morphism between Hodge structures of different weights.

In particular, the pure weight parts of each stratum generate the cohomology. They can be accessed by the Leray spectral sequence [Ara05] and reduced to the pure weight cohomologies of each factor in the fibration, which we determine in Section 4 and assemble in Section 5.2. Further, tracing the spectral sequence, one sees that relations among these generators correspond to the 'off-by-one' weight graded piece of the cohomology[2] $\mathrm{gr}^W_{\star+1} H^\star(-)$, which we give partial descriptions along the way and summarise in Section 6. Thus, we arrive at a set of generators and relations of the cohomology groups. In the calculation, we explore the cohomology of the interior $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ and the universal Picard group $\mathrm{Pic}^d_{1,n}$ in Section and radially aligned genus zero maps, which might be of independent interest.

1.1.3. *Parametrised geometry of derivatives.* Compared to thestable maps moduli space $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, a novel technicality in describing the strata of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ is the smoothability condition imposed on the stable

[1] It generalises the standard excision long exact sequence and is also known as the Vassiliev spectral sequence [Vas01].

[2] Continuing the analogy, $\mathrm{gr}^W_{\star+k} H^\star(-)$ for $k > 1$, seem to record 'higher homotopies' among the generators in a way that is (Koszul) dual to the weight spectral sequence associated to the compactification.

maps that cuts down the excess components of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ for an irreducible moduli space. This has been interpreted in [RSPW19] as the *factorisation property*: informally, a point in $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is given by a diagram

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\ \varphi\ } & C \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \overline{C} & \dashrightarrow & \mathbb{P}^r \end{array}$$

where $f : C \to \mathbb{P}^r$ is a stable map, $\varphi : \widetilde{C} \to C$ is a semistable model, $\psi : \widetilde{C} \to \overline{C}$ a contraction to an elliptic singularity, such that the composition $f \circ \varphi$ factors through $\widetilde{C} \xrightarrow{\psi} \overline{C}$: this is the closed condition known as the factorisation property. It has been shown to be equivalent to requiring that the derivatives on the maps from certain genus zero components have a non-vanishing linear dependency [BNR21, Corollary 2.3].

This characterisation leads to a concrete description of the strata in $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ satisfying the factorisation property; in particular, they have been shown to be connected in [KS24a, §3]. Section 4.1 of this work studies the factorisation geometry along this line. Let $\mathrm{Map}^*_d(\mathbb{P}^1, \mathbb{P}^r)$ be the space of pointed, degree d maps from $\mathbb{P}^1$ to $\mathbb{P}^r$. Picking the coordinates of the distinguished points, there is an explicit open embedding $\mathrm{Map}^*_d(\mathbb{P}^1, \mathbb{P}^r) \subset \mathbb{A}^{d(r+1)}$ as a basepoint free collection of polynomials, so that we may calcuate the derivatives via linear projection maps. Up to curve automorphisms, the factorisation strata are closed subschemes in $\prod_{i=1}^k \mathrm{Map}^*_{d_i}(\mathbb{P}^1, \mathbb{P}^r)$ consisting of maps with linearly dependent derivatives along the distinguished points. We approcah its topology in two directions:

- Given a tangent vector on the distinguished point $p \in \mathbb{P}^r$, say $v \in T_p\mathbb{P}^r$, we determine the space of pointed maps with derivative v in Section 4.1.1. Analogous to the results in [FW19] concering $\mathrm{Map}^*_d(\mathbb{P}^1, \mathbb{P}^r)$, they are largely independent of the degree of the map, in which case their cohomology is isomorphic to that of $\mathbb{A}^r \setminus \mathrm{pt}$.
- We describe the space of admissible tangent vectors: they are tuples of tangent vectors in $T_p\mathbb{P}^r$ satisfying suitable linear dependencies.

The base and fibre directions combine to imply in Corollary 4.17 that the pure weight cohomology of the space of parametrised maps with factorisation property is given only by $H^0 \cong \mathbb{Q}$. Passing to quotients, this imply that the only pure weight cohomology classes contributed by the factorisation property are the ψ-classes along bivalent or univalent vertices, which are tautological classes. In other words, imposing the closed condition of factorisation property does not lead to new cohomology classes in $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$.

1.2. **Related work.** The strategy of understanding weight filtrations within a stratified space is far from being new. The ideas can be traced back to the earlier work of Arbarello–Cornalba [AC98] on $\overline{\mathcal{M}}_{g,n}$ and have been utilised to great effect by Petersen [Pet14] on $\overline{\mathcal{M}}_{1,n}$ with its dual graph stratification, which inspires the present work. Unlike the moduli of stable curves, the strata in the mapping spaces are in general nontrivial fibrations instead of products. Thus, the application of the Leray spectral sequence together with its weights structure [Ara05] is essential, which has been pursued by Petersen in [Pet14] to $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$. The new ingredient featured in the present work is the combination between weight filtrations, Leray spectral sequence and the explicit genus zero geometry of rational functions [Seg79, CCMM91, FW19], in particular the explicit description of the pointed maps $\mathrm{Map}^*_d(\mathbb{P}^1, \mathbb{P}^r)$ in [FW19].

As mentioned earlier, this work fits into the investigation on the topology and combinatorics of genus one mapping spaces set out in [KS24a, KS24b]. On the other hand, the technique in this work is complementary to previous work of Fontanari [Fon07] who probed the cohomology of the Vakil–Zinger moduli space via torus localization; the interaction between the Białynicki-Birula cells and the tautological cycles seems to be an interesting general question.

Reduced genus one Gromov–Witten invariants [Zin08, CM18] are integrals on $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$. Thus, the presentation of cohomology in this work should provide a theoretical means to calculate the invariants and relate them to the graphical stratification of the moduli space.

1.3. **Future directions.** The combination of stratification and weight filtrations is a general tool applicable to a wide range of moduli spaces. In particular, ongoing work of the author [Son25] returns to the generators and relations of $H^*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))$ using the finer stratification. The results obtained from this technique will be closer in spirit to the motivic recursion formula given by Getzler–Pandharipande [GP06] than to the ones given by Mustata–Mustata [MM06, MM07, MM08] using wall-crossing techniques; comparing the two seems non-obvious and is worth pursuing.

Regarding the interior $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$, the Betti numbers as well as the full weight filtration remain unknown: this work concerns the pure and off-by-one pieces, and the top weight cohomology is known to vanish from [KS24a] using boundary complex techniques. We note that the geometric Batryev–Manin conjecture gives predictions on the finite field point counts of the mapping space, which sheds light on the full mixed Hodge structure. It also remains to be seen whether $H^\star(\mathcal{M}_{1,n}(\mathbb{P}^r, d))$ is *independent* of d for suitably large d: this is known to hold in genus zero [Pan96] and is an unstable analogue of homological stability questions on the topology of mapping spaces [CCMM91, Ban22, Aum24].

While the present results do not yield closed formulas for Betti numbers, ongoing work with Siddarth Kannan aims to calculate the S_n-equivariant Grothendieck ring class of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, which specialises to S_n-equivariant Serre characteristics and recovers the Betti numbers. This will involve the recursive combinatorics of rational tails as explored in previous calculations on $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ [KS24b] and the explicit geometric description of the factorisation condition.

We also note that the work of Battistella–Carocci [BC23] constructs a normal crossings compactification of $\mathcal{M}_{2,n}(\mathbb{P}^r, d)$ that extends the perspective of [RSPW19] with the theory of genus two Gorenstein singularities and admissible covers. The techniques employed in this work, combined with Petersen's previous work on the cohomology of $\mathcal{M}_{2,n}$ [Pet15, Pet16], should lead to a description of the cohomology of their construction.

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Convention. All (co)homology groups are taken with $\mathbb{Q}$-local systems and carry mixed Hodge structures unless stated otherwise. 'Smooth' means smooth as a Deligne–Mumford stack, of which the coarse moduli space may acquire singularities. For M a smooth DM stack of dimension m, we use $[M] \in H_c^{2m}(M)$ to denote its fundamental class.

2. General tools on weight filtration

In this expository section, we review technical results on weight filtration and spectral sequences that are used throughout later parts of the work. The key reference is the discussion in Petersen [Pet14].

2.1. **Pure weight cohomology.** The pure weight cohomology of the strata forms generators of the cohomology. We define them more carefully and recall their basic functoriality properties.

Definition 2.1. Let X be a DM stack. Its *pure weight cohomology* in degree k is the associated graded piece $\mathrm{gr}_k^W H^k(X) = W_k H^k(X)/W_{k-1}H^k(X)$. We make similar definitions for compactly supported cohomology and Borel–Moore homology as $\mathrm{gr}_k^W H_c^k(X)$ and $\mathrm{gr}_{-k}^W H_k^{\mathrm{BM}}(X)$, which are dual by the universal coefficient theorem.

Remark 2.2. When X is proper, $\mathrm{gr}_k^W H^k(X) = H^k(X)/W_{k-1}H^k(X)$; when X is smooth of dimension n, $\mathrm{gr}_k^W H^k(X) = W_k H^k(X) \subset H^k(X)$. Under Poincaré duality, $(W_k H^k(X))^\vee \cong \mathrm{gr}_{2n-k}^W H_c^{2n-k}(X)$.

Lemma 2.3. [Opr06b, §1.2] Let U be an irreducible smooth DM stack, and let V be a smooth DM stack such that $U \subset V$ is open. Then the pullback map $W_\star H^\star(V) \to W_\star H^\star(U)$ is surjective.

Lemma 2.4. (1) [Del74, Proposition 8.2.5] Let V be a proper variety, and let $\widetilde{V}$ be a smooth projective variety with a proper, surjective map $p : \widetilde{V} \to V$. Then the image

$$H^\star(V) \xrightarrow{p^*} H^\star(\widetilde{V})$$

is isomorphic to the pure weight quotient $\mathrm{gr}_\star^W H^\star(V)$.

(2) Let V be a variety, and let $\widetilde{V}$ be a smooth variety with a proper, surjective map $p : \widetilde{V} \to V$. Then the pullback $p^* : \mathrm{gr}_\star^W H_c^\star(V) \to \mathrm{gr}_\star^W H_c^\star(\widetilde{V})$ is injective.

Corollary 2.5. Let $U \subset V$ be an open immersion, and let $\widetilde{B}$ be a smooth variety with a proper, surjective map $\widetilde{B} \to V \setminus U$, so that the composition $\pi : \widetilde{B} \to V \setminus U \to V$ is proper. Then $\mathrm{gr}_k^W H_c^k(U) = \ker(\mathrm{gr}_k^W H_c^k(V) \xrightarrow{\pi^*} \mathrm{gr}_k^W H_c^k(\widetilde{B}))$.

Proof. Consider the following long exact sequence

$$\cdots \to H_c^{k-1}(V \setminus U) \to H_c^k(U) \to H_c^k(V) \to H_c^k(V \setminus U) \to \cdots$$

Maps in the sequence are compatible with their mixed Hodge structures. As $\mathrm{gr}_k^W(-)$ is an exact functor, we have a long exact sequence

$$\cdots \to \mathrm{gr}_k^W H_c^{k-1}(V \setminus U) \to \mathrm{gr}_k^W H_c^k(U) \to \mathrm{gr}_k^W H_c^k(V) \to \mathrm{gr}_k^W H_c^k(V \setminus U) \to \cdots$$

Because $H_c^{k-1}(V \setminus U)$ has weights $\leq (k-1)$, $\mathrm{gr}_k^W H_c^{k-1}(V \setminus U) = 0$, so

$$\mathrm{gr}_k^W H_c^k(U) = \ker(\mathrm{gr}_k^W H_c^k(V) \to \mathrm{gr}_k^W H_c^k(V \setminus U)).$$

The claim follows from composing this with the injective map $\mathrm{gr}_k^W H_c^k(V \setminus U) \to \mathrm{gr}_k^W H_c^k(\widetilde{B})$. □

2.2. **Leray spectral sequence.** Let $f : X \to Y$ be a morphism of algebraic varieties. The Leray spectral sequence for the singular cohomology of X is given by

$$E_2^{p,q} = H^p(Y, \mathrm{R}^q f_\star \mathbb{Q}_X) \Rightarrow H^{p+q}(X).$$

The higher direct image $\mathrm{R}^q f_\star \mathbb{Q}_X$ is a priori a locally constant sheaf on X with stalk over $y \in Y$ isomorphic to $H^\star(X_y)$. It admits a canonical enrichment to a mixed Hodge module [Sai89] that is compatible with the Hodge structures on $H^\star(X)$ and $H^\star(X_y)$.

Theorem 2.6. [Sai89, dCM10] For a morphism of algebraic varieties $f : X \to Y$, the second page of the Leray spectral sequence carries mixed Hodge structures such that the differentials are morphisms of mixed Hodge structures, compatible with that on the limit $H^\star(X)$.

Remark 2.7. The case when f is projective has been proven by Arapura [Ara05]; the approach, as well as the one in [dCM10], is independent of mthe formalism of mixed Hodge modules. See also Ivorra–Morel [IM24] for a modern treatment based on perverse motives.

Using proper direct image functors on the maps $X \xrightarrow{f} Y \to \mathrm{pt}$ in place of direct image, we have a Leray spectral sequence for the compactly supported cohomology of X given by

$$E_2^{p,q} = H_c^p(Y, \mathrm{R}^q f_! \mathbb{Q}_X) \Rightarrow H_c^{p+q}(X),$$

where $\mathrm{R}^q f_! \mathbb{Q}_X$ are mixed Hodge modules compatible with the isomorphisms $(\mathrm{R}^q f_! \mathbb{Q}_X)_y \cong H_c^q(X_y)$.

We are interested in relating the pure weight cohomology $W_\star H^\star(X)$ with the Leray spectral sequence of $f : X \to Y$. The following is a direct consequence of the compatibility between Leray spectral sequence and mixed Hodge structure.

Corollary 2.8. In the Leray spectral sequences for $f : X \to Y$ computing cohomology (resp. compactly supported cohomology) of X, only $\mathrm{gr}^W_{p+q} E_2^{p,q}$ survives to the associated graded piece $\mathrm{gr}^W_{p+q} H^{p+q}(X)$ (resp. $\mathrm{gr}^W_{p+q} H_c^{p+q}(X)$).

Recall that as $R^q f_* \mathbb{Q}_X$ and $R^q f_! \mathbb{Q}_X$ are mixed Hodge modules on Y, their associated graded pieces $\mathrm{gr}^W_k R^q f_* \mathbb{Q}_X$ and $\mathrm{gr}^W_k R^q f_! \mathbb{Q}_X$ are mixed Hodge modules which are pure of weight k.

We specialise to the situation of an étale locally trivial $f : X \to Y$ with fibres homeomorphic to F. In this case, $R^i f_* \mathbb{Q}_X$ resp. $R^i f_! \mathbb{Q}_X$ is a variation of mixed Hodge structures with stalks isomorphic (as vector spaces) to $H^i(F)$ resp. $H_c^i(F)$.

Lemma 2.9. With the same notation as earlier, suppose that $\mathrm{gr}^W_q R^q f_* \mathbb{Q}_X$ has trivial monodromy, then the E_2 page of the Leray spectral sequence for cohomology is

$$\mathrm{gr}^W_{p+q} E_2^{p,q} \cong \mathrm{gr}^W_p H^p(Y) \otimes \mathrm{gr}^W_q H^q(F).$$

Similarly, if $\mathrm{gr}^W_q R^q f_! \mathbb{Q}_X$ has trivial monodromy, then the E_2 page of the Leray spectral sequence computing the compactly supported cohomology is given by

$$\mathrm{gr}^W_{p+q} E_2^{p,q} \cong \mathrm{gr}^W_p H_c^p(Y) \otimes \mathrm{gr}^W_q H_c^q(F).$$

Remark 2.10. Combined with the corollary above, the pure weight cohomology of a suitable fibration comes from that of the base and of the fibre. This is well-known when the fibration is smooth and proper, in which case the Leray spectral sequence degenerates.

2.3. **Stratification.**

We consider a stratified DM stack

$$\varnothing = T_{-1} \subsetneq T_0 \subsetneq \cdots \subsetneq T_{k-1} \subsetneq X = T_k,$$

where each T_i is closed in X. Let $T_p^\circ := T_p \setminus T_{p-1}$ be the locally closed strata. Associated to the stratification is a spectral sequence

$$E_1^{p,q} = H_c^{p+q}(T_p^\circ) \Longrightarrow H_c^{p+q}(X).$$

Arapura [Ara05] proves that the differentials in the spectral sequence are morphisms of mixed Hodge structures. This fact imposes strong constraints on the behaviour of the spectral sequence: in nice cases, they lead to a description of generators and relations of the cohomology.

Lemma 2.11. [Pet14] For each k, the spectral sequence exhibits an inclusion

$$\mathrm{gr}_k^W H_c^k(X) \hookrightarrow \bigoplus_p \mathrm{gr}_k^W H_c^k(T_p^\circ).$$

The cokernel of the inclusion is isomorphic to a direct sum of subquotients of $\bigoplus_r \mathrm{gr}_k^W H_c^{k+1}(T_r^\circ)$, namely the off-by-one weights of the strata.

Dually, the spectral sequence for the Borel–Moore homology, gives a surjection

$$\bigoplus_{p+q=k} \mathrm{gr}_{-k}^W H_k^{\mathrm{BM}}(T_p^\circ) \twoheadrightarrow \mathrm{gr}_{-k}^W H_k^{\mathrm{BM}}(X),$$

and the kernel is isomorphic to a direct sum of subquotients of $\bigoplus_r \mathrm{gr}_{-k}^W H_{k+1}^{\mathrm{BM}}(T_r^\circ)$.

Proof. Because taking the weight graded pieces is exact, we have

$$\begin{aligned}\mathrm{gr}_{p+q}^W E_{r+1}^{p,q} &= \mathrm{gr}_{p+q}^W \frac{\ker\left(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}\right)}{\mathrm{im}\left(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q-r}\right)} \\ &= \frac{\ker\left(d_r : \mathrm{gr}_{p+q}^W E_r^{p,q} \to \mathrm{gr}_{p+q}^W E_r^{p+r,q-r+1}\right)}{\mathrm{im}\left(d_r : \mathrm{gr}_{p+q}^W E_r^{p-r,q+r-1} \to \mathrm{gr}_{p+q}^W E_r^{p,q}\right)}\end{aligned}$$

Since $E_r^{p-r,r-1}$ is a subquotient of $H_c^{p+q-1}(T_{p-r}^\circ)$, it has weight less than or equal to $p+q-1$. Hence $\mathrm{gr}_{p+q}^W E_r^{p-r,q+r-1} = 0$, and so

$$\mathrm{gr}_{p+q}^W E_{r+1}^{p,q} = \ker\left(d_r : \mathrm{gr}_{p+q}^W E_r^{p,q} \to \mathrm{gr}_{p+q}^W E_r^{p+r,q-r+1}\right).$$

Thus, composing the injections $\mathrm{gr}_{p+q}^W E_{r+1}^{p,q} \hookrightarrow \mathrm{gr}_{p+q}^W E_r^{p,q}$, we have $\mathrm{gr}_{p+q}^W E_\infty^{p,q} \hookrightarrow \mathrm{gr}_{p+q}^W E_1^{p,q}$. Taking the direct sum, there is the desired injection $\mathrm{gr}_k^W H_c^k(X) \hookrightarrow \bigoplus_p \mathrm{gr}_k^W H_c^k(T_p^\circ)$.

The cokernels are given by successive quotients

$$\mathrm{gr}_{p+q}^W E_r^{p,q}/\mathrm{gr}_{p+q}^W E_{r+1}^{p,q} = \mathrm{im}(d_r : \mathrm{gr}_{p+q}^W E_r^{p,q} \to \mathrm{gr}_{p+q}^W E_r^{p+r,q-r+1}) \subset \mathrm{gr}_{p+q}^W E_r^{p+r,q-r+1},$$

which is a subquotient of $\bigoplus_r \mathrm{gr}_k^W H_c^{k+1}(T_r^\circ)$.

The analogous claims on Borel–Moore homology groups follow from dualising. ☐

Remark 2.12. As observed by Petersen [Pet14, §1], the differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ maps from subquotients of $\mathrm{gr}_k^W H_c^k(T_p^\circ)$ to subquotients of $\mathrm{gr}_k^W H_c^{k+1}(T_{p+r}^\circ)$ by construction, where $k = p+q$. Therefore, the kernel of d_r specifies the classes on codimension r relations among the strata in T_p°, which are r steps deeper in the stratification.

With suitable assumptions on X and its strata, the above result leads to more concrete statements on ordinary cohomology via Poincaré duality.

Corollary 2.13. Following the notation as earlier, suppose X is smooth, and each T_p° is smooth of codimension n_p. Then there is a surjection

$$\bigoplus_p \mathrm{gr}_{k-2n_p}^W H_c^{k-2n_p}(T_p^\circ) \twoheadrightarrow \mathrm{gr}_k^W H^k(X).$$

The relations among the generators are subquotients of $\bigoplus_r \mathrm{gr}_{k-2n_p+1}^W H^{k-2n_p}(T_r^\circ)$. If T_p is smooth, then the surjection factors as $H_c^{k-2n_p}(T_p^\circ) \to H^{k-2n_p}(T_p) \to H^k(X)$, where the second map is the Gysin homomorphism.

Remark 2.14. The class $i_p(\alpha)$ can be visualised as the cycle $T_p \subset X$ decorated with the class α, which might be supported on one of its connected components. When X is smooth and proper, the above corollary then says that the classes $i_p(\alpha)$ additively span $H^\star(X)$.

Further, in case of X being smooth and proper, the limit $H^\star(X)$ is of pure weight, hence for all $k \neq 0$, $\bigoplus_{p+q=\star} \mathrm{gr}^W_{\star+k} E^{p,q}_\infty = 0$. This implies that, up to taking subquotients, the off by one weight cohomology groups $\bigoplus_p \mathrm{gr}^W_{\star-1} H^\star_c(T^\circ_p)$ are either matched with the off by two weight pieces or supply relations among the pure weight classes. The same reasoning gives in principle a recursion among the higher weight graded pieces of the strata as well.

For this reason, our calculations on the cohomology of the strata concern only the pure and the off-by-one weight graded pieces $\mathrm{gr}^W_\star H^\star(-), \mathrm{gr}^W_{\star+1} H^\star(-)$ and dually the graded pieces on compactly supported cohomology $\mathrm{gr}^W_\star H^\star_c(-), \mathrm{gr}^W_{\star-1} H^\star_c(-)$.

3. Stratification of mapping spaces

3.1. Stable map dual graphs.

We recall the decorated dual graph stratification on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.

Definition 3.1. [KS24a, Definition 1.1] Let $g, n, d \geq 0$.

A (g, n, d)-graph is a tuple $\mathbf{G} = (G, w, \delta, m)$ where:

- G is a connected graph;
- $w : V(\mathbf{G}) \to \mathbb{Z}_{\geq 0}$ is called the *genus function*;
- $\delta : V(\mathbf{G}) \to \mathbb{Z}_{\geq 0}$ is called the *degree function*;
- $m : \{1, \dots, n\} \to V(\mathbf{G})$ is called the *marking function*.

These data are required to satisfy:

(1) $\dim_{\mathbb{Q}} H_1(G, \mathbb{Q}) + \sum_{v \in V(\mathbf{G})} w(v) = g$;
(2) $\sum_{v \in V(\mathbf{G})} \delta(v) = d$.

A (g, n, d)-graph is called *stable* if for all vertices $v \in V(\mathbf{G})$ with $\delta(v) = 0$, we have

$$2w(v) - 2 + \mathrm{val}(v) + |m^{-1}(v)| > 0,$$

where $\mathrm{val}(v)$ means the graph valence of v.

The automorphism group of a (g, n, d)-graph is a graph automorphism that commutes with the genus, degree, and marking decoration functions.

Given a (g, n, d)-graph $\mathbf{G}$ and an edge $e \in E(\mathbf{G})$, we define $\mathbf{G}/e$ as a (g, n, d)-graph whose underlying graph is the quotient graph $\mathbf{G}/e$, and whose decorations are defined as:

- If e is not a loop, the two vertices v_1 and v_2 incident to e are combined into a new vertex v'. Then
$$\delta(v') = \delta(v_1) + \delta(v_2), w(v') = w(v_1) + w(v_2),$$
and $m^{-1}(v') = m^{-1}(v_1) \cup m^{-1}(v_2)$.
- If e is a loop, we increase w on the vertex supporting e by 1. The degree and marking functions are retained.

The resulting (g, n, d)-graph $\mathbf{G}/e$ is denoted as the edge contraction of $\mathbf{G}$ by the edge e.

The set of (g, n, d)-graphs forms a category with morphisms generated by isomorphisms and edge contractions.

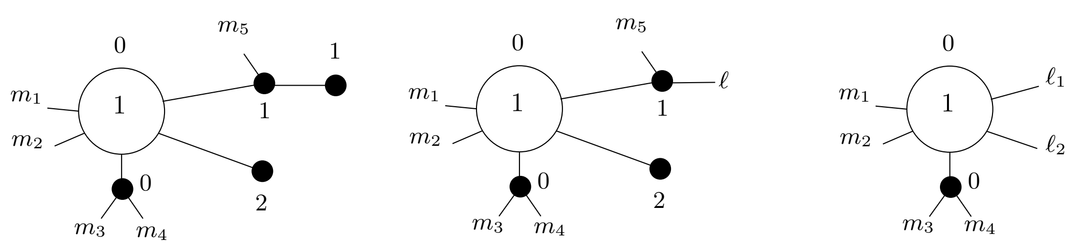
Definition 3.2. Let $\mathbf{G}$ be a $(1, n, d)$-graph. The *core* of $\mathbf{G}$ is the minimal genus one subgraph $\mathbf{G}^c$ with restricted degree and marked point decoration; further, we add an extra leg for each edge connecting $\mathbf{G}^c$ to $\mathbf{G} \smallsetminus \mathbf{G}^c$ and call this set $L'(\mathbf{G}^c)$. There is a poset structure on $(V(\mathbf{G}) \setminus V(\mathbf{G}^c)) \cup L'(\mathbf{G}^c) \cup (L(\mathbf{G}) \setminus L(\mathbf{G}^c))$ where $f_1 \leq f_2$ if the unique path from f_2 to $\mathbf{G}^c$ passes through f_1.

The *contraction core* of $\mathbf{G}$, denoted as $\mathbf{G}^{cc}$ is defined as:

- if the core has positive total degree, then set $\mathbf{G}^{cc} := \mathbf{G}^c$,
- if the core has total degree zero, then let $\mathbf{G}^{cc}$ be the minimal subgraph that contains each vertex in $V(\mathbf{G}) \setminus V(\mathbf{G}^c)$ that has positive degree and is minimal with respect to $\leq$; such a vertex is called a contraction vertex and denoted as $V_c(\mathbf{G}^{cc})$.

We similarly add an extra leg for each edge connecting $\mathbf{G}^{cc}$ to $\mathbf{G} \smallsetminus \mathbf{G}^{cc}$ and call this set $L'(\mathbf{G}^{cc})$; note that $V_c(\mathbf{G}^{cc})$ is in bijection to $L'(\mathbf{G}^{cc})$. In the second case, we define the internal contraction core $\mathbf{G}^{icc}$ as the maximal connected stable dual graph that has total degree zero and contain $\mathbf{G}^c$; we introduce the cut-off legs on $\mathbf{G}^{icc}$ which are in bijection to $L'(\mathbf{G}^{cc})$.

Example 3.3. The following figure shows a $(1, n, d)$-graph $\mathbf{G}$, its contraction core $\mathbf{G}^{cc}$ and core $\mathbf{G}^c$,



$\mathbf{G}$ $\quad$ $\mathbf{G}^{cc}$ $\quad$ $\mathbf{G}^c$

We now coarsen the dual graph stratification and forget the dual graph outside of the contraction core, apart from their total degree and marked point distributions. This will lead to the stratification on $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ and $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ that we work with.

Definition 3.4. Two $(1, n, d)$-graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ are *contraction equivalent* if there is an isomorphism $\phi : \mathbf{G}_1^{cc} \xrightarrow{\simeq} \mathbf{G}_2^{cc}$ as decorated dual graphs such that for each $\ell \in L'(\mathbf{G}^{cc})$,

$$\sum_{w \in V(\mathbf{G}_1): w \geq \ell} \delta_{\mathbf{G}_1}(w) = \sum_{w' \in V(\mathbf{G}_2): w' \geq \phi(\ell)} \delta_{\mathbf{G}_2}(w'),$$

$$m_{\mathbf{G}_1}^{-1}(\{w \in V(\mathbf{G}_1) \mid w \geq \ell\}) = m_{\mathbf{G}_2}^{-1}(\{w' \in V(\mathbf{G}_2) \mid w' \geq \phi(\ell)\}).$$

Equivalence classes under the equivalence relation are called contraction classes. The union of $(1, n, d)$-graph strata in the same contraction class $[\mathbf{G}]$ is called a contraction stratum and denoted as $\mathcal{M}_{[\mathbf{G}]}$.

Each contraction class $[\mathbf{G}]$ has a pair of well-defined integer partitions $(\boldsymbol{\delta}_{L'([\mathbf{G}])}, \boldsymbol{m}_{L'([\mathbf{G}])})$ indexed by $L'(\mathbf{G})$ recording the degrees

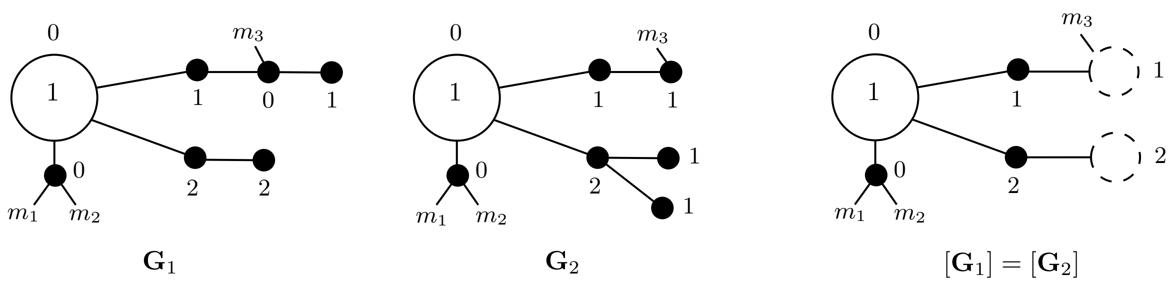
$$L'(\mathbf{G}) \ni \ell \mapsto \boldsymbol{\delta}_\ell := \sum_{w \in V(\mathbf{G}): w \geq \ell} \delta_{\mathbf{G}}(w)$$

and number of marked points

$$\ell \mapsto \boldsymbol{m}_\ell := \# m_{\mathbf{G}}^{-1}(\{w \in V(\mathbf{G}) \mid w \geq \ell\})$$

respectively.

Example 3.5. The figures below show two $(1,3,6)$-graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ that are contraction equivalent. It is convenient to illustrate the contraction class $[\mathbf{G}_1] = [\mathbf{G}_2]$ as their contraction cores together with dotted circles indicating the sum of degrees and marked points associated to the legs on the contraction cores.



Definition 3.6. Each class $[\mathbf{G}]$ has a unique element $\mathbf{G}^{\min}_{[\mathbf{G}]}$ that is minimal under edge contractions and characterised by $\mathbf{G}^{\min}_{[\mathbf{G}]} \setminus \mathbf{G}^{cc}$ consisting of vertices in bijection to $L'(\mathbf{G})$, each having degree and marked point distribution specified by $(\boldsymbol{\delta}_{L'([\mathbf{G}])}, \boldsymbol{m}_{L'([\mathbf{G}])})$.

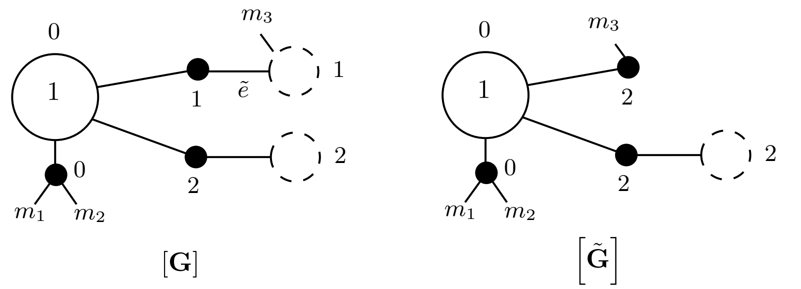
The automorphism group of a contraction class $[\mathbf{G}]$, denoted as $\mathrm{Aut}([\mathbf{G}])$, is the automorphism group of $\mathbf{G}^{\min}_{[\mathbf{G}]}$ as a $(1,n,d)$-graph.

Remark 3.7. The automorphism groups of contraction classes permute unmarked rational tails of the same degrees [KS24b, Lemma 3.6]. To be more explicit, let $L_0'([\mathbf{G}]) \subset L([\mathbf{G}])$ be the set of legs ℓ such that $|\boldsymbol{m}(\ell)| = 0$. The set $L_0'([\mathbf{G}]) \subset L([\mathbf{G}])$ is partitioned by $\boldsymbol{\delta}|_{L_0'([\mathbf{G}])}$, and the automorphism group of a contraction class $[\mathbf{G}]$, denoted as $\mathrm{Aut}([\mathbf{G}])$, is the Young subgroup of the symmetric group $S_{L'(\mathbf{G})}$ associated to the partition $\boldsymbol{\delta}|_{L_0'([\mathbf{G}])}$.

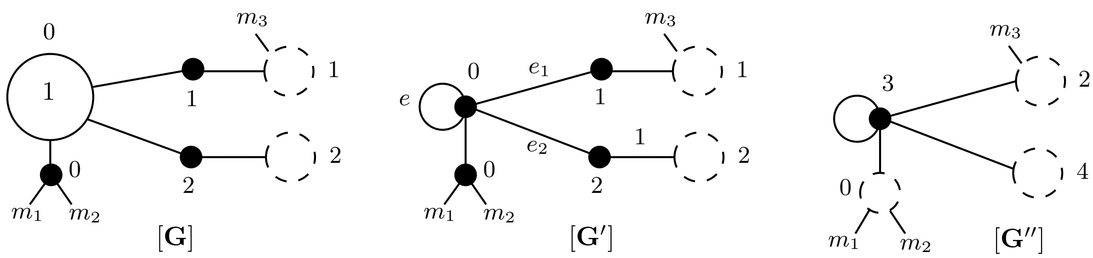
Definition 3.8. Given a contraction class of $(1,n,d)$-graphs, let $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]} \subset \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ be the locally closed subscheme of stable maps with their $(1,n,d)$-dual graphs in the class $[\mathbf{G}]$. It is a union of standard $(1,n,d)$ graph strata in $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, which we may denote as $\mathcal{M}^{\mathrm{st}}_{\mathbf{G}}$.

The poset structure on contraction classes of $(1,n,d)$-graphs is defined as the quotient of the poset of all $(1,n,d)$-graphs by the equivalence relation.

Example 3.9. We illustrate the edge contractions among the contraction classes of $(1,n,d)$-graphs in the following two examples. In the figure shown below, the class $[\tilde{\mathbf{G}}]$ is obtained by contracting the edge $\tilde{e}$ of $[\mathbf{G}]$, in which the total degree and marked points past $\tilde{e}$ are absorbed into the vertex on the contraction radius.



In the following figure, $[\mathbf{G}]$ is obtained from $[\mathbf{G}']$ by contracting the core edge e, and $[\mathbf{G}'']$ is obtained from $[\mathbf{G}']$ by contracting the edges e_1, e_2 which both connects the core to the vertices in the contraction core. Notice that with e_1, e_2 contracted, the core degree becomes positive, so the contraction core of $[\mathbf{G}'']$ agrees with its own core.



Remark 3.10. [CCUW20, §3.4] explains that the cone stack $\mathcal{M}_{g,n}^{\mathrm{trop}}$ of tropical curves can be recovered by the functor from a suitable category of graphs $J_{g,n}$ to the category FI of finite sets and injections by $G \mapsto E(G)$. In a similar fashion, one may consider the map from all $(1, n, d)$-graphs to the poset recording the $(1, n', d')$-type of the core and the total degrees of rational tails. This map records the coarsening of the stratification as desired. The same comment applies to the centrally aligned $(1, n, d)$-graphs that we introduce next.

3.2. **Centrally aligned dual graphs.** In this subsection, we define the centrally aligned dual graphs, which are $(1, n, d)$-graphs with a partially defined ordering on their vertices. The locally closed strata of the normal crossings boundary $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \setminus \mathcal{M}_{1,n}(\mathbb{P}^r, d)$ are labeled by centrally aligned dual graphs.

The definitions below follow [RSPW19] and are adapted from the exposition in [KS24a], which primarily focuses on the finer stratification of radially aligned $(1, n, d)$-graphs.

Definition 3.11. A centrally aligned $(1, n, d)$-graph is:

- a $(1, n, d)$-graph where the core has positive total degree, or
- a $(1, n, d)$-graph where the core has total degree zero, a proper subset $N(\mathbf{G}, \rho) \subsetneq V_c(\mathbf{G}^{cc})$, and a surjective map of posets $\rho : V(\mathbf{G}^{cc}) \setminus (V(\mathbf{G}^c) \cup N(\mathbf{G}, \rho)) \to \{1, \cdots, |\rho|\}$. We formally extend the map to $\rho : V(\mathbf{G}) \to \{0, 1, \cdots, |\rho|\}$ by sending $(V(\mathbf{G}^c) \cup N(\mathbf{G}, \rho))$ to 0.

The internal contraction core $\mathbf{G}^{icc}$ is defined as for non-aligned $(1, n, d)$-graphs, and let ρ^{icc} be the restriction of ρ to $\mathbf{G}^{icc}$. The automorphism group of a centrally aligned $(1, n, d)$-graph consists of automorphisms of the underlying $(1, n, d)$-graph that commutes with the map ρ.

An isomorphism of centrally aligned $(1, n, d)$-graphs is an isomorphism of the underlying $(1, n, d)$-graphs that commutes with the alignment functions.

Let $(\mathbf{G}, \rho)$ be centrally aligned $(1, n, d)$-graph. Given an integer $i \in \{1, \ldots, |\rho|\}$, we define the *radial merge* of $(\mathbf{G}, \rho)$ along i, denoted as $(\mathbf{G}_{\setminus i}, \rho_{\setminus i})$ as follows:

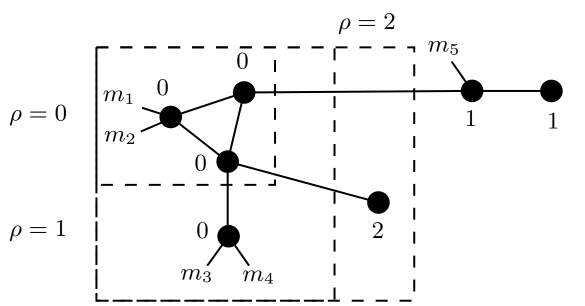
(1) post-compose ρ with the surjection $\{0, \ldots, k\} \to \{0, \ldots, k-1\}$ which decreases all $j \geq i$ by 1;
(2) whenever $v, w \in V(\mathbf{G})$ with $v \in f^{-1}(i-1)$ and $w \in f^{-1}(i)$, such that there is an edge e between v and w, perform the edge contraction of e as for $(1, n, d)$-graphs.

Contraction equivalence of centrally aligned $(1, n, d)$-graphs is defined in the same way as for $(1, n, d)$-graphs. We use $[\mathbf{G}, \rho]$ to denote the contraction class of $(\mathbf{G}, \rho)$. Similar to the case of $(1, n, d)$-graphs, each class $[\mathbf{G}, \rho]$ has a unique minimal element $(\mathbf{G}^{\min}_{[\mathbf{G},\rho]}, \rho^{\min})$, and $\mathrm{Aut}([\mathbf{G}, \rho])$ is defined as the automorphism group of $(\mathbf{G}^{\min}_{[\mathbf{G},\rho]}, \rho^{\min})$.

Remark 3.12. The complement $C(\mathbf{G}, \rho) := V_c(\mathbf{G}^{cc}) \setminus N(\mathbf{G}, \rho)$ is the subset of the contraction vertices that are specified as the ones closest to the core.

Remark 3.13. The central alignment defined above is the same as [RSPW19, §4.6] with δ equal to the contraction radius of the $(1, n, d)$-dual graph. Note that this is distinct from a radial alignment defined in [RSPW19, §3.3], which is more refined than a central alignment.

Example 3.14. The figure below shows a centrally aligned $(1, n, d)$-graph with $n = 5$ and $d = 4$. The core has total degree zero, and the only the contraction radius vertices are assigned the alignment function ρ.



$(\mathbf{G},\ \rho)$

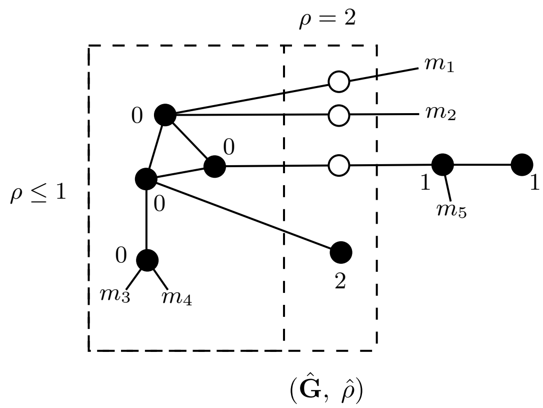
Definition 3.15. [RSPW19, §4], [KS24a, Definition 1.10] Let $[\mathbf{G}, \rho]$ be a (contraction class of) centrally aligned $(1, n, d)$-graphs with partitions $(\boldsymbol{\delta}, \boldsymbol{m})$. The *contraction radius* of $[\mathbf{G}, \rho]$ is defined as $\mathrm{rad}(\mathbf{G}, \rho) := \min\{m \in \mathbb{N} \mid \sum_{v \in \rho^{-1}(m)} \delta(v) > 0\}$, and the *contraction degree* is $d_{\min}(\mathbf{G}, \rho) = \sum_{v \in \rho^{-1}(\mathbf{G},\rho)} \delta(v)$.

Definition 3.16. [KS24a, Definition 1.7] The subdivision at the contraction radius is the prestable centrally aligned $(1, n, d)$-graph $[\hat{\mathbf{G}}, \hat{\rho}]$ obtained from $[\mathbf{G}, \rho]$ by subdividing each edge e directed from vertices v_1 to v_2 with $\rho(v_1) < \mathrm{rad}(\mathbf{G}, \rho)$ and $\rho(v_2) > \mathrm{rad}(\mathbf{G}, \rho)$ and subdividing each half-edge directed from a vertex v with $\rho(v) < \mathrm{rad}(\mathbf{G}, \rho)$. The newly added bivalent vertices are assigned degree zero and are not stable; $\hat{\rho}$ is the unique extension of ρ that assigns $\mathrm{rad}(\mathbf{G}, \rho)$ to each such bivalent vertex.

Similar to the contraction cores of non-centrally aligned $(1, n, d)$-graphs, we define the contraction radius core $\mathbf{G}^{rc}$, the subset of contraction vertices $V_{rc}(\mathbf{G}^{rc})$, and the set of legs $L'(\mathbf{G}^{rc})$ that connects the contraction radius core to the rest of the graph.

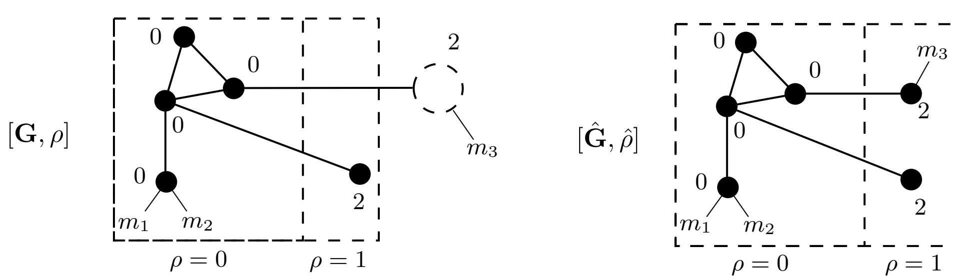
To the contraction radius core subdivision $[\hat{\mathbf{G}}, \hat{\rho}]$, the same constructions are applied with the bivalent vertices and their associated legs added to the core. They are denoted as $\hat{\mathbf{G}}^{rc}$, $\hat{V}_{rc}(\hat{\mathbf{G}}^{rc})$, and $\hat{L}'(\hat{\mathbf{G}}^{rc})$ respectively.

Example 3.17. The subdivision at the contraction radius of the example above is shown as follows. The contraction radius is 2. Bivalent, degree zero vertices are indicated as hollow circles.



Definition 3.18. The set of centrally aligned $(1, n, d)$-graphs forms a category where the morphisms are generated by isomorphisms, edge contractions among edges on $\rho^{-1}(0)$, radial merges within the contraction radius, and edge contractions along the contraction core subdivision outside of the contraction radius. In particular, we disallow certain edge contractions within the contraction radius. These morphisms pass to define a category on the set of contraction equivalence classes of centrally aligned $(1, n, d)$-graphs.

Example 3.19. The following figures are a contraction equivalence class $[\mathbf{G}, \rho]$ and $[\hat{\mathbf{G}}, \hat{\rho}]$, which results from contracting an edge outside of the contraction core. To go from $[\mathbf{G}, \rho]$ to $[\hat{\mathbf{G}}, \hat{\rho}]$, we first perform the contraction core subdivision and then contract the dotted circle along the subdivided edge.



After the combinatorial constructions, we describe the strata $\widetilde{\mathcal{M}}_{(\mathbf{G},\rho)}$ specified by centrally aligned dual graphs. They are modified from the stable maps strata $\mathcal{M}_{\mathbf{G}}$ by taking a torus fibre bundle over $\mathcal{M}_{\mathbf{G}}$ specified by combinatorial data from ρ and then imposing a closed condition known as the factorisation property [RSPW19, Definition 4.1]. The condition is defined in loc. lit. in terms of elliptic singularities and presented in later work [BNR21, §2.4], [KS24a, Lemma 3.10] as a linear dependency condition on certain derivatives of the stable maps which we now explain.

Definition 3.20. Let $[\mathbf{G}, \rho]$ be as above. Consider the product of pointed mapping spaces

$$\prod_{\ell \in L'([\mathbf{G},\rho])} \overline{\mathcal{M}}^{*}_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r, \boldsymbol{\delta}_\ell)$$

where $\ell \in L'$ specifies the frozen marked point in each factor. Let

$$\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]} \to \prod_{\ell \in L'([\mathbf{G},\rho])} \overline{\mathcal{M}}^{*}_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r, \boldsymbol{\delta}_\ell)$$

be the vector bundle recording the direct sum of tangent spaces at the marked points corresponding to $\hat{L}'(\hat{\mathbf{G}}^{rc})$, including the bivalent vertices added after the contraction radius subdivision. Let $\mathbb{P}^\circ\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]} \subset \mathbb{P}\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]}$ be the $\mathbb{G}_m^{|\hat{L}'|-1}$-torsor in the projectivisation of $\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]}$.

The factorisation rational tail locus, denoted $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$, is the closed subscheme in $\mathbb{P}^\circ\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]}$ consisting of a collection of stable maps and a *non-vanishing* linear dependency on the derivatives along the frozen marked points, up to common $\mathbb{G}_m$-multiplication; the derivatives along the bivalent, degree zero vertices are declared to be zero.

The image of $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$ under the map $\mathbb{P}^\circ\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]} \subset \mathbb{P}\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]}$, denoted as $\overline{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$, is the locally closed subscheme in $\prod_{\ell\in L'([\mathbf{G},\rho])}\overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell)$ consisting of stable maps where the derivatives along the frozen marked points have some non-vanishing linear dependency.

Let $(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})_{L'(\mathbf{G}^{cc})}$ be the degree and marked point partition associated to each contraction vertex and let $\mathbb{P}^\circ\mathcal{V}^{\hat{L}',(r)}_{[\mathbf{G},\rho]}$ be the $\mathbb{G}_m^{|\hat{L}'|-1}$-torsor over $\prod_{\ell\in L'([\mathbf{G}^{cc}])}\mathcal{M}^*_{0,\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}^{(r)}_\ell)$ recording a collection of tangent vectors up to rescaling. The factorisation radius mapping locus, denoted $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$, is the closed subscheme in $\mathbb{P}^\circ\mathcal{V}^{\hat{L}',(r)}_{[\mathbf{G},\rho]}$ (here each ℓ specifies each frozen marked point) consisting of a collection of maps from smooth rational curves and a *non-vanishing* linear dependency on the derivatives along the frozen marked points, up to common $\mathbb{G}_m$-multiplication.

We introduce the following variant of $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ that will be useful for later.

Definition 3.21. Define $\overline{\mathcal{M}}^{\mathrm{tar}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ as the closed subscheme in $\prod_{\ell\in L'([\mathbf{G}^{cc}])}\overline{\mathcal{M}}_{0,\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}^{(r)}_\ell)$ such that all the marked points corresponding to each $\ell\in L'([\mathbf{G}^{cc}])$ are mapped to the same point on the target. The evaluation map along the marked points in $\ell\in L'([\mathbf{G}^{cc}])$ defines a Zariski locally trivial fibration $\mathrm{ev}:\overline{\mathcal{M}}^{\mathrm{tar}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}\to\mathbb{P}^r$ with fibres isomorphic to $\prod_{\ell\in L'([\mathbf{G}^{cc}])}\overline{\mathcal{M}}^*_{0,\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}^{(r)}_\ell)$.

Let $\mathcal{V}^{\hat{L}',\mathrm{tar}}_{[\mathbf{G},\rho]}$ be the vector bundle on $\overline{\mathcal{M}}^{\mathrm{tar}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ defined analogously to $\mathcal{V}^{\hat{L}'}_{[\mathbf{G},\rho]}$. Let $\widetilde{\mathcal{M}}^{\mathbf{F},\mathrm{tar}}_{(\boldsymbol{\delta},\boldsymbol{m})}\subset\mathbb{P}^\circ\mathcal{V}^{\hat{L}',\mathrm{tar}}_{[\mathbf{G},\rho]}$ be the closed subscheme parametrising maps and a non-vanishing linear dependency on their derivatives, up to common rescaling. This is well-defined since the frozen marked points are mapped to the same point on the target. The evaluation map along the frozen marked points $\widetilde{\mathcal{M}}^{\mathbf{F},\mathrm{tar}}_{(\boldsymbol{\delta},\boldsymbol{m})}\to\mathbb{P}^r$ is Zariski locally trivial with fibre isomorphic to $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$.

Lemma 3.22. With the same notation as above, let $\widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}$ parametrise the corresponding contraction strata together with an ordering on $L'([\mathbf{G},\rho])$, then there is a fibration

$$\begin{array}{ccc}\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})} & \longrightarrow & \widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}\\ & & \downarrow\\ & & \widetilde{\mathcal{M}}_{\mathbf{G}^{icc}}\end{array}$$

and $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}\cong\widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}/S_{L'([\mathbf{G},\rho])}$. The space $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$ itself is a fibration

$$\begin{array}{ccc}\prod_{\ell\in L'([\mathbf{G}^{cc}])}\overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell-\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell-\boldsymbol{\delta}^{(r)}_\ell) & \longrightarrow & \widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}\\ & & \downarrow\\ & & \mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}\end{array}$$

Remark 3.23. The fibration structures of the strata are essential for reducing their cohomology groups to those of vertices and rational tails. However, as we see above, the fibre product structures only exists on a finite cover of the strata that is specified by an ordering of the rational tails. A clean way to describe the fibre product and finite group quotients is *plethysms* of $\mathbb{S}$-varieties [GP06], which specialise to the more familiar plethysms of symmetric sequences upon taking cohomology.

4. Building blocks

As explained in the introduction, the pure weight cohomology groups of the locally closed strata generate the cohomology of the desingularised mapping space $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ or the pure weight cohomology of the main component $\overline{\mathcal{M}}_{1,n}^{\mathrm{main}}(\mathbb{P}^r, d)$. Relations among the generators are given by the differentials in the stratification spectral sequence that map pure weight cohomology classes to *off-by-one* weight classes $\mathrm{gr}_k^W H_c^{k+1}(-)$; they are dual to subquotients of $\mathrm{gr}_{k'+1}^W H^{k'}(-)$ when the strata are smooth.

In this section, we give descriptions these two weight graded pieces of the cohomology groups of the following spaces.

- $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)}, \boldsymbol{m}^{(r)})}$, collection of maps from a ordered collection of smooth rational curves, together with a non-vanishing linear dependency up to common rescaling from Definition 3.20,
- $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$ parametrising maps from smooth, n-pointed elliptic curves,
- $\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}}$ as maps from cycles of rational curves, where $\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}$ denotes the decorated dual graph of a k-cycle with degree decoration $\mathbf{d} \in \mathbb{Z}^k_{\geq 0}$ and markings $\mathbf{m}$.

These spaces represent either vertices in (aligned) $(1, n, d)$-graphs or the collection of maps on the contraction radii, in which case the linear dependency condition implies that we cannot isolate contributions from individual vertices.

We approach maps from genus one curves - both smooth and nodal - via their linear systems. On the other hand, the first two spaces entirely concern maps from genus zero curves. For these spaces, our strategy is to:

- present them as quotients of *parametrised* mapping spaces by automorphism groups of the curve;
- via evaluation maps to $\mathbb{P}^r$, reduce the pure weight cohomology of parametrised mapping spaces to that of pointed parametrised mapping spaces together with pullback from $\mathbb{P}^r$ along the evaluation;
- utilise explict parametrisations of pointed parametrised mapping spaces, such as the one provided by Farb–Wolfson [FW19].

4.1. **Genus zero maps with factorisation.** The goal of this section is to prove Corollary 4.17, which states that when the factorisation condition is non-trivial, the pure weight cohomology group of $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)}, \boldsymbol{m}^{(r)})}$ is given by $H^0 \cong \mathbb{Q}$. In other words, the presence of linear dependency or factorisation condition does not contribute new cycles to the cohomology of the compact mapping spaces. We base our calculation on the concrete geometry of *parametrised* mapping spaces.

Definition 4.1. Let $\mathrm{Map}_d^*(\mathbb{P}^1, \mathbb{P}^r) = \{f : \mathbb{P}^1 \to \mathbb{P}^r \mid f(\infty) = [1 : 1 : \cdots : 1] =: p \in \mathbb{P}^r\}$ be the space of parametrised degree-d maps from $\mathbb{P}^1$ to $\mathbb{P}^r$ satisfying the pointedness condition. It is identified with the space of basepoint free polynomials as an open subset of $\mathbb{A}^{(r+1)d}$:

$$\{(f_0(t), \cdots, f_r(t)) \in \mathbb{A}^{(r+1)d} \mid f_i(t) \in \mathbb{C}[t] \text{ monic of degree } d, \bigcap_{i=0}^{r} f_i^{-1}(0) = \varnothing\}.$$

Definition 4.2. Let $\boldsymbol{\delta} \in \mathbb{Z}_{\geq 0}^k$ be a degree vector. Define

$$\mathrm{Map}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r) \subset \prod_{i=1}^{k} \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1, \mathbb{P}^r)$$

as the subspace consisting of tuples of parametrised pointed maps such that the images of tangent lines $d^{(i)}_\infty : T_\infty \mathbb{P}^1 \to T_p \mathbb{P}^r$ satisfies some non-vanishing linear dependency: for any choice of basis vectors $v^{(i)} \in T_\infty \mathbb{P}^1$, there exists $(\alpha_1, \dots, \alpha_k) \in (\mathbb{C}^\star)^k$ such that $\sum_{i=1}^k \alpha_i d^{(i)}_\infty(v^{(i)}) = 0$. Let $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r) \to \mathrm{Map}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)$ parametrise in addition the data of the non-vanishing linear dependency itself up to common rescaling.

Remark 4.3. Up to permutation of the entries, the mapping space $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)}, \boldsymbol{m}^{(r)})}$ is related to their corresponding parametrised mapping spaces in a standard way:

- for $\boldsymbol{m}^{(r)}_i = 0, 1$ take quotient of the parametrised mapping space by $\mathrm{Aut}(\mathbb{P}^1, *)$ (a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C})$) or $\mathbb{C}^\star$ respectively, which acts on parametrised maps from the corresponding univalent or bivalent component;
- for $\boldsymbol{m}^{(r)}_i \geq 2$, take (non-canonical) product of the parametrised mapping space with an ordered configuration space $\mathrm{Conf}^{\boldsymbol{m}^{(r)}-2}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$.

We note that the numerics are one off from the standard isomorphism $\mathrm{Map}_d(\mathbb{P}^1, \mathbb{P}^r) \cong \mathcal{M}_{0,3}(\mathbb{P}^r, d)$ because of the presence of a frozen marked point.

4.1.1. *Pointed maps with prescribed tangent lines.* Our first step of understanding $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)$ is to study pointed mapping spaces with a prescribed tangent line along the marked point. We shall see that, via the open embedding $\mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1, \mathbb{P}^r) \subset \mathbb{A}^{\boldsymbol{\delta}_i(r+1)}$, prescribing a tangent line is equivalent to taking an affine linear subspace in $\mathbb{A}^{\boldsymbol{\delta}_i(r+1)}$, where the coordinates on the ambient affine space $\mathbb{A}^{\boldsymbol{\delta}_i(r+1)}$ are the coefficients of polynomials. The open embedding fixes a basis vector $v \in T_\infty \mathbb{P}^1$, and taking the derivative at v gives the map $d_\infty : \prod_{i=1}^k \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1, \mathbb{P}^r) \to \prod_{i=1}^k T_p \mathbb{P}^r$, which is the product of maps $d^{(i)}_\infty : \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1, \mathbb{P}^r) \to T_p \mathbb{P}^r$.

An elementary calculation gives the following:

Lemma 4.4. [KS24a, Lemma 4.1] The map $d^{(i)}_\infty$ factors through

$$\mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1, \mathbb{P}^r) \to \mathbb{A}^{\boldsymbol{\delta}_i(r+1)} \xrightarrow{p_{\boldsymbol{\delta}_i - 1}} \mathbb{A}^{r+1} \to T_p \mathbb{P}^r,$$

where the projection $p_{\boldsymbol{\delta}_i - 1}$ records the $z^{\boldsymbol{\delta}_i - 1}$-coefficient of the $(r+1)$ polynomials, and the second map is the linear map $\mathbb{C}^{r+1} \to \mathbb{C}^{r+1}/\mathbb{C} \cdot (1, 1, \dots, 1) \cong T_p \mathbb{P}^r$.

Definition 4.5. Let $w \in T_p \mathbb{P}^r$, define $\mathrm{Map}^{*,w}_\delta(\mathbb{P}^1, \mathbb{P}^r) := \{f \in \mathrm{Map}^*_\delta(\mathbb{P}^1, \mathbb{P}^r) \mid d_\infty(v) = w\}$. When there is no risk of confusion, we may use $\mathrm{Map}^{*,w}_\delta$ to denote $\mathrm{Map}^{*,w}_\delta(\mathbb{P}^1, \mathbb{P}^r)$ for the sake of brevity.

Remark 4.6. By Lemma 4.4, $\mathrm{Map}^{*,w}_\delta(\mathbb{P}^1, \mathbb{P}^r)$ is the intersection of the open subset $\mathrm{Map}^*_\delta(\mathbb{P}^1, \mathbb{P}^r) \subset \mathbb{A}^{\delta(r+1)}$ and the affine subspace $L^w_\delta := \{(f_0, \cdots, f_r) \mid [(p_{\delta-1}(f_i))^r_{i=0}] = [w] \in \mathbb{C}^{r+1}/\langle(1, 1, \cdots, 1)\rangle \cong T_p \mathbb{P}^r\}$, so it is in particular an open in an affine space isomorphic to $\mathbb{A}^{\delta(r+1)-r}$.

We now determine the cohomology of $\mathrm{Map}^{*,w}_\delta(\mathbb{P}^1, \mathbb{P}^r)$ following the techniques in [FW19], where the authors determine the cohomology of $\mathrm{Map}^*_\delta(\mathbb{P}^1, \mathbb{P}^r)$. Similar to the results in loc. lit., these cohomology groups turn out to be well-behaved and lead to simplification of the ensuing calculations.

Lemma 4.7. When $\delta \geq 2$, for all w,

$$H^i(\mathrm{Map}_\delta^{*,w}) = \begin{cases} \mathbb{Q}(-r), & i = 2r-1 \\ \mathbb{Q}(0), & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

When $\delta = 1$ and $w \neq 0$, $\mathrm{Map}_\delta^{*,w} \cong \mathbb{A}^1$, whereas the space is empty when $\delta = 1$ and $w = 0$. Because $\mathrm{Map}_\delta^{*,w}$ is smooth, there is Poincaré duality

$$H_c^i(\mathrm{Map}_\delta^{*,w}) \cong H^{2((\delta-1)(r+1)+1)-i}(\mathrm{Map}_\delta^{*,w})^\vee$$

compatible with the weight filtration.

Proof. When $w \neq 0$ and $\delta = 1$, the space $\mathrm{Map}_1^{*,w}(\mathbb{P}^1, \mathbb{P}^r)$ is identified with the space of monic polynomials $\{(t+\alpha_0, \dots, t+\alpha_r) \in \mathbb{A}^{r+1} \mid [(\alpha_0, \dots, \alpha_r)] = w \in \mathbb{C}^{r+1}/\langle(1, \dots, 1)\rangle\}$ and is hence identified with $\mathbb{A}^1$.

When $\delta \geq 2$, we use induction on δ. The inductive hypothesis is that

$$H_c^i(\mathrm{Map}_\delta^{*,w}) = \begin{cases} \mathbb{Q}(2r - \delta(r+1)), & i = 2(\delta(r+1) - 2r) + 1 \\ \mathbb{Q}(r - \delta(r+1)), & i = 2(\delta(r+1) - r) \\ 0, & \text{otherwise} \end{cases}$$

and that $H_c^{2\delta(r+1)+1}(\mathrm{Map}_\delta^{*,w})$ is identified with the fundamental class of the basepoint locus under the coboundary map. The base case is split into cases $w = 0$ and $w \neq 0$.

When $w = 0, \delta = 2$, $\mathrm{Map}_2^{*,0}(\mathbb{P}^1, \mathbb{P}^r) = L_0 \setminus \mathbb{A}^2$, where $\mathbb{A}^2$ is identified with the space of monic quadratic polynomials and $\mathbb{A}^2 \hookrightarrow L_2^0$ is the diagonal map. From the excision exact sequence,

$$H_c^i(\mathrm{Map}_2^{*,0}(\mathbb{P}^1, \mathbb{P}^r)) = \begin{cases} \mathbb{Q}(-2), & i = 5 \\ \mathbb{Q}(-(r+2)), & i = 2(r+2) \\ 0, & \text{otherwise} \end{cases}$$

which satisfies the inductive hypothesis.

When $w \neq 0, \delta = 2$, $\mathrm{Map}_2^{*,0} = L_2^w \setminus (\mathbb{A}^1 \times \mathrm{Map}_1^{*,w})$ where $\mathbb{A}^1 \times \mathrm{Map}_1^{*,w} \to L_2^w$ is given by multiplication

$$(z, (t+\alpha_0, \dots, t+\alpha_r)) \mapsto ((t+z)(t+\alpha_0), \dots, (t+z)(t+\alpha_r))$$

and is an isomorphism onto the complement $L_w \setminus \mathrm{Map}_2^{*,0}$. As $\mathrm{Map}_1^{*,w} \cong \mathbb{A}^1$ from earlier, we have that $\mathrm{Map}_2^{*,0} \cong L_2^w \setminus \mathbb{A}^2$ and the same calculation that is done for $w = 0$ applies here.

For the inductive step, we stratify L_δ^w by the number of base points: let $L_{\delta,-k}^w \subset L_\delta^w$ be the closed subspace of tuples of polynomials with at least k basepoints and let $L_{\delta,-k}^{w,\circ} \subset L_w$ be the locally closed subspace consisting of tuples of polynomials with precisely k base points. From [FW19, §3], the multiplication map $\mathbb{A}^k \times \mathrm{Map}_{\delta-k}^{*,w} \to L_{\delta,-k}^{w,\circ}$ given by $(f, (g_0, \dots, g_k)) \mapsto (fg_0, \dots, fg_k)$ is proper and a *homeomorphism*, so the pullback is an isomorphism on compactly supported cohomology. While loc. lit. points out that the map is not an isomorphism in general, the pullback is a morphism (hence isomorphism) of mixed Hodge structures and determines that of $L_{\delta,-k}^{w,\circ}$.

Now we apply the stratification spectral sequence associated to the filtration[3]

$$\cdots \subsetneq L_{\delta,-(k+1)}^w \subsetneq L_{\delta,-k}^w \subsetneq L_{\delta,0}^w = L_\delta^w,$$

[3]When $w = 0$, $L_{\delta,-(d-1)}^w = L_{\delta,-d}^w$, in which case we formally delete the item $L_{\delta,-d}^w$ from the filtration.

which is given by $E_1^{-k,q} = H_c^{-k+q}(L^{w,\circ}_{\delta,-k}) \Rightarrow H_c^{-k+q}(L^w_{\delta,-d})$. Recall that the inductive hypothesis states that for all $\delta' < \delta$, the compactly supported cohomology classes in $\mathrm{Map}^{*,w}_{\delta'}$ are the fundamental class and the image of the fundamental class of $\mathrm{Map}^{*,w}_{\delta'-1}$ under the coboundary map; for $k > 0$, taking the homeomorphisms between $L^{w,\circ}_{\delta,-k}$ and $\mathbb{A}^k \times \mathrm{Map}^{*,w}_{\delta-k}$, a similar description holds for $L^{w,\circ}_{\delta,-k}$. Because the d_1 differentials among the E_1-pages $H_c^{-k+q}(L^{w,\circ}_{\delta,-k}) \to H_c^{-(k-1)+q}(L^{w,\circ}_{\delta,-(k-1)})$ are given by coboundary maps, they are isomorphisms for all $k > 1$. Therefore, $E_2^{-k,q}$ vanishes for all $k > 1$.

It remains to determine d_1 on the terms $E_1^{-1,q}$ and $E_1^{0,q}$. Because the spectral sequence is in the fourth quadrant, $E_2^{-1,q} = E_\infty^{-1,q}$, which is a subquotient of $H_c^{-1+q}(L^w_\delta) \cong H_c^{-1+q}(\mathbb{A}^{\delta(r+1)-r})$, and similarly $E_2^{0,q} = E_\infty^{0,q}$. Recall that $H_c^\star(\mathbb{A}^{\delta(r+1)-r})$ vanishes apart from $\star = 2(\delta(r+1)-r)$, in which case it is $\mathbb{Q}(-(\delta(r+1)-r))$ generated by the fundamental class. Therefore, $E_2^{-1,q} = 0$ for all q, and $E_2^{0,q} = 0$ for $q \neq 2(\delta(r+1)-r)$, and $E_2^{-1,2(\delta(r+1)-r)} = \mathbb{Q}(-(\delta(r+1)-r))$. In other words, $d_1^{-1,q}$ are isomorphisms for all q, and $d_1^{0,q}$ are isomorphisms for all $q \neq 2(\delta(r+1)-r)$, and $E_1^{0,2(\delta(r+1)-r)} = E_2^{0,2(\delta(r+1)-r)} = \mathbb{Q}(-(\delta(r+1)-r))$. We deduce that the compactly supported cohomology of $\mathrm{Map}^{*,w}_\delta$ is given by

$$\begin{cases} \mathbb{Q}(2r-\delta(r+1)), & i = 2(\delta(r+1)-2r)+1 \\ \mathbb{Q}(r-\delta(r+1)), & i = 2(\delta(r+1)-r) \\ 0, & \text{otherwise} \end{cases}$$

and $H_c^{2(\delta(r+1)-2r)+1}(\mathrm{Map}^{*,w}_\delta)$ is identified to $H_c^{2((\delta-1)(r+1)-r+1)}(\mathbb{A}^1 \times \mathrm{Map}^{*,w}_{\delta-1})$ under the coboundary map. This finishes the inductive step. □

Remark 4.8. To summarise, for $\delta \geq 2$, the cohomology of $\mathrm{Map}^{*,w}_\delta(\mathbb{P}^1, \mathbb{P}^r)$ together with its mixed Hodge structure agrees with that of $\mathbb{A}^r \setminus \mathrm{pt}$ and are *independent* of δ and w. This is in analogy to the result of Farb–Wolfson [FW19, Theorem 1.2] that when $\delta > 0$, the cohomology of $\mathrm{Map}^*_\delta(\mathbb{P}^1, \mathbb{P}^r)$ agrees with that of $\mathbb{A}^r \setminus \mathrm{pt}$ and is independent of δ.

4.1.2. *Spaces of linearly dependent vectors.* After describing pointed maps from $\mathbb{P}^1$ to $\mathbb{P}^r$ with prescribed tangent line at the marked point, we turn to the spaces of tangent vectors that may arise from maps with linearly dependent derivatives, parametrised by strata in $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, and determine their pure and off-by-one graded pieces of compactly supported cohomology. In particular, the space of interest to $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ has pure weight compactly supported cohomology generated by its fundamental class and off-by-one graded piece pulled back from $H^1(\mathbb{C}^\star) = \mathbb{Q}(-1)$.

Definition 4.9. Let V be an r-dimensional vector space over $\mathbb{C}$: for our purposes, $V = T_p\mathbb{P}^r$. Denote $\mathsf{D}^*_k \subset V^k$ as the locus of vectors that admit some non-vanishing linear dependency. Let $\pi_\sim : \widetilde{\mathsf{D}}^*_k \to \mathsf{D}^*_k$ be the $(\mathbb{C}^\star)^{k-1}$-torsor parametrising the non-vanishing linear dependency up to common rescaling. Abusing notation, we may replace the subscript k by an indexing set of size k, such as $[k] = \{1, \ldots, k\}$.

Let $\boldsymbol{\delta}$ be a length k degree vector. Define

$$\mathsf{D}^*_{\boldsymbol{\delta}} = \{\mathbf{v} \in \mathsf{D}^*_k \mid v_i \neq 0 \text{ if } \boldsymbol{\delta}_i = 1; v_i = 0 \text{ if } \boldsymbol{\delta}_i = 0\} \subset \mathsf{D}_k,$$

and define $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}} := \pi_\sim^{-1}(\mathsf{D}^*_{\boldsymbol{\delta}}) \subset \widetilde{\mathsf{D}}^*_k$.

Remark 4.10. From the description in Lemma 4.7, the spaces $\mathsf{D}^*_{\boldsymbol{\delta}}$ and $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ describe the realisable tangent vectors in the parametrised mapping space $\mathrm{Map}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)$ and $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)$ introduced above.

Remark 4.11. It is helpful to parametrise D_k (resp. D_k^*) as

$$\{(v_1,\ldots,v_k)\in V^{\oplus k}\mid \exists\alpha_1,\ldots,\alpha_{k-1}\in\mathbb{C}\text{ (resp. }\mathbb{C}^\star) : v_k=\sum_{i=1}^{k-1}\alpha_i v_i\},$$

which implies that both D_k and D_k^* are irreducible. The parametrisation also implies that $\widetilde{\mathsf{D}}_k^*$ is isomorphic to $V^{\oplus k-1}\times(\mathbb{C}^\star)^{k-1}$: this isomorphism distinguishes the last vector. It is helpful to present the cohomology of $\widetilde{\mathsf{D}}_k^*$ as $H^\star((\mathbb{C}^\star)^k/\mathbb{C}^\star\cdot\mathrm{Id})\cong\bigwedge\left(\mathbb{Q}\{\alpha_1,\ldots,\alpha_k\}/\langle\sum_{i=1}^k\alpha_i\rangle\right)$, where each $\alpha_i\in H^1((\mathbb{C}^\star)^k/\mathbb{C}^\star)$ is the generator of the cohomology the corresponding factor in $(\mathbb{C}^\star)^k$.

Definition 4.12. For a multi-degree vector $\boldsymbol{\delta}$, define $I_{\boldsymbol{\delta}}^{(1)}:=\{i\in[k]\mid\boldsymbol{\delta}_i=1\}$ and $I_{\boldsymbol{\delta}}^{(0)}:=\{i\in[k]\mid\boldsymbol{\delta}_i=0\}$. Denote $[k]_{\boldsymbol{\delta}_{\setminus 0}}:=[k]\setminus I_{\boldsymbol{\delta}}^{(0)}$.

Remark 4.13. Let $\boldsymbol{\delta}_{\setminus 0}$ be the multi-degree vector on the set $[k]_{\boldsymbol{\delta}_{\setminus 0}}$, then the projection map of $D_{\boldsymbol{\delta}}$ away from the $I_{\boldsymbol{\delta}}^{(0)}$ coordinates (on which the vectors are required to vanish) gives an isomorphism $\mathsf{D}_{\boldsymbol{\delta}}\xrightarrow{\cong}\mathsf{D}_{\boldsymbol{\delta}_{\setminus 0}}\subset\mathsf{D}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}$. Similarly, the projection map $\widetilde{\mathsf{D}}_{\boldsymbol{\delta}}^*\to\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*$ gives an isomorphism $\widetilde{\mathsf{D}}_{\boldsymbol{\delta}}^*\cong\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*\times(\mathbb{C}^\star)^{I_{\boldsymbol{\delta}}^{(0)}}$.

By definition, $\widetilde{\mathsf{D}}_{\boldsymbol{\delta}}^*$ is an open subspace of $(V\times\mathbb{C}^\star)^{k-1}$ as a complement of a union of linear subspaces. Using excision, we have the following description of the weight graded pieces of $H_c^\star(\widetilde{\mathsf{D}}_{\boldsymbol{\delta}}^*)$.

Lemma 4.14. We use the short hand $N_{\boldsymbol{\delta}_{\setminus 0}}$ to denote $\dim\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*=(r+1)(k-|I_{\boldsymbol{\delta}}^{(0)}|-1)$.

The pure weight compactly supported cohomology of $\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*$ is spanned by its fundamental class $[\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*]\in H_c^{2N_{\boldsymbol{\delta}_{\setminus 0}}}(\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*)$.

The off-by-one weight graded pieces $\mathrm{gr}_{\star+1}^W H_c^\star(\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*)$ are given by:

- images of the fundamental classes of the closed subspaces $\{(\boldsymbol{v},\boldsymbol{\alpha})\in\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}^*\mid\boldsymbol{v}_j=0\}\cong\mathbb{C}^\star\times\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}\setminus\{j\}}^*$ under the coboundary map, which gives
$$\mathrm{gr}_{2(N_{\boldsymbol{\delta}_{\setminus 0}}-r)}^W H_c^{2(N_{\boldsymbol{\delta}_{\setminus 0}}-r)+1}(\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*)\cong\mathbb{Q}(-(N_{\boldsymbol{\delta}_{\setminus 0}}-r))^{\oplus|I_{\boldsymbol{\delta}}^{(1)}|},$$
- the torus factors coming from the ambient space $(V\times\mathbb{C}^\star)^{k-1}$:
$$\mathrm{gr}_{2(N_{\boldsymbol{\delta}_{\setminus 0}}-1)}^W H_c^{2N_{\boldsymbol{\delta}_{\setminus 0}}-1}(\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*)\cong\mathbb{Q}(-(N_{\boldsymbol{\delta}_{\setminus 0}}-1))^{\oplus k-|I_{\boldsymbol{\delta}}^{(0)}|-1}.$$

Proof. We again consider the open embedding $\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*\subset\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}^{(0)}}^*$. The ambient space $\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}^*$ is isomorphic to $(\mathbb{C}^\star\times V)^{|[k]_{\boldsymbol{\delta}_{\setminus 0}}|-1}$, from which we can read off that its pure weight cohomology is the fundamental class, and its off-by-one cohomology is given by $H_c^{2N_{\boldsymbol{\delta}_{\setminus 0}}-1}((\mathbb{C}^\star\times V)^{|[k]_{\boldsymbol{\delta}_{\setminus 0}}|-1})$ as claimed above. On the other hand, the the complement $\partial\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*:=\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}^*\setminus\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*$ receives a proper, surjective map from the disjoint union of closed subspaces

$$\bigsqcup_{j\in I_{\boldsymbol{\delta}}^{(1)}}\{(\boldsymbol{v},\boldsymbol{\alpha})\in\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}^*\mid\boldsymbol{v}_j=0\}.$$

Projection map away from j-th coordinate identifies the subspace (corresponding to j) with $\mathbb{C}^\star\times\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}\setminus\{j\}}^*$, and its only pure weight compactly supported cohomology is given by the fundamental class.

Now consider the excision long exact sequence associated to the pair:

$$\cdots\to H_c^{q-1}(\partial\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*)\to H_c^q(\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*)\to H_c^q(\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}^*)\to H_c^q(\widetilde{\mathsf{D}}_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}^*\setminus\widetilde{\mathsf{D}}_{\boldsymbol{\delta}_{\setminus 0}}^*)\to\cdots$$

Truncating at gr^W_q implies that the pure weight compactly supported cohomology of $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}$ is given by its fundamental class, and truncating at gr^W_{q-1} gives the long exact sequence

$$\cdots \to \mathrm{gr}^W_{q-1} H^{q-1}_c(\partial \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}) \to \mathrm{gr}^W_{q-1} H^q_c(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}) \to \mathrm{gr}^W_{q-1} H^q_c(\widetilde{\mathsf{D}}^*_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}) \to \mathrm{gr}^W_{q-1} H^q_c(\partial \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}) \to \cdots$$

and both terms adjacent to $\mathrm{gr}^W_{k-1} H^q_c(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}})$ have been determined. Further, because $\dim \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}} - \dim \partial \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}} = r > 0$, the proper pullback maps

$$\mathrm{gr}^W_{q-1} H^{q-1}_c(\widetilde{\mathsf{D}}^*_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}) \to \mathrm{gr}^W_{q-1} H^{q-1}_c(\partial \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}),$$

$$\mathrm{gr}^W_{q-1} H^{q}_c(\widetilde{\mathsf{D}}^*_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}) \to \mathrm{gr}^W_{q-1} H^{q}_c(\partial \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}})$$

both vanish for all q. Therefore, we have the short exact sequence

$$0 \to \mathrm{gr}^W_{q-1} H^{q-1}_c(\partial \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}) \to \mathrm{gr}^W_{q-1} H^q_c(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}) \to \mathrm{gr}^W_{q-1} H^q_c(\widetilde{\mathsf{D}}^*_{[k]_{\boldsymbol{\delta}_{\setminus 0}}}) \to 0$$

which determines $\mathrm{gr}^W_{q-1} H^q_c(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}})$ as stated above. □

We approach the topology of the parametrised mapping space $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)$ via the maps d_∞ to $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ respectively. Indeed, the following lemma implies that over a point $((v_1, \ldots, v_k), [\alpha_1, \ldots, \alpha_k]) \in \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$, the fibres are all isomorphic to $\prod_{i=1}^k \mathrm{Map}^{*,v_i}_{\delta}(\mathbb{P}^1, \mathbb{P}^r)$.

Lemma 4.15. The parametrised mapping space $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)$ is the fibre product:

$$\begin{array}{ccc} \widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r) & \longrightarrow & \prod_{i=1}^k \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1, \mathbb{P}^r) \\ \downarrow & & \downarrow \\ \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}} & \longrightarrow & (T_p\mathbb{P}^r)^{\oplus k} \end{array}$$

Lemma 4.16. The pure and off-by-one weight graded pieces of $H^\star_c(\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r))$ satisfy Künneth formula for the fibration $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r) \to \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$, namely

$$\mathrm{gr}^W_q H^q_c\left(\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)\right) \cong \mathrm{gr}^W_q \left(\bigoplus_{s+t=q} H^s_c(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}) \otimes H^t_c(\prod_{i=1}^k \mathrm{Map}^{*,w}_{\boldsymbol{\delta}_i}) \right),$$

$$\mathrm{gr}^W_{q-1} H^q_c\left(\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1, \mathbb{P}^r)\right) \cong \mathrm{gr}^W_{q-1} \left(\bigoplus_{s+t=q} H^s_c(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}) \otimes H^t_c(\prod_{i=1}^k \mathrm{Map}^{*,w}_{\boldsymbol{\delta}_i}) \right).$$

Proof. Consider the proper direct image of the constant sheaf $\mathbb{Q}$ along any of the vertical maps in the above diagram. Because the rightmost vertical map is restriction of a linear map

$$\prod_{i=1}^k \mathbb{A}^{\boldsymbol{\delta}_i(r+1)} \to \prod_{i=1}^k \mathbb{C}^{r+1} \to (T_p\mathbb{P}^r)^{\oplus k}$$

to the open subset $\prod_{i=1}^k \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1, \mathbb{P}^r)$, the locally constant sheaf has trivial monodromy. Pulling back along the fibre diagrams, the same applies to the rest of the vertical maps.

Therefore, the Leray spectral sequences for the three vertical maps at the left are given by tensor product of the compactly supported cohomology groups of the base and the fibre. For convenience, in the remainder of the proof we use B to denote the base and use $M_{\boldsymbol{\delta}}$ to denote $\prod_{i=1}^k \mathrm{Map}^{*,w}_{\boldsymbol{\delta}_i}$.

The differentials in the Leray spectral sequence are compatible with the weight filtration, so the pure weight part $\mathrm{gr}^W_{s+t} E_2^{s,t} = \mathrm{gr}^W_s H^s_c(B) \otimes \mathrm{gr}^W_t H^t_c(M_{\boldsymbol{\delta}})$ is mapped under d_2 to the off-by-one weighted graded part of $E_2^{s+2,t-1}$, which is given by

$$\mathrm{gr}^W_{s+t} E_2^{s+2,t-1} = \left(\mathrm{gr}^W_{s+1} H^{s+2}_c(B) \otimes \mathrm{gr}^W_{t-1} H^{t-1}_c(M_{\boldsymbol{\delta}})\right) \oplus \left(\mathrm{gr}^W_{s+2} H^{s+2}_c(B) \otimes \mathrm{gr}^W_{t-2} H^{t-1}_c(M_{\boldsymbol{\delta}})\right).$$

Similar statements hold for the differentials on higher pages, with cohomology groups of the base and fibres replaced by their subquotients.

From Lemma 4.7, the ordinary cohomology of $M_{\boldsymbol{\delta}}$ is isomorphic as a mixed Hodge structure to some power of $\mathbb{A}^r \setminus 0$. By Künneth formula for $M_{\boldsymbol{\delta}} = \prod_{i=1}^k \mathrm{Map}^{*,w}_{\boldsymbol{\delta}_i}$, the non-vanishing cohomological degrees in the ordinary cohomology of $M_{\boldsymbol{\delta}}$ differ by multiples of $(2r-1)$. Dualizing, the same statement holds for compactly supported cohomology.

The cohomological degrees that can support non-vanishing $\mathrm{gr}^W_\star H^\star_c$ and $\mathrm{gr}^W_{\star-1} H^\star_c$ of $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ have been described in Lemmas 4.14. By inspecting the cohomological degrees and weights, we see that all differentials from $\mathrm{gr}^W_q\left(\bigoplus_{s+t=q} E_2^{s,t}\right)$ and $\mathrm{gr}^W_{q-1}\left(\bigoplus_{s+t=q} E_2^{s,t}\right)$ must vanish. Hence, the Künneth formulas on the pure and off-by-one weight graded pieces hold.

□

Corollary 4.17. Denote $\dim_{M_{\boldsymbol{\delta}}}$ as the dimension of $\prod_{i=1}^k \mathrm{Map}^{*,w_i}_{\boldsymbol{\delta}_i}$ for any choice of $(w_i)_{i=1}^k$ such that the product is non-empty. The pure weight compactly supported cohomology groups are given by

$$\mathrm{gr}^W_\star H^\star_c(\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r)) = \mathbb{Q}\cdot[\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r)],$$

The off-by-one weight graded pieces are given by

$$\mathrm{gr}^W_{\star-1} H^\star_c(\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r)) = \left(\underbrace{\mathrm{gr}^W_{\star-1} H^\star_c(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}} \times (\mathbb{C}^\star)^{|I_{\boldsymbol{\delta}^{(0)}}|-1})}_{\text{Remark 4.13, Lemma 4.14}} \otimes \mathbb{Q}\cdot[\prod_{i=1}^k \mathrm{Map}^{*,w}_{\boldsymbol{\delta}_i}]\right)$$
$$\oplus \left(\mathbb{Q}\cdot[\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}] \otimes \underbrace{\mathbb{Q}(-2(\dim_{M_{\boldsymbol{\delta}}} - r))^{\oplus k - |I^{(0)}_{\boldsymbol{\delta}}| - |I^{(1)}_{\boldsymbol{\delta}}|}}_{\mathrm{gr}^W_{\star-1} H^\star_c(\prod_{i=1}^k \mathrm{Map}^{*,w}_{\boldsymbol{\delta}_i}),\ \text{Lemma 4.7}}\right)$$

Another helpful way to rephrase the Künneth formula in Lemma 4.16 for $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r)$ is via the map $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r) \to \left(\prod_{i=1}^k \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1,\mathbb{P}^r)\right) \times \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ given by the product of the two natural maps. As $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r)$ is smooth, we use Poincaré duality to match the weight graded pieces of compactly supported cohomology with ordinary cohomology.

Corollary 4.18. The pullback map $H^\star\left(\left(\prod_{i=1}^k \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1,\mathbb{P}^r)\right) \times \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}\right) \to H^\star(\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r))$ is surjective on the pure and off-by-one weight graded pieces.

Proof. With Lemma 4.16, it suffices to prove that the closed embedding $\mathrm{Map}^{*,w}_\delta(\mathbb{P}^1,\mathbb{P}^r) \to \mathrm{Map}^*_\delta(\mathbb{P}^1,\mathbb{P}^r)$ induces a surjection on cohomology pullback, which amounts to proving that the map $H^{2r-1}(\mathrm{Map}^*_\delta(\mathbb{P}^1,\mathbb{P}^r)) \to H^{2r-1}(\mathrm{Map}^{*,w}_\delta(\mathbb{P}^1,\mathbb{P}^r))$ does not vanish when the target is non-zero. This is true because when they are non-zero, both classes are dual to the (coboundary image of) fundamental class of the basepoint locus. □

4.1.3. *Quotienting by automorphisms.* After describing the pure and off-by-one weight graded pieces of $H_c^\star(\widetilde{\mathrm{Map}}_{\boldsymbol{\delta}}^{*,\mathsf{F}}(\mathbb{P}^1,\mathbb{P}^r))$, we turn to the moduli spaces $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ that record maps satisfying the factorisation property without fixing the isomorphism class of the domain curves. From Remark 4.3, the parametrised mapping space is the fibre of the forgetful map

$$\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})} \to \prod_{i=1}^{k} \mathcal{M}_{0,\boldsymbol{m}_i}.$$

We determine the weight graded pieces of $H_c^\star(\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})})$ using the Leray spectral sequence associated to the forgetful maps. Along the way, we describe how the automorphism groups $\mathrm{Aut}(\mathbb{P}^1,*)$ and $\mathbb{C}^\star$ act on tuples of polynomials in $\mathrm{Map}^*_\delta(\mathbb{P}^1,\mathbb{P}^r)$.

To begin with, we set the notation for ψ-classes on $\mathsf{M}_{0,1},\mathsf{M}_{0,2}$. These tautological classes are a source of pure weight classes on strata of genus zero stable maps in general, and on the strata $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ and later $\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}}$ as a consequence.

Definition 4.19. Let $\psi\in H^2(\mathsf{M}_{0,1})$ or $\psi\in H^2(\mathsf{M}_{0,2})$ be the standard generator under the isomorphism of stacks $\mathsf{M}_{0,1}\cong B\mathrm{Aut}(\mathbb{P}^1,*)$ and $\mathsf{M}_{0,2}\cong B\mathbb{G}_m$. They agree with the psi classes of the marked points up to a sign, which suffices for our purposes.

Let v be a univalent or bivalent vertex in a $(1,n,d)$-graph, possibly with central alignment. The psi class ψ_v of vertex v is the cohomology class in the corresponding mapping space stratum pulled back from $\psi\in H^2(\mathsf{M}_{0,1})$ or $\psi\in H^2(\mathsf{M}_{0,2})$.

The groups $\mathrm{Aut}(\mathbb{P}^1,*)$ and $\mathbb{C}^\star$ act on $\mathrm{Map}^\star_\delta(\mathbb{P}^1,\mathbb{P}^r)$ and $T_p\mathbb{P}^r$, making $d_\infty:\mathrm{Map}^\star_\delta(\mathbb{P}^1,\mathbb{P}^r)\to T_p\mathbb{P}^r$, where recall that the marked point on $\mathbb{P}^1$ is $\infty=[1:0]$, and the specified point on the target is $p=[1:\cdots:1]\in\mathbb{P}^r$ and $T_p\mathbb{P}^r\cong\mathbb{C}^{r+1}/\mathbb{C}\cdot(1,\cdots,1)$. Recall the identification $\mathrm{Map}^\star_\delta(\mathbb{P}^1,\mathbb{P}^r)\subset\mathbb{A}^{\delta(r+1)}$ as tuples of monic degree δ polynomials $(f_0,\ldots,f_r)$ that does not share a common root; write $f_i=z^\delta+f_{i,\delta-1}z^{\delta-1}+\cdots+f_{i,0}$. Applying change of variables and rearranging, we get the following formulas for the group actions.

Lemma 4.20. $\left[\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}\right]\in\mathrm{Aut}(\mathbb{P}^1,*)$ acts on $\mathrm{Map}^\star_\delta(\mathbb{P}^1,\mathbb{P}^r)$ by

$$f_{i,\delta-k}\mapsto\sum_{\ell=0}^{k}\binom{\delta}{k-\ell}\frac{b^{k-\ell}c^\ell}{a^k}f_{i,\delta-\ell},$$

where we declare $f_{i,d}=1$ for all i. In particular, it acts on $T_p\mathbb{P}^r$ via multiplication by c/a.

Restricting to $\mathbb{C}^\star\subset\mathrm{Aut}(\mathbb{P}^1,*)$, $c/a=t\in\mathbb{C}^\star$ acts on $\mathrm{Map}^\star_\delta(\mathbb{P}^1,\mathbb{P}^r)$ by $f_{i,\delta-k}\mapsto t^kf_{i,\delta-k}$ and on $T_p\mathbb{P}^r$ via multiplication by t.

Lemma 4.21. The pure weight ordinary cohomology of $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ has a basis given by the psi classes at the bivalent or univalent vertices $\{1,\psi^i_j\mid\boldsymbol{m}^{(r)}_j<2, i=1,\ldots,r-1\}$. In particular, the forgetful map to univalent and bivalent vertices

$$\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}\to\prod_{\substack{1\le i\le k\\ m_i<2}}\mathcal{M}_{0,\boldsymbol{m}_i+1}$$

is surjective on pure weight cohomology.

The off-by-one weight graded part of the cohomology is generated as a $W_\star H^\star(\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})})$-module by:

- the basepoint loci of the pointed mapping space for univalent and bivalent vertices, given by $\bigoplus_{j\in[k]_{\boldsymbol{\delta}_{\setminus 0}}:\boldsymbol{m}_j^{(r)}\geq 2,\boldsymbol{\delta}_j>1} H^{2r-1}(\mathrm{Map}^{*,w}_{\boldsymbol{\delta}_j})$,
- pullback from genus zero vertices with valency greater or equal to three, given by

$$\bigoplus_{j\in[k]_{\boldsymbol{\delta}_{\setminus 0}}|\boldsymbol{m}_j^{(r)}\geq 2} H^1(\mathcal{M}_{0,\boldsymbol{m}_j^{(r)}+1}),$$

and classes pulled back from the base $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ (Lemma 4.14), which are

- the fundamental classes of the boundary linear subspaces, given by

$$\mathbb{Q}(-r)^{\oplus\{j\in[k]_{\boldsymbol{\delta}_{\setminus 0}}|\boldsymbol{m}_j^{(r)}\geq 2\}} \subset \mathrm{gr}^W_{2r} H^{2r-1}(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}}),$$

- the torus factors in the isomorphism $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}} \cong \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}} \times (\mathbb{C}^\star)^{|I^{(0)}_{\boldsymbol{\delta}}|-1}$, given by $\mathrm{gr}^W_{\star+1} H^\star((\mathbb{C}^\star)^{|I^{(0)}_{\boldsymbol{\delta}}|-1})$.

Proof. Consider the Leray spectral sequence for ordinary cohomology associated to

$$\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})} \to \prod_{1\leq i\leq k} \mathcal{M}_{0,\boldsymbol{m}_i+1}.$$

For vertices i such that $|\boldsymbol{m}_i| \geq 2$, there are greater or equal to three special points on the corresponding domain curve together with the frozen marked point $\star_i$, so the contribution from the vertex can be rigidified as a parametrised map. This implies that the composition with the projection map

$$\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})} \to \prod_{\substack{1\leq i\leq k\\ |\boldsymbol{m}_i^{(r)}|\geq 2}} \mathcal{M}_{0,\boldsymbol{m}_i+1}$$

satisfies Künneth formula for ordinary cohomology. Therefore, the off-by-one weight graded pieces of the parametrised mapping space associated to the vertex survive to the limit, which are the first three items in the above list of generators - the third item comes from the torus factor $(\mathbb{C}^\star)^{|I^0_{\boldsymbol{\delta}}|-1}$ in the isomorphism $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}} \cong \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}_{\setminus 0}} \times (\mathbb{C}^\star)^{|I^0_{\boldsymbol{\delta}}|-1}$. In the base direction, the off-by-one weight cohomology of $\prod_{\substack{1\leq i\leq k\\ |\boldsymbol{m}_i^{(r)}|\geq 2}} \mathcal{M}_{0,\boldsymbol{m}_i+1}$, which agrees with its first cohomology, survives to the limit.

We now turn to the classes contributed by the univalent and bivalent vertices, which correspond to $|\boldsymbol{m}_i^{(r)}| = 0$ and $|\boldsymbol{m}_i^{(r)}| = 1$ respectively. This amounts to understanding the non-zero differentials in forgetful map in the other direction

$$\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})} \to \prod_{\substack{1\leq i\leq k\\ |\boldsymbol{m}_i^{(r)}|<2}} \mathcal{M}_{0,\boldsymbol{m}_i^{(r)}\cup\star_i} = B(\prod_{\substack{1\leq i\leq k\\ m_i<2}} \mathrm{Aut}(\mathbb{P}^1, \boldsymbol{m}_i^{(r)}\cup\star_i)),$$

where we note that the base is simply connected with cohomology given by $\bigotimes_{\substack{1\leq i\leq k\\ |\boldsymbol{m}_i^{(r)}|<2}} \mathbb{Q}[\psi_i]$.

From the surjectivity in Corollary 4.18 and functoriality of Leray spectral sequence, it suffices to determine the differentials of spectral sequence associated to

$$\left[\left(\prod_{i=1}^k \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1,\mathbb{P}^r)\right) \times \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}} / \prod_{\substack{1\leq i\leq k\\ m_i<2}} \mathrm{Aut}(\mathbb{P}^1, \boldsymbol{m}_i^{(r)}\cup\star_i)\right] \to B\left(\prod_{\substack{1\leq i\leq k\\ m_i<2}} \mathrm{Aut}(\mathbb{P}^1, \boldsymbol{m}_i^{(r)}\cup\star_i)\right).$$

The advantage of considering the group action on the ambient space $\left(\prod_{i=1}^k \mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1,\mathbb{P}^r)\right) \times \widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ is that the action is diagonalised to each product factor: the factors $\mathrm{Aut}(\mathbb{P}^1, \boldsymbol{m}_i^{(r)}\cup\star_i)$ only act non-trivially on

$\mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1,\mathbb{P}^r)$. We can read off that the following differentials are isomorphisms between rank one $\mathbb{Q}$-vector spaces - each of them concerns torus actions on open subspaces of affine spaces:

$$d_2 : H^\star(B\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i))\otimes H^1((\mathbb{C}^\star\times V)^{[k]_{\boldsymbol{\delta}\setminus 0}-1})_i\to H^{\star+2}(B\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i)),$$

where the subscript $H^1((\mathbb{C}^\star\times V)^{[k]_{\boldsymbol{\delta}\setminus 0}-1})_i$ denote the rank one subspace corresponding to the i-th factor of the torus $(\mathbb{C}^\star)^{[k]_{\boldsymbol{\delta}\setminus 0}}/\mathbb{C}^\star$.

$$d_{2r} : H^\star(B\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i))\otimes H^{2r-1}(\mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1,\mathbb{P}^r))\to H^{\star+2r}(B\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i)),$$
$$d_{2r} : H^\star(B\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i))\otimes H^{2r-1}(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}\setminus 0})_{I^{(1)}_{\boldsymbol{\delta}}}\to H^{\star+2r}(B\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i)),$$

and the subscript $H^{2r-1}(\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}\setminus 0})_{I^{(1)}_{\boldsymbol{\delta}}}$ denote the classes that are dual to the (image under the coboundary map of) the fundamental classes of $\bigsqcup_{j\in I^{(1)}_{\boldsymbol{\delta}}}\{(\boldsymbol{v},\boldsymbol{\alpha})\in\widetilde{\mathsf{D}}^*_{[k]_{\boldsymbol{\delta}\setminus 0}}\mid \boldsymbol{v}_j=0\}$.

The description of the first differential comes from standard free action of $\mathbb{C}^\star$ on itself, which can be read off from the Leray spectral sequence associated to the $\mathbb{C}^\star$-torsor given by $\mathrm{pt}\to B\mathbb{C}^\star$. For the second two differentials, we simply observe that if the differentials were non-zero, the cohomology groups of the quotients $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}/\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i)$ and $\mathrm{Map}^*_{\boldsymbol{\delta}_i}(\mathbb{P}^1,\mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^1,\boldsymbol{m}_i^{(r)}\cup\star_i)$ will have non-zero cohomology in unbounded cohomological degrees, which is a contradiction.

As the Leray spectral sequence is multiplicative and the weight graded pieces of $\widetilde{\mathrm{Map}}^{*,\mathsf{F}}_{\boldsymbol{\delta}}(\mathbb{P}^1,\mathbb{P}^r)$ satisfies the Künneth formulas, the cup products of the above non-zero differentials give all the non-vanishing differentials in the spectral sequence. Therefore, the off-by-one weight pieces coming from the domains of the above differentials will not survive to the limit, and the ψ classes of the univalent or bivalent vertices have $\psi^r=0$ as claimed. □

Remark 4.22. We notice that because the classes ψ_v are pulled back from $\mathsf{M}_{0,1}$ or $\mathsf{M}_{0,2}$, they are tautological cycles. In particular, the classes are pure of Hodge type $(1,1)$.

4.2. **Maps from smooth elliptic curves.** Let $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ be the moduli space of degree d maps from an n-pointed, smooth elliptic curve to $\mathbb{P}^r$. $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ is empty when $d=1$ and non-empty and smooth for all $d>1$; when $d=0$, we have $\mathcal{M}_{1,n}(\mathbb{P}^r,0)\cong\mathcal{M}_{1,n}\times\mathbb{P}^r$. Let $\mathrm{Pic}^d_{1,n}$ be the universal degree-d Picard group over $\mathcal{M}_{1,n}$; it is the $\mathbb{G}_m$-rigidification of the universal Picard stack $\mathfrak{Pic}^d_{1,n}\to\mathcal{M}_{1,n}$.

The forgetful map $\mathcal{M}_{1,n}(\mathbb{P}^r,d)\to\mathcal{M}_{1,n}$ factors through $\mathcal{M}_{1,n}(\mathbb{P}^r,d)\to\mathrm{Pic}^d_{1,n}$ and the fibre over $(C,\mathbf{p},L)$ is

$$U_{r,d}(L):=\{[s_0,\dots,s_r]\in\mathbb{P}(H^0(E,L)^{\oplus r+1})|\bigcap_{i=0}^r s_i^{-1}(0)=\emptyset\}.$$

From this description, it is helpful to describe a partial compactification of $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ via complete linear systems as follows. Let $\mathcal{P}_d$ be the Poincaré line bundle over $\mathrm{Pic}^d_{1,n}\times_{\mathcal{M}_{1,n}}\mathcal{C}_{1,n}$, and let $\pi:\mathrm{Pic}^d_{1,n}\times_{\mathcal{M}_{1,n}}\mathcal{C}_{1,n}\to\mathrm{Pic}^d_{1,n}$ be the projection map to the first factor, then Riemann–Roch implies that $\pi_*\mathcal{P}_d^{\oplus r+1}$ forms a vector bundle of rank $d(r+1)$ over $\mathrm{Pic}^d_{1,n}$.

Definition 4.23. Let $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)\to\mathrm{Pic}^d_{1,n}$ be the total space of the projective bundle $\mathbb{P}(\pi_*\mathcal{P}^{\oplus r+1})$, and let $H_{\hat{\mathcal{Q}}}\in H^2(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d))$ denote its hyperplane class.

Let $\mathcal{B}_{1,n}(\mathbb{P}^r,d)$ denote the complement $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)\setminus\mathcal{M}_{1,n}(\mathbb{P}^r,d)$.

By construction, the fibre of $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)\to\mathrm{Pic}^d_{1,n}$ over $(C,\mathbf{p},L)$ is $\mathbb{P}(H^0(C,L)^{\oplus r+1})$. $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)$ contains $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ as the dense open set consisting of basepoint free tuples of sections.

Remark 4.24. While the notation is chosen to resemble that of the moduli space of quasimaps [MOP11], the basepoints in $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)$ are not required to be disjoint from the marked points. In this way, the space $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)$ can be considered as a quasimaps moduli space where each marked point has zero weight.

To understand both the pure and off-by-one weight graded pieces of $H_c^\star(\mathcal{M}_{1,n}(\mathbb{P}^r, d))$ by the partial compactification $\mathcal{M}_{1,n}(\mathbb{P}^r, d) \subset \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)$, we would like to approach the topology of the complement $\mathcal{B}_{1,n}(\mathbb{P}^r, d)$ via the following result.

Lemma 4.25. The complement $\mathcal{B}_{1,n}(\mathbb{P}^r, d)$ receives a proper, surjective map from $\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d-2)$.

Proof. We identify $\mathrm{Sym}^2\mathcal{C}_{1,n} \cong \mathbb{P}_{\mathrm{Pic}^2_{1,n}}(\mathcal{P}_2)$. The map $\mathcal{P}_2 \boxtimes \mathcal{P}_{d-2}^{\oplus r+1} \to \mathcal{P}_d^{\oplus r+1}$ over the multiplication map $\otimes : \mathrm{Pic}^2_{1,n} \times \mathrm{Pic}^{d-2}_{1,n} \to \mathrm{Pic}^d_{1,n}$ then projectivises to a map $\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d-2) \to \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)$. Concretely, the map is given by multiplying sections of line bundles. The above description shows that the map is projective, hence proper.

By construction, the image of the map has basepoints, so it lands in $\mathcal{B}_{1,n}(\mathbb{P}^r, d)$. To see surjectivity, it suffices to work over the fibre over a marked point. Given a tuple of sections $[s_0 : \cdots : s_r]$ with basepoints, the degree of the basepoint divisor B is greater or equal to 2. Thus, pick any degree two effective divisor $D \leq B$ together with a choice of setction s_D (unique up to scalar multiplication), then $[s_0 : \cdots : s_r]$ is the image of $(D, [s_0/s_D : \cdots : s_r/s_D])$. ☐

Definition 4.26. Let $\mathrm{Pic}^{d_1,d_2}_{1,n}$ denote the fibre product $\mathrm{Pic}^{d_1}_{1,n} \times_{\mathcal{M}_{1,n}} \times \mathrm{Pic}^{d_2}_{1,n}$.

Remark 4.27. The $\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d-2) \to \mathrm{Pic}^{d_1,d_2}_{1,n}$ is a box product of projective bundles; in particular, it has fibres isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{(r+1)(d-2)}$. The multiplication map $m : \mathrm{Pic}^{d_1,d_2}_{1,n} \to \mathrm{Pic}^d_{1,n}$ is smooth, projective of relative dimension one.

Lemma 4.28. $\mathrm{gr}^W_\star H_c^\star(\mathcal{M}_{1,n}(\mathbb{P}^r, d))$ is given by $\mathrm{gr}^W_\star H_c^\star(\mathrm{Pic}^d_{1,n}) \otimes H^i_{\hat{\mathcal{Q}}}$ for $d(r+1) - 2r + 1 \leq i \leq d(r+1)$.

Dually, $\mathrm{gr}^W_\star H^\star(\mathcal{M}_{1,n}(\mathbb{P}^r, d))$ is given by $\mathrm{gr}^W_\star H^\star(\mathrm{Pic}^d_{1,n}) \otimes H^i_{\hat{\mathcal{Q}}}$ for $0 \leq i \leq 2r-1$.

Proof. The proper and surjective map $\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d-2) \to \mathcal{B}_{1,n}(\mathbb{P}^r, d)$ leads to an exact sequence

$$\mathrm{gr}^W_k H^k(\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d-2)) \to \mathrm{gr}^W_k H^k(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)) \to \mathrm{gr}^W_k H^k(\mathcal{M}_{1,n}(\mathbb{P}^r, d)) \to 0,$$

where the first map is The desired claim then follows from the projective bundle formula for $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d) \to \mathrm{Pic}^d_{1,n}$. Consider the excision sequence for the open immersion $\mathcal{M}_{1,n}(\mathbb{P}^r, d) \subset \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)$, truncated[4] at weight k:

$$0 \to \mathrm{gr}^W_k H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r, d)) \to \mathrm{gr}^W_k H^k_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)) \to \mathrm{gr}^W_k H^k_c(\mathcal{B}_{1,n}(\mathbb{P}^r, d)) \to \cdots$$

The terms to the left of $\mathrm{gr}^W_k H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r, d))$ vanish because H^ℓ_c(-) has weights in $0, \ldots, \ell$. Therefore,

$$\mathrm{gr}^W_k H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r, d)) = \ker(\mathrm{gr}^W_k H^k_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)) \to \mathrm{gr}^W_k H^k_c(\mathcal{B}_{1,n}(\mathbb{P}^r, d))).$$

The proper, surjective map $\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d-2) \to \mathcal{B}_{1,n}(\mathbb{P}^r, d)$ induces an injective proper pullback map on pure weight compactly supported cohomology

$$\mathrm{gr}^W_k H^k_c(\mathcal{B}_{1,n}(\mathbb{P}^r, d)) \hookrightarrow \mathrm{gr}^W_k H^k_c(\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d-2)).$$

[4] Recall that the functors gr^W_k are exact and hence preserve long exact sequences.

Composing the two maps, we see that

$$\mathrm{gr}^W_k H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r,d)) = \ker(\mathrm{gr}^W_k H^k_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)) \to \mathrm{gr}^W_k H^k_c(\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d-2))).$$

The map $H^k_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)) \to H^k_c(\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d-2))$ can be understood via cohomology of local systems over $\mathrm{Pic}^d_{1,n}$ as follows. By a version of projective bundle formula for compactly supported cohomology, there is

$$H^\star_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)) \cong H^\star_c(\mathrm{Pic}^d_{1,n}) \otimes H^\star(\mathbb{P}^{d(r+1)}),$$

$$H^\star_c(\mathrm{Sym}^2\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d-2)) \cong H^\star_c(\mathrm{Pic}^{2,d-2}_{1,n}) \otimes H^\star(\mathbb{P}^1 \times \mathbb{P}^{(d-2)(r+1)}).$$

Further, the Leray spectral sequence for $m : \mathrm{Pic}^{2,d-2}_{1,n} \to \mathrm{Pic}^d_{1,n}$ degnerates to give $H^\star_c(\mathrm{Pic}^{2,d-2}_{1,n}) \cong H^\star_c(\mathrm{Pic}^d_{1,n}, \mathrm{R}m_!\mathbb{Q}_{\mathrm{Pic}^{2,d-2}_{1,n}})$. Assembling the isomorphisms, the proper pullback reads as

$$H^\star_c(\mathrm{Pic}^d_{1,n}) \otimes H^\star(\mathbb{P}^{d(r+1)}) \to H^\star_c(\mathrm{Pic}^d_{1,n}, \mathrm{R}m_!\mathbb{Q}_{\mathrm{Pic}^{2,d-2}_{1,n}}) \otimes H^\star(\mathbb{P}^1 \times \mathbb{P}^{(d-2)(r+1)})$$

which is injective on $H^\star_c(\mathrm{Pic}^d_{1,n}) \otimes H^j_{\hat{\mathcal{Q}}}(\mathbb{P}^{d(r+1)})$ for $j \leq 2(2+(r+1)(d-2))$ and the map vanishes for higher j because of cohomological degree reasons. Therefore, the kernel is as claimed in the statement, and it gives the pure weight cohomology $W_\star H^\star(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$. □

The class $H_{\hat{\mathcal{Q}}} \in H^2(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$ comes from the projective bundle formula and records the hyperplane class in the linear system. We relate it to hyperplane classes pulled back from $\mathbb{P}^r$.

Lemma 4.29. When $n \geq 1$, $d \geq 2$, $H_{\hat{\mathcal{Q}}} \in H^2(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$ agrees with ev_i^*H where $H \in H^2(\mathbb{P}^r)$ is a hyperplane class and $\mathrm{ev}_i : \mathcal{M}_{1,n}(\mathbb{P}^r,d) \to \mathbb{P}^r$ is any evaluation map.

Proof. Let $\hat{\mathcal{C}}_{1,n}(\mathbb{P}^r,d)$ be the universal curve over $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)$. Namely, it is the fibre product

$$\begin{array}{ccc} \hat{\mathcal{C}}_{1,n}(\mathbb{P}^r,d) & \xrightarrow{\pi_{\hat{\mathcal{Q}}}} & \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d) \\ \downarrow{\scriptstyle \pi_{\mathcal{C}}} & \square & \downarrow \\ \mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \mathrm{Pic}^d_{1,n} & \longrightarrow & \mathrm{Pic}^d_{1,n} \end{array}$$

Let $\mathcal{C}_{1,n}(\mathbb{P}^r,d) \to \mathcal{M}_{1,n}(\mathbb{P}^r,d)$ be the universal curve over $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$, then $\mathcal{C}_{1,n}(\mathbb{P}^r,d)$ is open in $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)$ pulled back from the open immersion $\mathcal{M}_{1,n}(\mathbb{P}^r,d) \subset \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)$.

Let $\mathrm{ev} : \mathcal{C}_{1,n}(\mathbb{P}^r,d) \to \mathbb{P}^r$ be the evaluation map, then $\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^r}(1) \cong (\pi_{\mathcal{C}}^*\mathcal{P}_d)|_{\mathcal{C}_{1,n}(\mathbb{P}^r,d)}$ where recall that $\mathcal{P}_d$ is the Poincaré line bundle on $\mathcal{C}_{1,n} \times_{\mathcal{M}_{1,n}} \mathrm{Pic}^d_{1,n}$. On the other hand, the line bundle $\mathcal{O}_{\pi_*\mathcal{P}_d^{\oplus r+1}}(1)$ on $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)$ satisfies that $\pi_{\hat{\mathcal{Q}}}\mathcal{O}_{\pi_*\mathcal{P}_d^{\oplus r+1}}(1) \cong (\pi_{\mathcal{C}}^*\mathcal{P}_d)|_{\mathcal{C}_{1,n}(\mathbb{P}^r,d)}$. Therefore,

$$\pi_{\hat{\mathcal{Q}}}^* H_{\hat{\mathcal{Q}}}|_{\mathcal{C}_{1,n}(\mathbb{P}^r,d)} = \mathrm{ev}^*H$$

on $H^2(\mathcal{C}_{1,n}(\mathbb{P}^r,d))$. We pull back both sides along the section $s_i : \mathcal{M}_{1,n}(\mathbb{P}^r,d) \to \mathcal{C}_{1,n}(\mathbb{P}^r,d)$ corresponding to the marked point p_i: they are $s_i^*(\pi_{\hat{\mathcal{Q}}}^* H_{\hat{\mathcal{Q}}}|_{\mathcal{C}_{1,n}(\mathbb{P}^r,d)}) = (\pi_{\hat{\mathcal{Q}}} \circ s_i)^* H_{\hat{\mathcal{Q}}} = H_{\hat{\mathcal{Q}}}$ and $s_i^*\mathrm{ev}^*H = \mathrm{ev}_i^*H$. Hence $H_{\hat{\mathcal{Q}}} = \mathrm{ev}_i^*H$. □

It remains to understand the pure weight cohomology of $\mathrm{Pic}^d_{1,n}$. We adapt the methods of Canning–Larson–Payne [CLP24, §2.2] for the computation of $W_\star H^\star(\mathrm{Pic}^d_{1,n})$ and $W_\star H^\star(\mathrm{Pic}^{d_1,d_2}_{1,n})$, the latter of which is relevant to understanding the topology of the basepoint loci $\mathcal{B}_{1,n}(\mathbb{P}^r,d)$.

Lemma 4.30. (1) The above natural maps induces an isomorphism

$$W_\star H^\star(\mathrm{Pic}^d_{1,n}) \cong W_\star H^\star(\mathcal{M}_{1,n+1}) \oplus W_{\star-2}H^{\star-2}(\mathcal{M}_{1,n})(-1).$$

The Tate twist is given by $\Theta \in H^2(\mathrm{Pic}^d_{1,n})$ as the image of $1 \in H^0(\mathcal{M}_{1,n})(-1) \to H^2(\mathcal{M}_{1,n})$ under the Gysin pushforward; this is the class of a relative polarisation of $\mathrm{Pic}^d_{1,n} \to \mathcal{M}_{1,n}$.

(2) There is an isomorphism

$$W_\star H^\star(\mathrm{Pic}^{d_1,d_2}_{1,n}) \cong W_\star H^\star(\mathcal{M}_{1,n+2}) \oplus W_{\star-2}H^{\star-2}(\mathcal{M}_{1,n+1})(-1)^{\oplus 2} \oplus W_{\star-4}H^{\star-4}(\mathcal{M}_{1,n})(-2).$$

Similar to the above, the Tate twists are given by $\Theta_1, \Theta_2 \in H^2(\mathrm{Pic}^{d_1,d_2}_{1,n})$, and $\Theta_1 \cup \Theta_2 \in H^4(\mathrm{Pic}^{d_1,d_2}_{1,n})$.

Proof. Let $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$ be the universal curve. We recall that $R^0\pi_*\mathbb{Q} = \mathbb{Q}(0)$, $R^2\pi_*\mathbb{Q} = \mathbb{Q}(-1)$, and denote $\mathbb{V} := R^1\pi_*\mathbb{Q}$. Let $\pi^{(n)} : \mathcal{E}^n \to \mathcal{M}_{1,1}$ be its $(n+1)$-th fibre product over $\mathcal{M}_{1,1}$. The Picard group admits an open immersion $\mathrm{Pic}^d_{1,n} \subset \mathcal{E}^n$.

The Leray spectral sequence associated to $\pi^{(n)} : \mathcal{E}^n \to \mathcal{M}_{1,1}$ degenerates to

$$H^k(\mathcal{E}^n) = \bigoplus_{p+q=k} H^p(\mathcal{M}_{1,1}, R^q\pi^{(n)}_*\mathbb{Q}),$$

and by Künneth formula

$$R^q\pi^{(n)}_*\mathbb{Q} = \bigoplus_{i_2+\cdots+i_{n+1}=q} \left(\bigotimes_{j=2}^{n} R^{i_j}_*\mathbb{Q}\right) \otimes R^{i_{n+1}}\pi_*\mathbb{Q}$$

where the last summand is distinguished as the derived pushforward along $\mathrm{Pic}^d_{1,n} \cong \mathcal{C}_{1,n} \to \mathcal{M}_{1,n}$.

We may adapt the proof of [CLP24, Prop. 2.2] and consider the excision sequence $\mathrm{Pic}^d_{1,n} \subset \mathcal{E}^n$ and normalisation of the boundary by $\mathcal{E}^{n-2}$:

$$\bigoplus W_k H^k(\mathcal{E}^{n-1}) \to W_k H^k(\mathcal{E}^n) \to W_k H^k(\mathrm{Pic}^d_{1,n}) \to 0.$$

Loc. lit. applies the technique on the open embedding $\mathcal{M}_{1,n+1} \subset \mathcal{E}^n$ to prove that $W_kH^k(\mathcal{M}_{1,n+1})$ is generated by (the pure weight subspaces of) direct summands $(i_2, \ldots, i_n, i_{n+1})$ such that all $i_j \leq 1$. The same considerations imply that $W_kH^k(\mathrm{Pic}^d_{1,n})$ is generated by (the pure weight subspaces of) direct summands $(i_2, \ldots, i_n, i_{n+1})$ such that $i_j \leq 1$ for all $2 \leq j \leq n$.

Combining this with the descriptions of $W_kH^k(\mathcal{M}_{1,n+1})$, we see that the the subset of the summands contributing to $W_kH^k(\mathrm{Pic}^d_{1,n})$ in which $i_{n+1} \leq 1$ are precisely the pure weight cohomology of $W_kH^k(\mathcal{M}_{1,n+1})$ and the summands in which $i_{n+1} = 2$ are $W_{k-2}H^{k-2}(\mathcal{M}_{1,n}) \otimes \mathbb{Q}(-1)$. As the last tensor term corresponds to the map $\pi^{\mathrm{Pic}} : \mathrm{Pic}^d_{1,n} \to \mathcal{M}_{1,n}$, the second subset of summands is precisely $H^\star(\mathcal{M}_{1,n}, R^2\pi^{\mathrm{Pic}}_*\mathbb{Q})$, which in turn agrees with the Gysin pushforward as claimed.

Analogous calculations on the open embedding $\mathrm{Pic}^{d_1,d_2}_{1,n} \subset \mathcal{E}^{n+1}$ implies that among the summands

$$R^q\pi^{(n+1)}_*\mathbb{Q} = \bigoplus_{i_2+\cdots+i_{n+1}+i_{n+2}=q} \left(\bigotimes_{j=2}^{n} R^{i_j}_*\mathbb{Q}\right) \otimes R^{i_{n+1}}_*\mathbb{Q} \otimes R^{i_{n+2}}_*\mathbb{Q},$$

where again the last two terms correspond to the derived pushforward $\mathrm{Pic}^{d_1,d_2} \to \mathcal{M}_{1,n}$, the pure weight cohomology $W_kH^k(\mathrm{Pic}^{d_1,d_2}_{1,n})$ is generated by summands where $i_j \leq 1$ for $2 \leq n+1$. Grouping the summands into whether $i_{n+1}, i_{n+2} = 2$ gives the desired direct sum decomposition together with their interpretation under the Leray spectral sequence for $\mathrm{Pic}^{d_1,d_2} \to \mathcal{M}_{1,n}$. □

To understand the relations among cycles in $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, we also want to understand the off-by-one weight part $\mathrm{gr}^W_{k+1}H^k(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$. Consider the pair $\mathcal{M}_{1,n}(\mathbb{P}^r,d)\subset\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)$. Using the shorthand $\mathcal{B}$ for $\mathcal{B}_{1,n}(\mathbb{P}^r,d)$, we have the excision sequence in compactly supported cohomology:

$$\cdots\to H^{k-1}_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d))\xrightarrow{\phi}H^{k-1}_c(\mathcal{B})\to H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r,d))\to H^k_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d))\xrightarrow{\varphi}H^k_c(\mathcal{B})\to\cdots$$

Truncating at gr^W_{k-1}, there is a short exact sequence

$$0\to\frac{\mathrm{gr}^W_{k-1}H^{k-1}_c(\mathcal{B}_{1,n}(\mathbb{P}^r,d))}{\mathrm{im}(\phi)}\to\mathrm{gr}^W_{k-1}H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r,d))\to\ker(\mathrm{gr}^W_{k-1}\varphi)\to 0 \tag{1}$$

The following lemma uses the knowledge on the cohomology of $\mathcal{M}_{1,n}$ in [Pet14] to compute the off-by-one cohomology of the Picard groups $\mathrm{Pic}^d_{1,n}$: in odd cohomological degrees, they all reduce to the weight graded piece $\mathrm{gr}^W_4H^3(\mathcal{M}_{1,4})$, which corresponds to Getzler relation on $\overline{\mathcal{M}}_{1,4}$.

Lemma 4.31. For k odd, $\mathrm{gr}^W_{k+1}H^k(\mathrm{Pic}^d_{1,n})\neq 0$ only when $k=3$ and $n\geq 3$ in which case it is isomorphic to $\mathrm{gr}^W_4H^3(\mathcal{M}_{1,n+1})$ and when $k=5$ and $n\geq 4$ in which case it is isomorphic to $\mathrm{gr}^W_4H^3(\mathcal{M}_{1,n})(-1)$ under the Gysin pushforward along any section $\mathcal{M}_{1,n}\to\mathrm{Pic}^d_{1,n}$.

Proof. We recall that there is a non-canonical isomorphism $\mathrm{Pic}^d_{1,n}\cong\mathcal{C}_{1,n}$ over $\mathcal{M}_{1,n}$ and that $\mathcal{C}_{1,n}\setminus\mathcal{M}_{1,n+1}=\bigsqcup_{i=1}^n p_i(\mathcal{M}_{1,n})$, where $p_i:\mathcal{M}_{1,n}\to\mathcal{C}_{1,n}$ are the marked point sections. Consider the long exact sequence for the pair $\mathcal{M}_{1,n+1}\subset\mathrm{Pic}^d_{1,n}$:

$$\mathrm{gr}^W_{k-1}H^{k-2}(\mathcal{M}_{1,n})^{\oplus n}(-1)\to\mathrm{gr}^W_{k+1}H^k(\mathrm{Pic}^d_{1,n})\to\mathrm{gr}^W_{k+1}H^k(\mathcal{M}_{1,n+1})\to\mathrm{gr}^W_{k-1}H^{k-1}(\mathcal{M}_{1,n})^{\oplus n}(-1)$$

From [Pet14, Theorem 1.3], for k odd, $\mathrm{gr}^W_{k+1}H^k(\mathcal{M}_{1,n+1})$ is non-zero only when $k=3,n\geq 3$, and generated by pullbacks from the Getzler relation $\mathrm{gr}^W_4H^3(\mathcal{M}_{1,4})$ along forgetful maps $\mathcal{M}_{1,n+1}\to\mathcal{M}_{1,4}$. Thus, the only possibly non-zero terms of $\mathrm{gr}^W_{k+1}H^k(\mathrm{Pic}^d_{1,n})$ are

$$0\to\mathrm{gr}^W_4H^3(\mathrm{Pic}^d_{1,n})\to\mathrm{gr}^W_4H^3(\mathcal{M}_{1,n+1})\to\mathrm{gr}^W_2H^2(\mathcal{M}_{1,n})^{\oplus n}(-1)\to\cdots$$

$$\cdots\to\mathrm{gr}^W_4H^3(\mathcal{M}_{1,n})^{\oplus n}(-1)\to\mathrm{gr}^W_6H^5(\mathrm{Pic}^d_{1,n})\to 0$$

In the first exact sequence, the term $\mathrm{gr}^W_2H^2(\mathcal{M}_{1,n})^{\oplus n}(-1)$ vanishes due to [Pet14, Theorem 1.1], which states that the only even pure weight cohomology of $\mathcal{M}_{1,n}$ is in H^0. Therefore there is an isomorphism $\mathrm{gr}^W_4H^3(\mathrm{Pic}^d_{1,n})\cong\mathrm{gr}^W_4H^3(\mathcal{M}_{1,n+1})$.

On the other hand, the Leray spectral sequence for $\pi^{\mathrm{Pic}}:\mathrm{Pic}^d_{1,n}\to\mathcal{M}_{1,n}$ degenerates and reads

$$H^k(\mathrm{Pic}^d_{1,n})=H^k(\mathcal{M}_{1,n})\oplus H^{k-1}(\mathcal{M}_{1,n},\mathbb{V})\oplus H^{k-2}(\mathcal{M}_{1,n})(-1),$$

and the Gysin pushforward $H^{k-2}(\mathcal{M}_{1,n})(-1)\to H^k(\mathrm{Pic}^d_{1,n})$ is an isomorphism onto the summand

$$H^{k-2}(\mathcal{M}_{1,n},R^2\pi^{\mathrm{Pic}}_*\mathbb{Q})=H^{k-2}(\mathcal{M}_{1,n})(-1),$$

thus $\mathrm{gr}^W_6H^5(\mathrm{Pic}^d_{1,n})\cong\mathrm{gr}^W_4H^3(\mathcal{M}_{1,n})(-1)$. ∎

Remark 4.32. The last few results in this section give information on $\mathrm{gr}^W_{\star-1}H^\star_c(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$ as follows. Lemma 4.30 (2), together with the projective bundle formula, computes the pure weight compactly supported cohomology $\mathrm{gr}^W_\star H^\star_c(\mathcal{B}_{1,n}(\mathbb{P}^r,d))$ of the basepoint locus. On the other hand, the previous result determines the off-by-one graded piece $\mathrm{gr}^W_{\star-1} H^\star_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d))$ of the ambient space. The terms on the two ends of the short exact sequence (1)

$$0 \to \frac{\mathrm{gr}^W_{k-1} H^{k-1}_c(\mathcal{B}_{1,n}(\mathbb{P}^r, d))}{\mathrm{im}(\phi)} \to \mathrm{gr}^W_{k-1} H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r, d)) \to \ker(\mathrm{gr}^W_{k-1}\varphi) \to 0$$

are quotient and subspaces of the previous two groups respectively.

Therefore, while we have fully computed $\mathrm{gr}^W_{\star-1} H^\star_c(\mathcal{M}_{1,n}(\mathbb{P}^r, d))$, we have provided upper bounds and sources of the off-by-one graded pieces. A complete description involves understanding the proper pull-back map $H^\star_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)) \to H^\star_c(\mathcal{B}_{1,n}(\mathbb{P}^r, d))$ and will be taken up in [Son25].

The partial description also suffices for the following vanishing result that are useful for determining $\mathrm{Pic}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$.

Lemma 4.33. $\mathrm{gr}^W_2 H^1(\mathcal{M}_{1,n}(\mathbb{P}^r, d)) = 0$.

Proof. Recall that $\dim \mathcal{M}_{1,n}(\mathbb{P}^r, d) = n + d(r+1)$, which we denote as $\mathrm{d}_{n,r,d}$. By Poincaré duality this is equivalent to

$$\mathrm{gr}^W_{2(\mathrm{d}_{n,r,d}-1)} H^{2\mathrm{d}_{n,r,d}-1}_c(\mathcal{M}_{1,n}(\mathbb{P}^r, d)) = 0.$$

We inspect the short exact sequence

$$0 \to \frac{\mathrm{gr}^W_{k-1} H^{k-1}_c(\mathcal{B}_{1,n}(\mathbb{P}^r, d))}{\mathrm{im}(\phi)} \to \mathrm{gr}^W_{k-1} H^k_c(\mathcal{M}_{1,n}(\mathbb{P}^r, d)) \to \ker(\mathrm{gr}^W_{k-1}\varphi) \to 0$$

at $k = 2\mathrm{d}_{n,r,d} - 1$, where recall that $\varphi : H^k_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)) \to H^k_c(\mathcal{B})$. When $k - 1 > 2\dim \mathcal{B}_{1,n}(\mathbb{P}^r, d)$, the term $\mathrm{gr}^W_{k-1} H^{k-1}_c(\mathcal{B}_{1,n}(\mathbb{P}^r, d))$ vanishes for dimension reasons. From Lemma 4.31, $H^k(\mathrm{Pic}^d_{1,n})$ has off-by-one weight cohomology only when $k = 3$. Applying Poincaré duality and projective bundle formula, we see that

$$\mathrm{gr}^{2(\mathrm{d}_{n,r,d}-1)} H^{2\mathrm{d}_{n,r,d}-1}_c(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d)) = 0,$$

hence $\ker(\mathrm{gr}^W_{k-1}\varphi) = 0$ as well. The vanishing result follows. □

4.3. **Maps from nodal elliptic curves.** We turn to the cohomology of $\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}}$, where $\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}$ is a degree decorated k-cycle with multi-degree $\mathbf{d}$ and markings $\mathbf{m}$.

Definition 4.34. Define $\mathsf{C}_{k,\mathbf{m}}$ as the underlying marked dual graph. Let $\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}} \subset \mathfrak{M}_{1,|\mathbf{m}|}$ be the moduli of nodal curves of type $\mathsf{C}_{k,\mathbf{m}}$: it may be an Artin stack with torus stabilisers. Recall that $\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$ is isomorphic to $\left(\prod_{v\in V(\mathsf{C}_{k,\mathbf{m}})} \mathcal{M}_{0,\mathbf{m}(v)+2}\right)/\mathrm{Aut}(\mathsf{C}_{k,\mathbf{m}})$. Notice that $\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$ has positive dimensional stabilisers when there are vertices $v \in V(\mathsf{C}_{k,\mathbf{m}})$ with $|\mathbf{m}(v)| = 0$.

Let $\mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}} \to \mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$ be the universal Picard group with multi-degree $\mathbf{d}$, which is a $\mathbb{G}_m$-torsor. Let $\mathfrak{C}_{\mathsf{C}_{k,\mathbf{m}}} \to \mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$ be the universal curve, with $\mathcal{P}_{\mathbf{d}}$ the universal line bundle over $\mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}} \times_{\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}} \mathfrak{C}_{\mathsf{C}_{k,\mathbf{m}}}$. The projection map to the universal curve is denoted as

$$\pi^{(\mathbf{d})} : \mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}} \times_{\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}} \mathfrak{C}_{\mathsf{C}_{k,\mathbf{m}}} \to \mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}}.$$

Let $\hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}} \to \mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}}$ be the projective bundle $\mathbb{P}(\pi^{(\mathbf{d})}_* \mathcal{P}^{\oplus r+1}_{\mathbf{d}})$. This is the projectivisation of the $(r+1)$-th direct sum of the complete linear system analogous to $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r, d) \to \mathrm{Pic}^d_{1,n}$. As earlier, the mapping space $\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}}$ may be presented as the basepoint free locus in $\hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}} \to \mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}}$.

Lemma 4.35. When $\mathbf{m}$ satisfies that $|\mathbf{m}(v)| \geq 1$ for all vertices v on the cycle, the pure weight cohomology of $\mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}}$ is pulled back from the base $W_\star H^\star(\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}})$, which is isomorphic to

$$\left[\bigotimes_{\substack{v\in V(\mathsf{C}_{k,\mathbf{m}})\\ \mathbf{m}(v)=0}} H^\star\left(\mathsf{M}_{0,2}\right)\right]^{\mathrm{Aut}(\mathsf{C}_{k,\mathbf{m}})}$$

consisting of $\mathrm{Aut}(\mathsf{C}_{k,\mathbf{m}})$-invariant polynomials in the ψ_v-classes for bivalent vertices $v \in V(\mathsf{C}_{k,\mathbf{m}})$. In particular, when all $\mathbf{m}(v) \geq 1$, the pure weight cohomology of $\mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}}$ is given by $H^0 \cong \mathbb{Q}$.

Its off-by-one weight graded piece is generated as a $W_\star H^\star(\mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}})$ by pullbacks from the WDVV relation in $H^1(\mathcal{M}_{0,4})$ and

$$\left[\bigotimes_{\substack{v\in V(\mathsf{C}_{k,\mathbf{m}})\\ \mathbf{m}(v)=0}} H^\star\left(\mathsf{M}_{0,2}\right) \otimes H^1(\mathbb{G}_m)\right]^{\mathrm{Aut}(\mathsf{C}_{k,\mathbf{m}})}$$

where $H^1(\mathbb{G}_m)$ receives the sign representation of $\mathrm{Aut}(\mathsf{C}_{k,\mathbf{m}}) \subset D_k$.

Proof. Recall that over a cycle of rational curves C, the Picard group with multi-degree $\boldsymbol{d}$ is computed by the exact sequence

$$0 \to H^0(C, \mathcal{O}_C^*) \to H^0(C^\nu, \mathcal{O}_{C^\nu}^*) \to \bigoplus_{e\in E(C)} H^0(\mathcal{O}_{\nu_e}^*) \to \mathrm{Pic}^{\boldsymbol{d}}(C) \to 0,$$

which is isomorphic to $0 \to \mathbb{C}^\star \xrightarrow{\Delta} (\mathbb{C}^\star)^{V(C)} \to (\mathbb{C}^\star)^{E(C)} \to \mathrm{Pic}^{\boldsymbol{d}}(C) \to 0$.

On the other hand, $\mathrm{Aut}(C)$ fits in the exact sequence $1 \to D \to \mathrm{Aut}(C) \to \prod_{v\in V(C)} \mathbb{C}^\star \to 1$ where D is the dihedral group of the cycle. We may compute that the $\mathrm{Aut}(C)$-action on $\mathrm{Pic}^{\boldsymbol{d}}(C)$ factors through the sign representation of $\mathrm{Aut}(C) \to \mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{Pic}^{\boldsymbol{d}}(C) \cong \mathbb{C}^\star$ by inversion. A similar statement holds after assigning marked points to C.

Denote $\pi : \mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}} \to \mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$ as the natural map. The action of $\mathrm{Aut}(C)$ described above determines the monodromy of $R\pi_*\mathbb{Q}$ as a local system on $\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$. The claimed result now follows from a sheaf cohomology computation. □

We define the following auxiliary moduli space of pairs of line bundles to describe the basepoint locus $\hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}} \setminus \mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}}$.

Definition 4.36. Let $\mathrm{Pic}^{\mathbf{d}_1,\mathbf{d}_2}_{\mathsf{C}_{k,\mathbf{m}}} \to \mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$ be the fibre product of the universal Picard group $\mathrm{Pic}^{\mathbf{d}_1}_{\mathsf{C}_{k,\mathbf{m}}} \to \mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$ and $\mathrm{Pic}^{\mathbf{d}_2}_{\mathsf{C}_{k,\mathbf{m}}} \to \mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}}$.

Following the considerations analogous to the previous section, we have the following lemmas that lead to a description of the off-by-one weight graded piece of $H^\star_c(\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}})$.

Lemma 4.37. The pure weight cohomology of $\mathrm{Pic}^{\mathbf{d}_1,\mathbf{d}_2}_{\mathsf{C}_{k,\mathbf{m}}}$ is pulled back from the base $W_\star H^\star(\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}})$.

Lemma 4.38. Let $D(\mathsf{C}_{k,(\mathbf{d},\mathbf{m})})$ be the set of non-negative multi-degrees $\boldsymbol{\delta}$ on $\mathsf{C}_{k,\mathbf{m}}$ such that $\boldsymbol{\delta} \leq \mathbf{d}$ and $\sum_{v\in V(\mathsf{C}_{k,\mathbf{m}})} \boldsymbol{\delta}(v) = 2$. There is a proper, surjective morphism

$$\bigsqcup_{\boldsymbol{\delta}\in D(\mathsf{C}_{k,(\mathbf{d},\mathbf{m})})} \left(\mathbb{P}\pi_*^{(\boldsymbol{\delta})}\mathcal{P}_{\boldsymbol{\delta}} \boxtimes \hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d}-\boldsymbol{\delta},\mathbf{m})}}\right) \to \hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}} \setminus \mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}},$$

where the box product is over $\mathrm{Pic}^{\boldsymbol{\delta},\mathbf{d}-\boldsymbol{\delta}}_{\mathsf{C}_{k,\mathbf{m}}}$ and has fibres isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{(r+1)(d-2)-1}$.

Corollary 4.39. The pure weight cohomology of $\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}}$ is generated by $H^i_{\hat{\mathcal{Q}}}$, $d(r+1)-2r+1 \leq i \leq d(r+1)$, where $H_{\hat{\mathcal{Q}}} \in H^2(\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}})$ is the hyperplane class, and $W_\star H^\star(\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}})$. The off-by-one weight graded piece $\mathrm{gr}^W_{\star+1} H^\star(\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}})$ is generated as a $W_\star H^\star(\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}})$-module by $\mathrm{gr}^W_{\star+1} H^\star(\mathrm{Pic}^{\mathbf{d}}_{\mathsf{C}_{k,\mathbf{m}}})$ and the pure weight basepoint classes $\mathrm{gr}^W_\star H^\star(\hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}} \setminus \mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}})$, which receives a surjection

$$\bigoplus_{\boldsymbol{\delta}\in D(\mathsf{C}_{k,(\mathbf{d},\mathbf{m})})} \mathrm{gr}^W_\star H^\star(\mathbb{P}\pi^{(\boldsymbol{\delta})}_* \mathcal{P}_{\boldsymbol{\delta}} \boxtimes \hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d}-\boldsymbol{\delta},\mathbf{m})}}) \to \mathrm{gr}^W_\star H^\star(\hat{\mathcal{Q}}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}} \setminus \mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathbf{m})}}).$$

5. Generators and source of relations

In this section, we assemble the results from the previous section to describe the weight graded pieces of the compactly supported cohomology groups of strata $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ and $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$. We recall that each contraction class of central aligned $(1,n,d)$-graphs $[\mathbf{G},\rho]$ or of $(1,n,d)$-graphs $[\mathbf{G}]$ corresponds to a locally closed stratum $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]} \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ or $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]} \subset \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, respectively. We compute the weight graded pieces

$$\mathrm{gr}^W_\star H^\star_c(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}), \mathrm{gr}^W_\star H^\star_c(\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}) \text{ and } \mathrm{gr}^W_{\star-1} H^\star_c(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}), \mathrm{gr}^W_{\star-1} H^\star_c(\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}).$$

Because $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ is a normal crossings compactification, the strata $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}$ are smooth DM stacks, and similarly each locally closed stratum $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}$ is smooth as well. Therefore, their compactly supported cohomology groups are dual to the ordinary cohomology groups.

5.1. Künneth formulas from fibre products.

Firstly, when a contraction class of $(1,n,d)$-graph $[\mathbf{G}]$ has core of positive total degree, the contraction core coincides with the core, and the only central alignment on $\mathbf{G}$ is the trivial one and imposes trivial factoristion property. Thus, the locally closed strata $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]} \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ agrees with over $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]} \subset \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$. The following is the standard fibre product description of strata and strata closures in stable maps spaces [BM96, §3].

Lemma 5.1. Let $[\mathbf{G}]$ be as above and recall the partition $(\boldsymbol{\delta},\boldsymbol{m})_{L'(\mathbf{G}^c)}$ from Definition 3.4: this is the distribution of degrees and marked points along the legs $L'(\mathbf{G}^c)$ that connects the core $\mathbf{G}^c$ to $\mathbf{G} \setminus \mathbf{G}^c$. Let $\mathcal{M}^{\mathrm{st,ord}}_{[\mathbf{G}]} \to \mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}$ be the finite map parametrising a map together with an ordering of the legs $L'(\mathbf{G}^c)$, so that $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]} \cong \mathcal{M}^{\mathrm{st,ord}}_{[\mathbf{G}]}/\mathrm{Aut}([\mathbf{G}])$. The space $\mathcal{M}^{\mathrm{st,ord}}_{[\mathbf{G}]}$ is a fibration

$$\begin{array}{ccc} \prod_{\ell\in L'(\mathbf{G}^c)} \overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell) & \longrightarrow & \mathcal{M}^{\mathrm{st,ord}}_{[\mathbf{G}]} \\ & & \downarrow \\ & & \mathcal{M}_{\mathbf{G}^c} \end{array}$$

Similar descriptions hold in the presence of linear dependency or non-trivial central alignment. Recall from Lemma 3.22 that when the core has total degree zero, the Vakil–Zinger space stratum $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}$ is a finite quotient $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]} \cong \widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}/\mathrm{Aut}([\mathbf{G},\rho])$ where the cover $\widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}$ is a fibration given by

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})} & \longrightarrow & \widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]} \\ & & \downarrow \\ & & \widetilde{\mathcal{M}}_{\mathbf{G}^{icc},\rho^{icc}} \end{array}$$

where the fibres $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$ record the restriction of the map to the subcurve strictly outside the contracted core; because the internal contraction cores $\mathbf{G}^{icc}$ have total degree zero, the bases $\mathcal{M}_{\mathbf{G}^{icc}}$ resp. $\widetilde{\mathcal{M}}_{\mathbf{G}^{icc},\rho^{icc}}$ are the product of $\mathbb{P}^r$ and the locally closed strata in $\overline{\mathcal{M}}_{1,n'}$ resp. $\overline{\mathcal{M}}^{\mathrm{cen}}_{1,n'}$ specified by $\mathbf{G}^{icc}$ resp. $(\mathbf{G}^{icc},\rho^{icc})$. For both fibres, the maps on the innermost radius of vertices have constraints on their differentials that are denoted by $\mathbf{D}$ and $\mathbf{F}$ respectively, and the maps further from the radius are genus zero stable maps without constraints. Thus, they fit into fibration diagrams

$$\begin{array}{ccc}\prod_{\ell\in L'([\mathbf{G}^{cc}])}\overline{\mathcal{M}}^{*}_{0,\boldsymbol{m}_\ell-\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell-\boldsymbol{\delta}^{(r)}_\ell) & \longrightarrow & \widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})} \\ & & \downarrow \\ & & \mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}\end{array}$$

Lemma 5.2. The fibration diagram describing $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$ leads to Künneth formula

$$H^\star_c\left(\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}\right)\cong H^\star_c\left(\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}\right)\otimes H^\star\left(\prod_{\ell\in L'([\mathbf{G}^{cc}])}\overline{\mathcal{M}}^{*}_{0,\boldsymbol{m}_\ell-\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell-\boldsymbol{\delta}^{(r)}_\ell)\right).$$

Proof. The genus zero pointed mapping spaces $\overline{\mathcal{M}}^{*}_{0,\boldsymbol{m}_\ell-\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell-\boldsymbol{\delta}^{(r)}_\ell)$ are smooth and proper as DM stacks, hence the forgetful maps $\overline{\mathcal{M}}^{\mathbf{D}}_{(\boldsymbol{\delta},\boldsymbol{m})}\to\mathcal{M}^{\mathbf{D}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ and $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}\to\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}$ are smooth and proper. Therefore, the proper direct images along the maps agree with the derived pushforward, the relative hard Lefschetz is applicable, so their Leray spectral sequences for compactly supported cohomology degenerate. Further, the derived pushforwards are local systems with trivial monodromy, because they are pulled back from products of evaluation maps

$$\overline{\mathcal{M}}_{0,(\boldsymbol{m}_\ell-\boldsymbol{m}^{(r)}_\ell)\sqcup\star}(\mathbb{P}^r,\boldsymbol{\delta}_\ell-\boldsymbol{\delta}^{(r)}_\ell)\xrightarrow{\mathrm{ev}_\star}\mathbb{P}^r,$$

where the base is simply connected. Therefore, the E_2-pages of the Leray spectral sequences are tensor products of the cohomologies of the base and fibres, and they give the cohomology of the total spaces. □

A similar Künneth formula hold for the ordered stratum $\widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}$.

Lemma 5.3. The compactly supported cohomology of the ordered moduli space admits Künneth formula

$$\begin{aligned}H^\star_c(\widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]})\cong H^\star_c\left(\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)},\boldsymbol{m}^{(r)})}\right)&\otimes H^\star\left(\prod_{\ell\in L'([\mathbf{G}^{cc}])}\overline{\mathcal{M}}^{*}_{0,\boldsymbol{m}_\ell-\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell-\boldsymbol{\delta}^{(r)}_\ell)\right)\\ &\otimes H^\star_c(\underline{\mathcal{M}}_{\mathbf{G}^{icc},\rho^{icc}})\otimes H^\star(\mathbb{P}^r).\end{aligned}$$

Proof. The moduli of the internal contraction cores are isomorphic to products $\mathcal{M}_{\mathbf{G}^{icc}}\cong\underline{\mathcal{M}}_{\mathbf{G}^{icc}}\times\mathbb{P}^r$ and $\mathcal{M}_{(\mathbf{G}^{icc},\rho^{icc})}\cong\underline{\mathcal{M}}_{(\mathbf{G}^{icc},\rho^{icc})}\times\mathbb{P}^r$. Recall the space $\widetilde{\mathcal{M}}^{\mathbf{F},\mathrm{tar}}_{(\boldsymbol{\delta},\boldsymbol{m})}$ from Definition 3.21: this is the analogue of $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$ where the frozen marked points in $L'([\mathbf{G}^{cc}])$ are required to be mapped to the same but unspecified point on $\mathbb{P}^r$, so the evaluation map $\widetilde{\mathcal{M}}^{\mathbf{F},\mathrm{tar}}_{(\boldsymbol{\delta},\boldsymbol{m})}\to\mathbb{P}^r$ is Zariski locally trivial with fibres isomorphic to $\widetilde{\mathcal{M}}^{\mathbf{F}}_{(\boldsymbol{\delta},\boldsymbol{m})}$.

The composition of the forgetful map and the projection $\widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}\to\mathcal{M}_{\mathbf{G}^{icc},\rho^{icc}}\to\underline{\mathcal{M}}_{\mathbf{G}^{icc},\rho^{icc}}$ induces a product isomorphism $\widetilde{\mathcal{M}}^{\mathrm{ord}}_{[\mathbf{G},\rho]}\cong\underline{\mathcal{M}}_{\mathbf{G}^{icc},\rho^{icc}}\times\widetilde{\mathcal{M}}^{\mathbf{F},\mathrm{tar}}_{(\boldsymbol{\delta},\boldsymbol{m})}$. The claimed formulas now follow from the Künneth formula of the product isomorphism and the Zariski locally trivial fibration $\widetilde{\mathcal{M}}^{\mathbf{F},\mathrm{tar}}_{(\boldsymbol{\delta},\boldsymbol{m})}\to\mathbb{P}^r$. □

Remark 5.4. In view of genus zero stable maps spaces as a non-linear Grassmannian [Pan96], one may expect a Chow-theoretic or motivic version of the above Künneth formulas. As evidence, we prove that

$\overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r, d)$ satisfies the the Chow–Künneth generation property (CKgP) [Tot16], [BS23, §2], [CL24, §3]. To begin with, we observe that $\mathcal{M}^*_{0,m}(\mathbb{P}^r, d)$ is an open subset in an affine bundle of rank $d(r+1)$ over $\mathcal{M}_{0,m+1}$, with fibres given by $\mathrm{Map}^*_d(\mathbb{P}^1, \mathbb{P}^r) \subset \mathbb{A}^{d(r+1)}$. As $\mathcal{M}_{0,m}$ satisfies the CKgP, so is any affine bundle over it. Therefore, $\mathcal{M}^*_{0,m}(\mathbb{P}^r, d)$ satisfies the CKgP. The standard stratification of $\overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r, d)$ by $(0, n, d)$-graphs with one frozen leg implies that $\overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r, d)$ satisfies the CKgP, namely that $A_*(X) \otimes A_*(\overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r, d))$ surjects onto $A_*(X \times \overline{\mathcal{M}}^*_{0,n}(\mathbb{P}^r, d))$ for all X.

We now describe the cohomologies of the strata in $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ which are finite group quotients of the auxiliary ordered moduli spaces.

Corollary 5.5. When the total degree of the core $\mathbf{G}^c$ is positive[5],

$$H^\star_c(\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}) \cong \left(H^\star \left(\prod_{\ell \in L'(\mathbf{G}^c)} \overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r, \boldsymbol{\delta}_\ell) \right) \otimes H^\star_c(\mathcal{M}_{\mathbf{G}^c}) \right)^{\mathrm{Aut}([\mathbf{G}])}.$$

For a contraction class of centrally aligned $(1, n, d)$-graph $[\mathbf{G}, \rho]$, its stratum $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]} \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ has cohomology

$$\begin{aligned} H^\star_c(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}) \cong \Bigg[H^\star_c \left(\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)}, \boldsymbol{m}^{(r)})} \right) \otimes H^\star \left(\prod_{\ell \in L'([\mathbf{G}^{cc}])} \overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell - \boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r, \boldsymbol{\delta}_\ell - \boldsymbol{\delta}^{(r)}_\ell) \right) \\ \otimes H^\star_c(\underline{\mathcal{M}}_{\mathbf{G}^{icc}, \rho^{icc}}) \otimes H^\star(\mathbb{P}^r) \Bigg]^{\mathrm{Aut}([\mathbf{G},\rho])}. \end{aligned}$$

The automorphism groups of $[\mathbf{G}]$ and $[\mathbf{G}, \rho]$ act on $H^\star_c(\mathcal{M}_{\mathbf{G}^c})$ and $H^\star_c(\underline{\mathcal{M}}_{\mathbf{G}^{icc}, \rho^{icc}})$ by permuting the marked points corresponding to $L'(\mathbf{G})$ as a subgroup of $S_{L'(\mathbf{G}^c)}$ and of $S_{L'(\mathbf{G}^{icc})}$ respectively. In the other direction, $\mathrm{Aut}([\mathbf{G}, \rho])$ acts on $H^\star_c\left(\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)}, \boldsymbol{m}^{(r)})}\right)$, and $H^\star\left(\prod_{\ell \in L'([\mathbf{G}^{cc}])} \overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell - \boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r, \boldsymbol{\delta}_\ell - \boldsymbol{\delta}^{(r)}_\ell)\right)$ by permuting the factors as a subgroup of $S_{L'(\mathbf{G}^{cc})}$, where for the action on $\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)}, \boldsymbol{m}^{(r)})}$ we recall the closed embedding

$$\mathcal{M}^{\mathbf{F}}_{(\boldsymbol{\delta}^{(r)}, \boldsymbol{m}^{(r)})} \subset \mathbb{P}^\circ \mathcal{V}_{[\mathbf{G},\rho]^{L',(r)}} \to \prod_{\ell \in L'([\mathbf{G}^{cc}])} \mathcal{M}^*_{0,\boldsymbol{m}^{(r)}_\ell}(\mathbb{P}^r, \boldsymbol{\delta}^{(r)}_\ell).$$

Finally, the above formulas are compatible with the weight filtrations.

Remark 5.6. Since all strata in $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ admit the same fibre product descriptions, their compactly supported cohomology groups all satisfy the first formula in the above corollary.

Remark 5.7. We note that the automorphism group invariants in the above are computable: the characters of $H^\star(\mathcal{M}_{1,n})$ as an S_n-representation has been determined by Getzler [Get99]; see also [CLP24, Proposition 2.2], and the permutation representation of the product of genus zero mapping spaces are accessible as well. Continuing Remark 3.23, the invariants are plethysms of $\mathbb{S}$-modules, and it is feasible to continue the work in [GP06, KS24b] and give generating functions of the weight graded pieces of the strata as graded $\mathbb{S}$-modules.

Furthur, the formula in [CLP24, Proposition 2.2] implies that the automorphism group action will impose non-trivial constraints on the Hodge structures present on the pure weight cohomology groups of the strata. We will take up this discussion in Section 5.3.

[5] Recall that in which case the only central alignment on $[\mathbf{G}]$ is trivial, so the stratum $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ agrees with $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}$.

5.2. **Assembling the generators.** We now combine the Künneth formulas above and the descriptions of the weight graded pieces of each stratum component to give a precise statement of the pure weight cohomology groups of each stratum. This section is a summary of the previous calculations and serves to state the main result, Theorem A, on the cohomology generators.

Theorem 5.8. Let $[\mathbf{G}]$ be a contraction equivalence class of $(1,n,d)$-graphs, and let $[\mathbf{G},\rho]$ be a contraction equivalence class of centrally aligned $(1,n,d)$-graphs. Let $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]} \subset \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ be the stratum in the full stable maps space specified by $[\mathbf{G}]$, and let $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]} \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ be as earlier. In the following, we use the definitions of core and contraction core from Definition 3.2.

(1) When the core of $[\mathbf{G}]$ is a genus one vertex of positive degree, $\mathrm{gr}^W_\star H^\star_c(\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]})$ are $\mathrm{Aut}([\mathbf{G}])$-invariant tensor products of
- a class in[6] $W_\star H^\star_c(\mathcal{M}_{1,(L\cup L')(\mathbf{G}^c)}(\mathbb{P}^r,\boldsymbol{\delta}_{\mathbf{G}^c}))$, which is generated by $W_\star H^\star_c(\mathcal{M}_{1,(L\cup L')(\mathbf{G}^c)})$, Θ, and $H^i_{\hat{\mathcal{Q}}}$ for $d(r+1)-2r+1 \le i \le d(r+1)$ (Lemma 4.28), and
- a class in $\prod_{\ell\in L'(\mathbf{G}^c)} \overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell)$.

(2) When the core $[\mathbf{G}]$ is a cycle $\mathsf{C}_{k,(\mathbf{d},\mathrm{m})}$ of genus zero vertices with positive total degree, $\mathrm{gr}^W_\star H^\star_c(\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]})$ are $\mathrm{Aut}([\mathbf{G}])$-invariant tensor products of
- a class in $W_\star H^\star_c(\mathcal{M}_{\mathsf{C}_{k,(\mathbf{d},\mathrm{m})}})$, generated by $H_{\hat{\mathcal{Q}}}$ and $W_\star H^\star_c(\mathfrak{M}_{\mathsf{C}_{k,\mathbf{m}}})$, which are $\mathrm{Aut}(\mathsf{C}_{k,\mathbf{m}})$-invariant polynomials of ψ-classes of bivalent vertices (Lemma 4.38), and
- a class in $\prod_{\ell\in L'(\mathbf{G}^c)} \overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell)$.

The same formula holds for the stratum $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, with $\mathrm{Aut}(\mathbf{G},\rho)$ replacing $\mathrm{Aut}(\mathbf{G})$ in the case of $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}$.

(3) When the core has total degree zero and is a genus one vertex, $\mathrm{gr}^W_\star H^\star_c(\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]})$ are $\mathrm{Aut}([\mathbf{G}])$-invariant tensor products of
- $W_\star H^\star(\mathcal{M}_{1,(L\cup L')(\mathbf{G}^c)}) \otimes H^\star(\mathbb{P}^r)$, and
- a class in $\prod_{\ell\in L'(\mathbf{G}^c)} \overline{\mathcal{M}}^*_{0,\boldsymbol{m}_\ell}(\mathbb{P}^r,\boldsymbol{\delta}_\ell)$.

The formula for $\mathrm{gr}^W_\star H^\star_c(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]})$ is the invariant tensor products of the above together with $\psi_v^{i_v}, 1 \le i_v \le r-1$, where v runs over all bivalent vertices in $L'([\mathbf{G}^{cc}])$ (Lemma 4.21).

(4) When the core $[\mathbf{G}]$ is a cycle $\mathsf{C}_{k,(\mathbf{d},\mathrm{m})}$ of genus zero vertices with zero total degree, the pure weight cohomology groups of the strata $\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}$ and $\widetilde{\mathcal{M}}_{[\mathbf{G}]}$ are spanned by invariant tensor products of all the generators in the previous item apart from $W_\star H^\star(\mathcal{M}_{1,(L\cup L')(\mathbf{G}^c)})$.

These generators fit into the stratification spectral sequence where the E_1-pages are given by the compactly supported cohomology groups of the locally closed stratum, and which converges to the pure weight compactly supported cohomology of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ and $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ respectively. As both moduli spaces are proper, their compactly supported cohomology groups agree with the ordinary cohomology groups, and Poincaré duality applies to $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$. The general results from Lemma 2.11 and Corollary 4.18 implies the following generation result.

Corollary 5.9 (Theorem A)**.** With the same notation as above, there are injections and surjection

$$\mathrm{gr}^W_\star H^\star(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)) \hookrightarrow \bigoplus_{[\mathbf{G}]} \mathrm{gr}^W_\star H^\star_c(\mathcal{M}^{\mathrm{st}}_{[\mathbf{G}]}),$$

[6]The notation $(L\cup L')(\mathbf{G}^c)$ is a short hand for $L(\mathbf{G}^c)\cup L'(\mathbf{G}^c)$, namely both the legs assigned to $\mathbf{G}^c$ and the legs connecting $\mathbf{G}^c$ to the remainder of the graph.

$$\bigoplus_{[\mathbf{G},\rho]} \mathrm{gr}^W_\star H^\star(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}) \twoheadrightarrow \mathrm{gr}^W_\star H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)),$$

and the pure weight (compactly supported) cohomology groups of the strata are spanned by the list of classes from Theorem 5.8.

Remark 5.10. We acknowledge the mismatch in stating that we have obtained a set of 'generators' for $\mathrm{gr}^W_\star H^\star(\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$. A more precise statement is that we have identified the pure weight cohomology of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ as a linear subspace of tautological classes and pullback of $W_\star H^\star(\mathcal{M}_{1,n'})$.

5.3. **Degrees with odd cohomology.** The description of the generators points to the Hodge structures present on $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$, which in turn imply vanishing of certain cohomology groups.

In the following, let $\mathsf{L} = H^2(\mathbb{P}^1)$ be the Tate motive, and let $W_k H^k(\mathcal{M}_{1,k}) =: \mathsf{S}_{k+1}$ be the weight k Hodge structure corresponding to $\mathrm{SL}_2(\mathbb{Z})$-cusp forms of weight $k+1$ under the Eichler–Shimura isomorphism.

Corollary 5.11. The Hodge structures present in $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ are products of L and S_{k+1}.

Proof. The list of generators in Theorem 5.8 are either Chern classes with Hodge structure L or pullback from $W_\star H^\star(\mathcal{M}_{1,n})$. It has been proven in [CLP24, §2] that each $W_k H^k(\mathcal{M}_{1,n})$ is generated by pullback of the Hodge structures $W_k H^k(\mathcal{M}_{1,k}) \to W_k H^k(\mathcal{M}_{1,n})$ along the forgetful maps $\mathcal{M}_{1,n} \to \mathcal{M}_{1,k}$. Therefore, each $W_\star H^\star(\mathcal{M}_{1,n})$ has Hodge structure S_{k+1}. □

Recall that the Hodge structure $\mathsf{S}_{k+1} = W_k H^k(\mathcal{M}_{1,k})$ corresponds to $\mathrm{SL}_2(\mathbb{Z})$-cusp forms of weight $k+1$ by the Eichler–Shimura isomorphism. As the smallest weight for the first non-zero $\mathrm{SL}_2(\mathbb{Z})$-cusp form is 12, S_{k+1} is non-zero for the first time when $k = 11$. This recovers the following vanishing result by Fontanari [Fon07].

Corollary 5.12. For odd $k < 11$ and all n, d, we have $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)) = 0$.

Proof. Since $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is smooth and proper, $H^k(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ carries a Hodge structure of weight k. On the other hand, the above implies that the first odd weight Hodge structure that could be present on $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ is S_{12}, which is off weight 11. Therefore, no odd cohomology groups in degree less than 11 can be non-vanishing. □

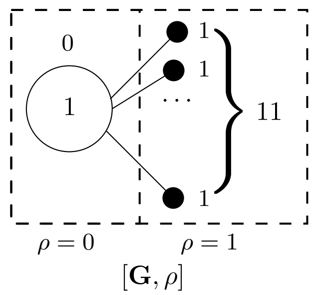
In fact, our description of the generators can refine the above result and ask for the lowest values of (n, r, d) for $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ to have odd cohomology and for the cohomological degree that witnesses the first such class. For instance, we can see that when $n \geq 11$, the pullback to $\mathcal{M}_{1,11}$ gives $H^{11}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)) \neq 0$. For a more interesting example, we fix $n = 0$ and ask:

Question. What is the smallest d such that $\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, d)$ has odd cohomology?

Considering the generator contribution from each dual graph stratum, we see that a necessary condition for a stratum to contribute Hodge structure of the form S_{12} is that it contains a genus one vertex with valency at least 11. In the absence of marked points, each subgraph incident to the genus one vertex needs to have positive degree, hence $d \geq 11$. We claim that this suffices to produce odd cohomology:

Corollary 5.13. When $r \geq 10$, the smallest d such that $\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, d)$ has odd cohomology is $d = 11$.

Proof. From the discussion above, $d = 11$ is a lower bound, so it suffices to show that the following centrally aligned type in $\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, 11)$ has non-vanishing odd cohomology when $r \geq 10$.



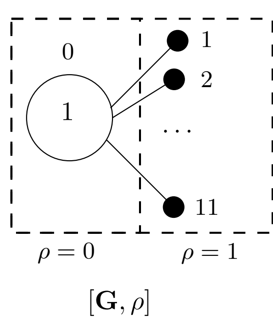
From Theorem 5.8, we see that the pure weight cohomology contribution from the stratum is

$$\left[W_\star H^\star(\mathcal{M}_{1,11}) \otimes \bigotimes_{j=1}^{11} \left(\bigoplus_{i=0}^{r-1} \psi_{v_j}^i \right) \right]^{S_{11}},$$

where $S_{11} \cong \mathrm{Aut}([\mathbf{G}, \rho])$. The odd cohomology all comes from setting $\star = 11$. It is known that as an S_{11}-representation, $W_{11}H^{11}(\mathcal{M}_{1,11}) \cong \mathrm{sgn}_{S_{11}} \otimes \mathsf{S}_{12}$, thus the odd cohomology simplifies to $\mathsf{S}_{12} \otimes \bigwedge^{11} \left(\bigoplus_{i=0}^{r-1} \psi^i \right)$, which is non-zero if and only if $\dim \left(\bigoplus_{i=0}^{r-1} \psi^i \right) \geq 11$, namely when $r \geq 11$. When $r = 11$, the corresponding cohomological degree on the stratum is $121 = 11 + 2 \cdot \binom{11}{2}$, which maps to $H^{123}(\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, 11))$ under the Gysin pushforward. Because the odd degree generators are of pure weight carry Hodge structures S_{12}, the differentials to and from them in the stratification spectral sequence for $H^\star(\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, 11))$ all vanish, so the generators survive to $H^\star(\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, 11))$. □

Remark 5.14. We give a more geometric picture of the above calculation. Pick distinct lines $\mathbf{L} = \{L_0, \ldots, L_{10}\}$ in $\mathbb{P}^r$ that satisfy a non-vanishing linear dependency and consider maps in the stratum specified by the above aligned graph that embed the rational tails into the lines. This specifies a subscheme $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho],\mathbf{L}}$ in which the rational tails are ordered by the lines, so that it admits a well-defined map $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho],\mathbf{L}} \to \mathcal{M}_{1,11}$. The Künneth formula in the proof then implies that when $r \geq 10$, pulling back $H^{11}(\mathcal{M}_{1,11})$ and applying Gysin pushforward defines a non-zero odd cohomology class in $\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, 11)$.

Example 5.15. The smallest odd cohomological degree k such that $H^k(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ is realized as $k = 13$ and $d = \binom{12}{2} = 66$. The relevant dual graph stratum is given by



which has trivial automorphism group. Therefore, the stratum contributes non-vanishing H^{11}, which survives in the spectral sequence and Gysin pushes forward to $H^{13}(\widetilde{\mathcal{M}}_{1,0}(\mathbb{P}^r, 66))$. On the other hand, we already know that the minimum k needs to be $k \geq 11$, but $\mathrm{gr}_{11}^W H^{11}(\mathcal{M}_{1,0}(\mathbb{P}^r, d))$ vanishes for all r, d. Therefore the smallest such k is $k = 13$.

6. TOWARDS RELATIONS

After stating the generators of the cohomology of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, we offer an informal discussion on their relations. From Lemma 2.11, they arise precisely as images of the differentials in the stratification spectral sequences. The weight graded parts of the differentials on E_1 page are given by:

$$0 \to \mathrm{gr}^W_{p+q} E_1^{p,q} = \bigoplus_{\substack{[\mathbf{G},\rho] \\ \dim \widetilde{\mathcal{M}}_{[\mathbf{G},\rho]} = p}} \mathrm{gr}^W_{p+q} H_c^{p+q}(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}) \to \bigoplus_{\substack{[\mathbf{G}',\rho'] \\ \dim \widetilde{\mathcal{M}}_{[\mathbf{G}',\rho']} = p+1}} \mathrm{gr}^W_{p+q} H_c^{p+q+1}(\widetilde{\mathcal{M}}_{[\mathbf{G}',\rho']}) \to \cdots$$

and it is diagonalised by maps of the form $\mathrm{gr}^W_{p+q} H_c^{p+q}(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}) \to \mathrm{gr}^W_{p+q} H_c^{p+q+1}(\widetilde{\mathcal{M}}_{[\mathbf{G}',\rho']})$ where $[\mathbf{G},\rho] \rightsquigarrow [\mathbf{G}',\rho']$ is one of the following morphisms described in 3.11:

- edge contraction within $\rho^{-1}(0)$,
- radial merge within the contraction radius,
- edge contraction outside of the contraction radius.

When the core of $[\mathbf{G},\rho]$ has positive total degree, the central alignment is trivial, and so $[\mathbf{G},\rho]$ can be contracted along any edge.

While the pure weight graded piece $\mathrm{gr}^W_{p+q} E_1^{p,q}$ has been described in the previous section, we give a brief summary on the classes in the successive term $\mathrm{gr}^W_{p+q} E_1^{p+1,q}$: $\mathrm{gr}^W_{\star-1} H_c^\star(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]})$ is generated as a $\mathrm{gr}^W_\star H_c^\star(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]})$-module by the following list of classes, which depend on the centrally aligned type $[\mathbf{G},\rho]$:

- pullback of Getzler relation $\mathrm{gr}^W_4 H^3(\mathcal{M}_{1,4})$ when $[\mathbf{G},\rho]$ contains a genus one vertex,
- pullback of the WDVV relation $\mathrm{gr}^W_2 H^1(\mathcal{M}_{0,4})$ and Picard group of a nodal elliptic curve (Lemma 4.35) when $[\mathbf{G},\rho]$ coontains a cycle of genus zero vertices,
- torus fibres and inadmissible tangent vectors from $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ when $[\mathbf{G},\rho]$ has non-trivial central alignment (Lemma 4.21),
- basepoint linear system classes when the core of $[\mathbf{G},\rho]$ has positive total degree (Remark 4.32, Corollary 4.39),
- the basepoint classes from genus zero vertices on contraction radius with valency greater or equal to three, when when $[\mathbf{G},\rho]$ has non-trivial central alignment (Lemma 4.21).

In general, the off-by-one graded pieces of the higher pages are cohomologies groups under the differentials, so they are successive subquotients of the classes listed above. The E_ℓ page differentials map between (subquotients of) cohomology groups of strata that are related by ℓ contractions. The images of each differential maps pure weight classes on the stata to the subquotients of off-by-one weight graded pieces, and the higher page of pure weight classes are the kernel of the map. The non-zero images of the differentials dualises to kernels under the successive surjective maps from the pure weight classes of the strata to $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$, which are precisely the relations among the generators. While we do not explicitly determine the differentials in this work, there are the following expectations.

- Because the WDVV and Getzler relations are known to lift to relations on $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{M}}_{1,4}$ respectively, the pullbacks of $\mathrm{gr}^W_2 H^1(\mathcal{M}_{0,4})$ and $\mathrm{gr}^W_2 H^1(\mathcal{M}_{0,4})$ will lead to non-zero differentials corresponding to pullbacks of the relations among the cycles.
- The cohomology classes coming from the torus factors on $\widetilde{\mathsf{D}}^*_{\boldsymbol{\delta}}$ are related to the locally closed strata of the projective bundles showing up in the strata blow-up $\mathfrak{M}^{\mathrm{cen}}_{1,n} \to \mathfrak{M}_{1,n}$ of prestable genus one curves [RSPW19] and in turn the desingularisation $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}^{\mathrm{main}}_{1,n}(\mathbb{P}^r, d)$. In this way, these

classes receive differentials that are controlled by the projective bundle formula; they are locally modeled by the calculation of $H^\star(\mathbb{P}^k)$ by its torus stratification.

- The classes from the basepoint loci on positive degree vertices (both genus zero and genus one) are expected to receive differentials coming from contracting edges that connect vertices with positive degrees.

These heuristics concern the classes contributed from individual vertices and contraction radii. Via natural pullback diagrams and Künneth formulas described in previous sections, they should assemble to describe non-zero differentials from the pure weight graded piece to off-by-one weight graded pieces and their subquotients.

We also observe that the basepoint classes have cohomological degree growing linearly with r. For instance, the class on $\mathrm{Map}_d^{*,w}(\mathbb{P}^1,\mathbb{P}^r)$ is $2r-1$, locally modeled by $\mathbb{A}^r\setminus\mathrm{pt}$, and the basepoint classes on genus one (both smooth and nodal) vertices with positive degrees come from the $(r+1)$-direct sum of the complete linear system of degree $(d-2)$, which has rank $2(r+1)$ less than that of degree d. Both type of classes hence contribute to differentials in cohomological degree that grow linearly with r. Thus, we expect relations that are not pulled back from the WDVV or Getzler relations on moduli of curves and reflect the linear geometry of basepoint free tuples of linear systems. The same consideration is valid for genus zero stable maps as well and should offer an explanation for the complexity of relations among tautological cycles as determined by Mustata–Mustata [MM06, MM07, MM08].

6.1. **Application to Picard group.** After outlining the general strategy to produce relations among the generators, we specialise to the low cohomological degree and determine $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$.

Set $\mathrm{d}_{n,r,d}=\dim\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$. After dualising to $H_c^{2\mathrm{d}_{n,r,d}-2}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ and inspecting the stratification spectral sequence, we observe that $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ is generated by the classes $\Theta, H_{\hat{Q}}$ on $\mathcal{M}_{1,n}(\mathbb{P}^r,d)$ (Lemma 4.28) and the fundamental classes of the boundary divisors.

Proposition 6.1 (Corollary C)**.** The classes listed above form a basis of $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$, and the cycle class map $A^1_{\mathbb{Q}}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))\to H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ is an isomorphism.

Proof. We calculate $H_c^{2\mathrm{d}_{n,r,d}-2}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ via the stratification spectral sequence.

For dimension reasons, the only E_1 pages that can possibly contribute are $\mathrm{gr}^W_{2(\mathrm{d}_{n,r,d}-1)}E_1^{\mathrm{d}_{n,r,d}-1,\mathrm{d}_{n,r,d}-1}$ and $\mathrm{gr}^W_{2(\mathrm{d}_{n,r,d}-1)}E_1^{\mathrm{d}_{n,r,d},\mathrm{d}_{n,r,d}-2}=H_c^{2(\mathrm{d}_{n,r,d}-1)}(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$. By considering the cohomological degree and weight, the latter survives to E_∞ page as a subquotient of $H_c^{2(\mathrm{d}_{n,r,d}-1)}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$.

The page $E_1^{\mathrm{d}_{n,r,d}-1,\mathrm{d}_{n,r,d}-1}$ is given by the direct sum of the fundamental classes of the boundary stratum and is hence pure of weight $2(\mathrm{d}_{n,r,d}-1)$. By construction of the stratification $E_\star^{\mathrm{d}_{n,r,d}-1-r,\mathrm{d}_{n,r,d}-1+(r-1)}$ has weights strictly less than $2(\mathrm{d}_{n,r,d}-1)$ for all $r>0$, so the term does not receive any non-zero differential in any page. On the other hand, $E_\star^{\mathrm{d}_{n,r,d}-1-r,\mathrm{d}_{n,r,d}-1+(r-1)}$ vanishes for all $r<-1$. Therefore, the only possibly non-zero differential will be $d_1: E_1^{\mathrm{d}_{n,r,d}-1,\mathrm{d}_{n,r,d}-1}\to E_1^{\mathrm{d}_{n,r,d},\mathrm{d}_{n,r,d}-1}$. For weight reason, the map d_1 is non-zero if and only if the composition

$$E_1^{\mathrm{d}_{n,r,d}-1,\mathrm{d}_{n,r,d}-1}\to E_1^{\mathrm{d}_{n,r,d},\mathrm{d}_{n,r,d}-1}\to\mathrm{gr}^W_{2(\mathrm{d}_{n,r,d}-1)}E_1^{\mathrm{d}_{n,r,d},\mathrm{d}_{n,r,d}-1}=\mathrm{gr}^W_{2(\mathrm{d}_{n,r,d}-1)}H_c^{2\mathrm{d}_{n,r,d}-1}(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$$

is non-zero. However, the latter is dual to $\mathrm{gr}^W_2H^2(\mathcal{M}_{1,n}(\mathbb{P}^r,d))$, which vanishes from Lemma 4.33. Therefore, the whole page $E_1^{\mathrm{d}_{n,r,d}-1,\mathrm{d}_{n,r,d}-1}$ survives to E_∞ and forms a subquotient of $H_c^{2(\mathrm{d}_{n,r,d}-1)}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$.

The basis of $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ follows from combining the two subquotients of $H_c^{2(\mathrm{d}_{n,r,d}-1)}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ and dualising.

We turn to the Picard group of the mapping spaces. Recall that[7] $A^1_{\mathbb{Q}}(\mathrm{Pic}^d_{1,n}) = \langle\Theta\rangle$, then by projective bundle formula on $\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d) \to \mathrm{Pic}^d_{1,n}$, we have that $A_{\mathrm{d}_{n,r,d}-1}(\hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)) \cong \langle\Theta, H_{\hat{Q}}\rangle$. Therefore, by excision sequences for the open embedding $\mathcal{M}_{1,n}(\mathbb{P}^r,d) \subset \hat{\mathcal{Q}}_{1,n}(\mathbb{P}^r,d)$ as well as $\mathcal{M}_{1,n}(\mathbb{P}^r,d) \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, we have that $A^1_{\mathbb{Q}}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ is generated by the boundary divisors, H and Θ.

Let $\mathcal{B}$ be the vector space freely generated by the set of boundary divisors, $H_{\hat{Q}}$, and Θ. The previous result on the basis of $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d))$ gives an isomorphism $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)) \cong \mathcal{B}$, then the cycle class map composes to the identity $\mathrm{id}_{\mathcal{B}}$,

$$\mathcal{B} \twoheadrightarrow A^1_{\mathbb{Q}}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)) \to H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)) \cong \mathcal{B},$$

hence the cycle class map is an isomorphism. □

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