Topology of genus one mapping spaces to projective space

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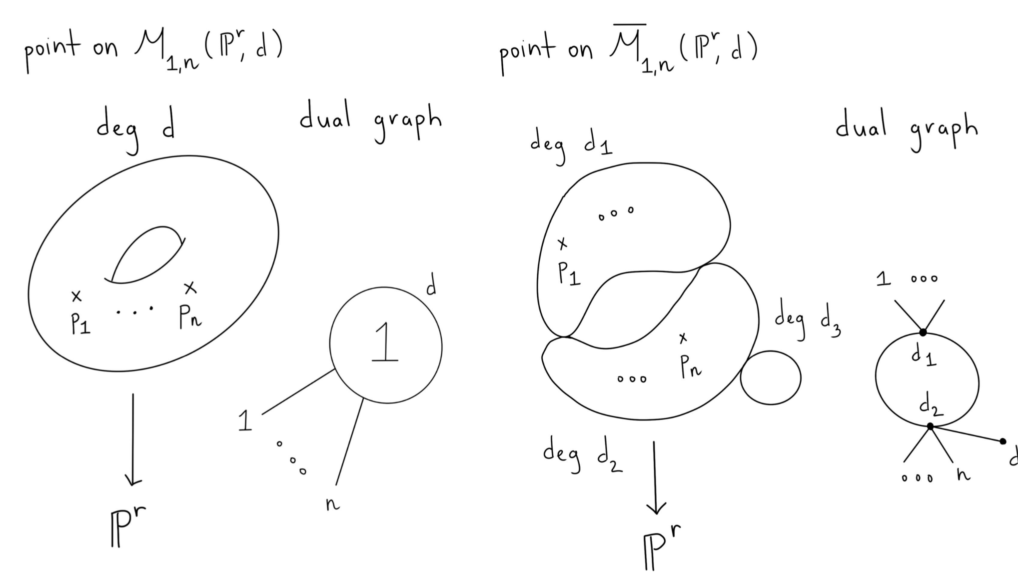
Summary

We study the topology of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, the Vakil–Zinger space of genus one maps to $\mathbb{P}^r$ [VZ08] by giving a list of generators of its rational cohomology $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ and classifying sources of potential relations.

Setup

- $\mathcal{M}_{1,n}(\mathbb{P}^r, d)$: moduli space of degree d maps from n-marked elliptic curves,
- $\mathcal{M}_{1,n}(\mathbb{P}^r, d) \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$: normal crossings compactification due to Vakil–Zinger [VZ08] as a desingularisation of the stable maps compactification $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$. It defines reduced genus one Gromov–Witten theory.



- Ranganathan–Santos–Parker–Wise [RSPW19]: $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is stratified by pairs $(\mathbf{G}, \rho)$. $\mathbf{G}$ a stable map dual graph and ρ a *central alignment* on vertices.

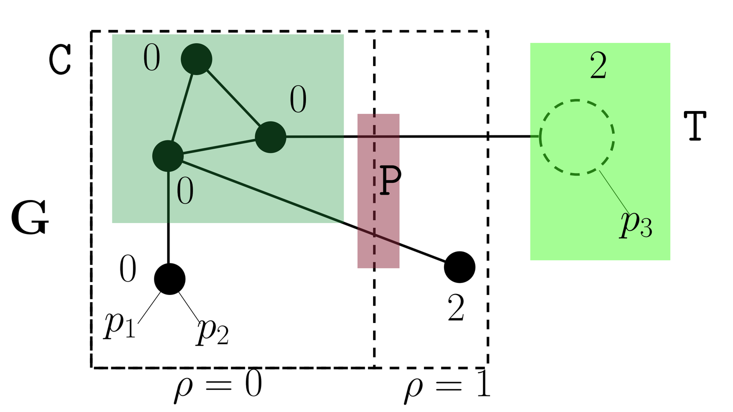


Figure 1. ρ assigns levels to vertices. The dotted circle denotes genus zero *stable maps*.

$\widetilde{\mathcal{M}}^\circ_{[\mathbf{G},\rho]} \subset \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ parametrises stable maps with dual graph $\mathbf{G}$ satisfying a smoothability condition imposed by ρ. The following classes live in $H^\star(\widetilde{\mathcal{M}}^\circ_{[\mathbf{G},\rho]})$:

C: Pullback from $H^\star(\mathcal{M}_{1,n})$, relative polarisation $\Theta \in H^2(\mathrm{Pic}^{d'}_{1,n})$ of the Picard group $\mathrm{Pic}^{d'}_{1,n} \to \mathcal{M}_{1,n}$, and $\hat{H} \in H^2(\mathcal{M}_{1,n}(\mathbb{P}^r, d))$ hyperplane class of the $(r+1)$-direct sum of complete linear system on $\mathrm{Pic}^{d'}_{1,n}$.

P: Pullback of ψ class on a univalent or bivalent genus zero vertex. ψ classes for vertices with higher valencies vanish on $\widetilde{\mathcal{M}}^\circ_{[\mathbf{G},\rho]}$.

T: Pullback of a class on the genus zero stable maps space.

Let $\overline{\mathcal{M}}_{[\mathbf{G},\rho]}$ be the closure of $\widetilde{\mathcal{M}}^\circ_{[\mathbf{G},\rho]}$. The above classes admit lifts to $H^\star(\overline{\mathcal{M}}_{[\mathbf{G},\rho]})$.

Generators of cohomology

Let $F \in H^\star(\overline{\mathcal{M}}_{[\mathbf{G},\rho]})$ be a polynomial of the classes listed previously, and let $\iota_{[\mathbf{G},\rho]} : H^\star(\overline{\mathcal{M}}_{[\mathbf{G},\rho]}) \to H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ be the Gysin pushforward.

Theorem. The cohomology $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ is generated by classes $\{\iota_{[\mathbf{G},\rho]} F\}$.

Further, we identify all sources of relations among these generators.

Corollary. $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ is generated by *tautological* cycles and $H^{2\star+1}(\overline{\mathcal{M}}_{1,n})$.

Applications: Hodge structures and Picard group

The Hodge structures in $H^\star(\overline{\mathcal{M}}_{1,n})$ are well understood by Getzler [Get99], Petersen [Pet14], and Canning–Larson–Payne [CLP24].

Corollary. The Hodge structures on $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ are products of $\mathsf{L} := H^2(\mathbb{P}^1)$ and $\mathsf{S}_{k+1} := W_k H^k(\mathcal{M}_{1,k})$ for $k \leq n + d$.

The cusp forms $\mathsf{S}_{k+1} \neq 0$ only for odd $k \geq 11$. By considering odd weight Hodge structures on $H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$, we recover the result of Fontanari [Fon07]:

Corollary. For $k < 11$ odd, $H^k(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ vanishes.

We also determine the Picard group of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$.

Theorem. $H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ has a *basis* given by Θ, $\hat{H}$, and the boundary divisors. The cycle class map $\mathrm{Pic}(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)) \to H^2(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d))$ is an isomorphism.

Step 1: stratification

- The stratification by $(\mathbf{G}, \rho)$ gives a spectral sequence

$$\underbrace{H^\star_c(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]})}_{\text{compact support}} \Rightarrow H^\star(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)).$$

- *Weight filtrations* on $H^\star(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]})$ are compatible with differentials, implying that:

  - Only pure weight quotient $\mathrm{gr}^W_\star H^\star_c(\mathcal{M}_{[\mathbf{G},\rho]})$ survives to the limit and gives generators.
  - Relations are images in the differentials, classified by subquotients of $\mathrm{gr}^W_{\star-1} H^\star_c(\mathcal{M}_{[\mathbf{G},\rho]})$.

The calculation of the $\mathrm{gr}^W_{\star-1} H^\star_c(\mathcal{M}_{[\mathbf{G},\rho]})$ suggests that for $k \ll r$, there are few relations among the generators of $H^k(\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, and that the cohomological degrees admitting relations grow *linearly* with r. We hope to derive a complete set of relations in future work.

Step 2: Weight graded pieces of strata $\mathrm{gr}^W_{\star(-1)} H^\star_c(\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]})$

- $\widetilde{\mathcal{M}}_{[\mathbf{G},\rho]}$ is a fibre product of maps from the genus one subcurve and *genus zero stable maps*: we hence separate the known classes from the latter.
- Maps from elliptic curves are handled via linear systems on their universal Picard groups, leveraging knowledge of $H^\star(\overline{\mathcal{M}}_{1,n})$ [Pet14, CLP24].
- Smoothability condition by ρ is equivalent to *linear dependency* conditions on differentials [BNR21, KS24a], which can be explicitly parametrised by adapting configuration space techniques on pointed maps from $\mathbb{P}^1$ to $\mathbb{P}^r$.
- The cohomology of fibre products and quotients are computed using the Leray spectral sequence, which is compatible with weight filtration [Ara05].

Related work

- We adapt stratification and weight filtration techniques of Arbarello–Cornalba [AC98], Petersen, and Canning–Larson–Payne to mapping spaces.
- The work is also a genus one extension of previous work on the topology of genus zero maps to the projective space [MM06, Opr06, Pan99], in which the complexity is already known to exceed that of $\overline{\mathcal{M}}_{0,n}$.

Future directions

- Joint with S. Kannan: calculate S_n-Euler characteristics of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, building on [KS24b, KK25] concerning $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ and aligned genus zero maps.
- Describe the cohomology of the genus two Vakil–Zinger style construction by Battistella–Carocci [BC23].
- Intersection theory calculations on $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r)$ analogous to [Pan99].

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