Graph enumeration for moduli spaces of curves and maps

Terry Song

University of Cambridge

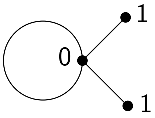
Joint work with Siddarth Kannan

Stratification on $\overline{\mathcal{M}}_{g,n}$

$\mathbb{G} = (G, w, m)$ decorated dual graph.
$\mathcal{M}_{\mathbb{G}} \subset \overline{\mathcal{M}}_{g,n}$ locally closed stratum.

$$\mathcal{M}_{\mathbb{G}} \cong \prod_{v \in V(G)} \mathcal{M}_{w(v),\mathrm{val}(v)} / \mathrm{Aut}(\mathbb{G}),$$

$\overline{\mathcal{M}}_{g,n}$ is stratified by $\mathcal{M}_{\mathbb{G}}$.
A stratum in $\overline{\mathcal{M}}_3$:



Task

Express cut-and-paste invariants of $\mathcal{M}_{\mathbb{G}}$ in terms of the invariants of each $\mathcal{M}_{w(v),\mathrm{val}(v)}$.

Serre characteristics

Task

Express cut-and-paste invariants of $\mathcal{M}_{\mathbb{G}} = \prod_{v \in V(G)} \mathcal{M}_{w(v),\mathrm{val}(v)}/\mathrm{Aut}(\mathbb{G})$ in terms of the invariants of each $\mathcal{M}_{w(v),\mathrm{val}(v)}$.

X a variety with S_n-action, then $H^i_c(X)$ is a mixed Hodge structure with S_n-action. The S_n-Serre (Hodge–Euler) characteristic

$$\mathrm{e}^{S_n}(X) := \sum_i (-1)^i [H^i_c(X)] \in K_0(\mathrm{MHS}_{S_n}) \cong K_0(\mathrm{MHS}) \otimes K_0(\mathrm{Rep}_{S_n})$$

They specialise to E-polynomials and Euler characteristics and their S_n-equivariant enrichments.

Generating functions

We work with the following generating functions.

Definition

$$\mathsf{b} = \sum_{g,n} \mathsf{e}^{S_n}(\overline{\mathcal{M}}_{g,n})t^{g-1}, \mathsf{b}_g = \sum_{n} \mathsf{e}^{S_n}(\overline{\mathcal{M}}_{g,n})t^{g-1},$$

$$\mathsf{a} = \sum_{g,n} \mathsf{e}^{S_n}(\mathcal{M}_{g,n})t^{g-1}, \mathsf{a}_g = \sum_{n} \mathsf{e}^{S_n}(\mathcal{M}_{g,n})t^{g-1}$$

take value $K_0(\mathsf{MHS}_{\mathbb{S}})(\!(t)\!)$ ($\mathbb{S} = \bigsqcup_{n\geq 0} S_n,$). Again,

$$K_0(\mathsf{MHS}_{\mathbb{S}}) \cong K_0(\mathsf{MHS}) \otimes \Lambda.$$

$\Lambda := \bigoplus_n K_0(\mathsf{Rep}_{S_n})$ is K_0 of symmetric sequences/$\mathbb{S}$-modules: collections $(\mathcal{V}(n))_{n\geq 0}$, $\mathcal{V}(n)$ an S_n-rep.

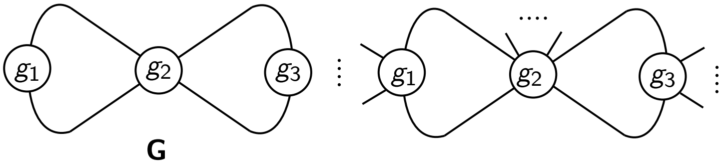
$$\Lambda \cong \mathbb{Q}\underbrace{[p_1, p_2, \dots,]}_{\text{power sums}} \cong \bigoplus_{\text{Integer partitions } \lambda} \mathbb{Q}\cdot s_\lambda \text{ (Schur functions)}$$

Graph strata formula

Let $\mathbf{G} = (G, w)$ be a genus decorated graph. Let

$$a_{\mathbf{G}} = \sum_{n \geq 0} e^{S_n} \Bigg(\bigsqcup_{\substack{\mathbb{G}=(G,w,m), \\ \#m=n}} \mathcal{M}_{\mathbb{G}} \Bigg) t^{g(\mathbf{G})-1}$$

be all strata in $\overline{\mathcal{M}}_{g,n}$ from attaching markings to $\mathbf{G}$.



Graph strata formula

Let $\mathbf{G} = (G, w)$ be a genus decorated graph. Let

$$\mathsf{a}_{\mathbf{G}} = \sum_{n \geq 0} \mathrm{e}^{S_n}\Big(\bigsqcup_{\substack{\mathbb{G}=(G,w,m),\\ \#m=n}} \mathcal{M}_{\mathbb{G}} \Big) t^{g(\mathbf{G})-1}$$

be all strata in $\overline{\mathcal{M}}_{g,n}$ from attaching markings to $\mathbf{G}$.

Theorem [Kannan–S. '25]

We explicitly define operators on symmetric functions $\mathfrak{D}_{\mathbf{G},\mathrm{Aut}(\mathbf{G})}$ such that

$$\mathsf{a}_{\mathbf{G}} = \mathfrak{D}_{\mathbf{G},\mathrm{Aut}(\mathbf{G})}(\{\mathsf{a}_{w(v)}, v \in V(G)\}).$$

The operators $\mathfrak{D}_{\mathbf{G},\mathrm{Aut}(\mathbf{G})} = \sum_{\sigma \in \mathrm{Aut}(\mathbf{G})} \mathfrak{D}_\sigma$ are graph-theoretic: each $\mathfrak{D}_\sigma$ only depends on its permutation cycle type on the map $H(G) \to V(G)$, where $H(G)$ = set of half-edges of G.

Formulas for $\overline{\mathcal{M}}_{g,n}$

Recall $\mathsf{b}_g = \sum_n \mathsf{e}^{S_n}(\overline{\mathcal{M}}_{g,n})t^{g-1}$, $\mathsf{a}_g = \sum_n \mathsf{e}^{S_n}(\mathcal{M}_{g,n})t^{g-1}$.

$$\mathsf{a}_{\mathbf{G}} = \mathfrak{D}_{\mathbf{G},\mathrm{Aut}(\mathbf{G})}(\{\mathsf{a}_{w(v)}, v \in V(G)\})$$

Grouping graphs $\mathbf{G}$ and automorphisms $\sigma \in \mathrm{Aut}(\mathbf{G})$ via the permutation cycle types of of $\mathrm{Aut}(\mathbf{G})$ on $H(G) \to V(G)$,

Theorem [Kannan–S.]

$$\mathsf{b}_g = \sum_{\underline{\Theta}:\text{cycle types}} K_g(\underline{\Theta}) \cdot t^{||\Theta||/2} \mathfrak{D}^{(\underline{\Theta})}(\mathsf{a}_0, \ldots, \mathsf{a}_g).$$

- The cycle types are recorded by tuples of maps $\{\text{integer partitions}\} \to \{\text{integer partitions}\}$.
- $\mathfrak{D}^{(\underline{\Theta})}$ are explicit operators on symmetric functions.
- $$K_g(\underline{\Theta}) := \sum_{\mathbf{G}} \frac{\#\{\sigma \in \mathrm{Aut}(\mathbf{G}) \text{ has type } \underline{\Theta}\}}{\mathrm{Aut}(\mathbf{G})}$$

  is a weighted count of graphs with their automorphisms.

Getzler–Kapranov formula

Theorem [Getzler–Kapranov, §8.13]

$$\mathrm{b} = \operatorname{Log}\left(\exp\left(\sum_{n\geq 1} t^n \left(\frac{n}{2}\frac{\partial^2}{\partial p_n^2} + \frac{\partial}{\partial p_{2n}}\right)\right)\operatorname{Exp}(\mathrm{a})\right),$$

$\operatorname{Log}, \operatorname{Exp}$, and $\partial/\partial p_n$ are operations on symmetric functions/representations.

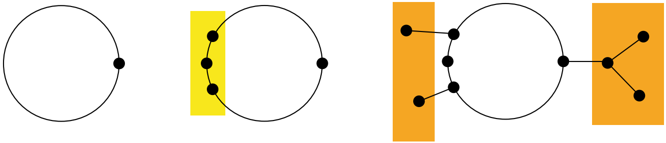
- Explicit formulas from Theorem: $\overline{\mathcal{M}}_{0,n}, \overline{\mathcal{M}}_{1,n}$ [Getzler '95-'98], $\overline{\mathcal{M}}_{2,n}$ [Diaconu '20], $\overline{\mathcal{M}}_{3,\leq 11}$ and $\overline{\mathcal{M}}_{4,\leq 3}$ [Faber, Bergström].
- Hard to extract formulas of each $\mathcal{M}_{\mathbb{G}}$.

Connection

We independently expand Getzler–Kapranov's formula and give the terms $K_g(\underline{\Theta})\mathfrak{D}^{(\underline{\Theta})}(\mathrm{a}_0, \ldots, \mathrm{a}_g)$ graph-theoretic interpretations.

Simplification

We may simplify our formulas by recording subdivisions and attachments by genus zero subgraphs (catepillars and rational tails)



After the steps, the formula for $\mathrm{b}_g = \sum_n \mathrm{e}^{S_n}(\overline{\mathcal{M}}_{g,n})$ is only in terms of the finitely many graphs in $\overline{\mathcal{M}}_g$ (rather than all the graphs underlying $\bigsqcup_n \overline{\mathcal{M}}_{g,n}$) and their automorphisms.

Applications

A large range of moduli spaces are recursively stratified by graphs: $(\bigsqcup_n \mathcal{M}_n) = \mathcal{M} \subset \overline{\mathcal{M}}$, and $\overline{\mathcal{M}} = \bigsqcup_{G\text{: graphs}} \mathcal{M}_G, \mathcal{M}_G = \left(\prod_{v \in V(G)} \mathcal{M}_{\mathrm{val}(G)}\right) / \mathrm{Aut}(G)$.

- Combinatorial subspaces of $\bigsqcup_{g,n} \overline{\mathcal{M}}_{g,n}$ such as stable curves of compact type (dual graph being a tree) $\bigsqcup_{g,n} \mathcal{M}^{\mathrm{ct}}_{g,n}$.
- Torus-fixed stable maps $\bigsqcup_{g,n,\beta} \overline{\mathcal{M}}_{g,n}(X,\beta)^{\mathbb{C}^\star}$, where X admits suitable $\mathbb{C}^\star$-action.
- Fulton–MacPherson spaces $\supset$ configuration spaces and trees of projective spaces of Chen–Gibney–Krashen.
- Admissible covers $\supset$ Hurwitz spaces.
- Compactified Jacobians $\supset$ universal Jacobian over $\bigsqcup_{g,n} \mathcal{M}_{g,n}$.

Sample calculations

For any $\mathbb{C}^\star$-space M (not necessarily smooth), $\chi(M) = \chi(M^{\mathbb{C}^\star})$.

g	$\chi\left(\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, 3)\right)$
0	$16\binom{r+1}{4} + 21\binom{r+1}{3} + 6\binom{r+1}{2}$
1	$216\binom{r+1}{4} + 247\binom{r+1}{3} + 55\binom{r+1}{2}$
2	$3160\binom{r+1}{4} + 3342\binom{r+1}{3} + 645\binom{r+1}{2}$
3	$44800\binom{r+1}{4} + 45114\binom{r+1}{3} + 8088\binom{r+1}{2}$
4	$630352\binom{r+1}{4} + 613213\binom{r+1}{3} + 104208\binom{r+1}{2}$

Sample calculations

Let $s_\lambda \in \Lambda$ denote the Schur function for the partition λ.
Let $\mathbb{L}$ be the Hodge structure on $H^2(\mathbb{P}^1) = H^2_c(\mathbb{A}^1)$. It is the specialisation of $[\mathbb{A}^1] \in K_0(\mathrm{Var})$.

n	$\mathrm{e}^{S_n}(\mathcal{M}^{\mathrm{ct}}_{3,n})$
0	$\mathbb{L}^6 + 2\mathbb{L}^5 + 2\mathbb{L}^4 + \mathbb{L}^3 + 1$
1	$(\mathbb{L}^7 + 4\mathbb{L}^6 + 7\mathbb{L}^5 + 4\mathbb{L}^4 + \mathbb{L}^3 + 1)s_1$
2	$(2\mathbb{L}^7 + 7\mathbb{L}^6 + 7\mathbb{L}^5 + \mathbb{L}^4 - 4\mathbb{L}^3 - 3\mathbb{L}^2)s_{11}$ $+(\mathbb{L}^8 + 6\mathbb{L}^7 + 17\mathbb{L}^6 + 17\mathbb{L}^5 + 6\mathbb{L}^4 - 2\mathbb{L}^2 + 1)s_2$
3	$(2\mathbb{L}^7 + 6\mathbb{L}^6 - 11\mathbb{L}^4 - 8\mathbb{L}^3 - 3\mathbb{L}^2 + 1)s_{111}$ $+(4\mathbb{L}^8 + 24\mathbb{L}^7 + 43\mathbb{L}^6 + 22\mathbb{L}^5 - 9\mathbb{L}^4 - 16\mathbb{L}^3 - 9\mathbb{L}^2 + 1)s_{21}$ $+(\mathbb{L}^9 + 8\mathbb{L}^8 + 32\mathbb{L}^7 + 52\mathbb{L}^6 + 32\mathbb{L}^5 + 6\mathbb{L}^4 - 6\mathbb{L}^3 - 5\mathbb{L}^2)s_3$

Sample calculations

n	$\mathrm{e}(\mathcal{M}_{3,n}^{\mathrm{ct}})$
0	$\mathbb{L}^6 + 2\mathbb{L}^5 + 2\mathbb{L}^4 + \mathbb{L}^3 + 1$
1	$\mathbb{L}^7 + 4\mathbb{L}^6 + 7\mathbb{L}^5 + 4\mathbb{L}^4 + \mathbb{L}^3 + 1$
2	$(2\mathbb{L}^7 + 7\mathbb{L}^6 + 7\mathbb{L}^5 + \mathbb{L}^4 - 4\mathbb{L}^3 - 3\mathbb{L}^2)$ $+(\mathbb{L}^8 + 6\mathbb{L}^7 + 17\mathbb{L}^6 + 17\mathbb{L}^5 + 6\mathbb{L}^4 - 2\mathbb{L}^2 + 1)$
3	$(2\mathbb{L}^7 + 6\mathbb{L}^6 - 11\mathbb{L}^4 - 8\mathbb{L}^3 - 3\mathbb{L}^2 + 1)$ $+(4\mathbb{L}^8 + 24\mathbb{L}^7 + 43\mathbb{L}^6 + 22\mathbb{L}^5 - 9\mathbb{L}^4 - 16\mathbb{L}^3 - 9\mathbb{L}^2 + 1) \cdot 2$ $+(\mathbb{L}^9 + 8\mathbb{L}^8 + 32\mathbb{L}^7 + 52\mathbb{L}^6 + 32\mathbb{L}^5 + 6\mathbb{L}^4 - 6\mathbb{L}^3 - 5\mathbb{L}^2)$

Sending $\mathbb{L} \mapsto uv \mapsto 1$ specialises to E-polynomials and Euler characteristics.