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SOLUTIONS TO SHEET I

Exercise 1. Notice that everything is on the level of $Z_\star(-)$.

(a) We factor $f \times g = (f \times \mathrm{id}_{Y'}) \circ (\mathrm{id}_{X'} \times g)$, as suggested by Fulton. This checks that $f \times g$ is proper/flat (with the correct relative dimension) when both maps are.

(b) and (c) The factorization also implies that it suffices to check the formulas (b), (c) assuming that one of f or g is the identity. Now we first check (c) and then (b).

(c) By linearity, we may assume $\alpha = [V], \beta = [W]$ for subvarieties $V \subset X', W \subset Y'$. We note that $i : W \hookrightarrow Y'$ satisfies that $(\mathrm{id}_{X'} \times i)_*([X'] \times [W]) = [X'] \times [W] \in Z_\star(X' \times Y)$ by definition. Therefore, by composing with $(\mathrm{id}_{X'} \times i)$ and $(\mathrm{id}_X \times i)$, we may assume $W = Y'$. Now consider the fibre diagram

$$\begin{tikzcd}
X' \times Y' \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\pi_{X'}"'] & X \times Y' \arrow[d, "\pi_X"] \\
X' \arrow[r, "f"'] & X
\end{tikzcd}$$

where the two vertical maps are the projections. We apply the projection formula (Prop 1.7) to $[W] \in Z_\star(X')$ so that $(f \times \mathrm{id})_*([W \times Y']) = [f(W) \times Y']$. By definition, $(f \times \mathrm{id})_*([W] \times [Y'])$ and $[f(W) \times Y'] = [f(W)] \times [Y']$, hence we are done.

(b) By definition, $(f \times \mathrm{id})^*([V] \times [W]) = (f \times \mathrm{id})^{-1}([V \times W]) = [f^{-1}(V) \times W]$. [By Lemma 1.7.1, the formula still makes sense when $V \times W$ is reducible, which can happen when k is not algebraically closed.]

(d) To check that the map is well-defined, it suffices to prove that $\alpha \sim 0 \Rightarrow \alpha \times \beta \sim 0$ and $\beta \sim 0 \Rightarrow \alpha \times \beta \sim 0$. By symmetry, we only need to check the first of the two. By linearity, suffices to assume $\beta = [W]$. Again, by composing with $W \hookrightarrow Y$, suffices to assume $W = Y$. In this case, $\alpha \times [Y] = \pi_X^{-1}(\alpha)$, and we know that $\alpha \sim 0$ implies that $\pi_X^{-1}(\alpha) \sim 0$ by Theorem 1.7.

Exercise 2. Let X admit a cell decomposition $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n$ and write $X_i \setminus X_{i-1} = \bigsqcup_j U_{ij}$ where $U_{ij} \cong \mathbb{A}^{n_{ij}}$ (abusing notation, we will simply use $\mathbb{A}^{n_{ij}}$ to denote U_{ij} in the following discussion). We perform induction on n. When $n = 0$, $X = X_0 \cong \bigsqcup_j \mathbb{A}^{n_{0j}}$, so $\bigoplus_{k+l=m} A_k X \otimes A_l Y$ is reduced to $\bigoplus_j A_{n_{0j}} \mathbb{A}^{n_{0j}} \otimes A_{m-n_{0j}} Y = \bigoplus_j \mathbb{Z} \cdot [\mathbb{A}^{n_{0j}}] \otimes A_{m-n_{0j}} Y$, and the product map is $\bigoplus_j A_{n_{0j}} \mathbb{A}^{n_{0j}} \otimes A_{m-n_{0j}} Y \to A_m(X \times Y)$ given by $[\mathbb{A}^{n_{0j}}] \otimes [V] \mapsto [\mathbb{A}^{n_{0j}}]$, which agrees with the flat pullback for $X \times Y \to Y$. From Prop 1.9, we know that the pullback is surjective.

For the induction, consider $X_{n-1} \subsetneq X$ such that $X \setminus X_{n-1}$ is a disjoint union of affine spaces (hence only one step of stratification) and X_{n-1} admits a cell decomposition with one fewer step $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1}$. Thus, both $X \setminus X_{n-1}$ and X_{n-1} satisfy the inductive hypothesis, so we get $\bigoplus_{k+l=m} A_k X_{n-1} \otimes A_l Y \to A_m(X_{n-1} \times Y)$ and $\bigoplus_{k+l=m} A_k(X \smallsetminus X_{n-1}) \otimes A_l Y \to A_m((X \smallsetminus X_{n-1}) \times Y)$ surjective. Now consider the excision sequences

$$\begin{tikzcd}
\bigoplus_{k+l=m} A_k(X_{n-1}) \otimes A_l(Y) \arrow[r] \arrow[d] & A_m(X_{n-1} \times Y) \arrow[d] \\
\bigoplus_{k+l=m} A_k(X) \otimes A_l(Y) \arrow[r] \arrow[d] & A_m(X \times Y) \arrow[d] \\
\bigoplus_{k+l=m} A_k(X \smallsetminus X_{n-1}) \otimes A_l(Y) \arrow[r] \arrow[d] & A_m((X \smallsetminus X_{n-1}) \times Y) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Diagram chase or snake lemma shows that the middle horizontal map is surjective as well.

[Compare this with Künneth formulas in homology or cohomology. Künneth decompositions (or even the surjection as in this statement) rarely hold for Chow groups.]

Exercise 3. The open subsets $\mathrm{G}_{T_{\underline{a}}}$ form an open cover of $\mathrm{Gr}(d, V)$: lying in $\mathrm{G}_{T_{\underline{a}}}$ is an open condition because it is non-vanishing of some determinant; they form a cover because every full rank $d \times m$ matrix can be rearranged to row echlon form using row operations to having some $d \times d$-minor equal to the identity matrix. Let $\underline{a}$ be the coordinates of the minor, then the point in $\mathrm{Gr}(d, V)$ represented by the $d \times m$ matrix lies in $\mathrm{G}_{T_{\underline{a}}}$. [Or simply: every full rank matrix has some minor with non-vanishing determinant.]

We have seen that each $\mathrm{G}_{T_{\underline{a}}}$ is isomorphic to some affine space, hence irreducible. To prove that $\mathrm{Gr}(d, V)$ is irreducible as well, it suffices to prove that the open subsets pairwise intersect. For any pair $\underline{a}_1, \underline{a}_2$, we can write down some $d \times m$-matrix such that both the $\underline{a}_1, \underline{a}_2$-minors have non-vanishing determinant. The point in $\mathrm{Gr}(d, V)$ represented by the $d \times m$ matrix then lies in $\mathrm{G}_{T_{\underline{a}_1}} \cap \mathrm{G}_{T_{\underline{a}_2}}$.

[Assuming the fact that $\mathrm{GL}(V)$ is smooth and irreducible, we can prove this by using the transitive $\mathrm{GL}(V)$-action on $\mathrm{Gr}(d, V)$.]

Exercise 4. *Abel–Jacobi argument.* Firstly, each Cartier divisor (section of $\mathscr{K}^*/\mathscr{O}^*$) is linearly equivalent to a Cartier divisor that is regular (lies in $\Gamma(\mathscr{O}^\star)$) in an open neighbourhood of the unique singular point p by rearranging local representatives. (We can avoid any single point on a scheme.) On the smooth curve $C \setminus p$, Weil divisors are the same as Cartier divisors, so we can always represent elements in $\mathrm{Pic}(C)$ by Weil divisors (zero cycles) supported on $C \setminus p$.

Fix some smooth inflection point $q_0 \in C$ and consider the map $C \setminus p \to \mathrm{Pic}^0(C)$ given by $q \mapsto q - q_0$. We claim that this is a bijection. Firstly, given $x \in C \setminus p$, we consider the line L_x between x and q_0 (when $x = q_0$, this is the tangent line). $L_x.C - x - q_0 = [o_x] \in A_0(C)$ where $[o_x]$ is the class of some smooth point $o_x \in C$. We have $o_x + x + q_0 = L_x.C \sim 3q_0$. In general, for three collinear points a, b, c on $C \setminus p$ (counted with multiplicity), we have $a + b + c \sim 3q_0$.

- Surjectivity: let $\sum_j n_j p_j \in \mathrm{Pic}^0(C)$, so that $\sum_j p_j = 0$, which implies that $\sum_j n_j p_j = \sum_j n_j (p_j - q_0)$. Using the relation $a + b + c \sim 3q_0$ for collinear a, b, c, we get $-(a - q_0) = q_0 - a \sim (b - q_0) + (c - q_0)$. Applying this relation on the negative items n_j, we may assume that all n_j is positive. Now we apply induction on the sum $\sum_j n_j$ to prove that for all $\{n_j \geq 0\}$: the base case is when there is only one $n_j = 1$, which is clear; given two items $(a - q_0) + (b - q_0)$, we consider the third point c of intersection of C with $L_{a,b}$, then consider o_c. By construction of o_c, $o_c + c + q_0 \sim 3q_0 \sim a + b + c$, from which we know that $a - q_0 + b - q_0 \sim o_c - q_0$, so we reduce the number of items by one.

- Injectivity: suppose $a - q_0 \sim b - q_0$, consider the o_b and the line between a and o_b, so let L_{a,o_b} intersect C at c. Then $a + o_b + c \sim 3q_0 \sim b + o_b + q_0$. Subtracting $o_b + 2q_0$ from both sides, $a - q_0 + c - q_0 \sim b - q_0$. Hence by assumption that $a - q_0 \sim b - q_0$, we have $c - q_0 \sim 0$. Suppose $c \neq q_0$, the fact that $c - q_0 \sim 0$ implies that there exists a degree one map to $\mathbb{P}^1$ (hence an iso), but we know that the two curves cannot be isomorphic. Therefore, $c = q_0$, so that $o_a = o_b$, then using the expression $a + o_a + q_0 \sim 3q_0 \sim b + o_b + q_0$, we get $a \sim b$. By the same rational function argument, $a \sim b$ implies that $a = b$, so the map is injective.

We will show in the next exercise that in the two examples, there is a group isomorphism between $C \smallsetminus p$ (which is isomorphic to some algebraic group over $\mathbb{C}$) and $\mathrm{Pic}^0(C)$ under the group law constructed above. This is true more generally, but we will not use or prove this fact.

Exercise 5. Describing the smooth locus.

Firstly, we prove that there is a unique singular point on both curves: by taking the partial derivatives $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ and setting them to zero for the two equations $F(x, y, z) = y^2z - x^2z - x^3$ and $F(x, y, z) = y^2z - x^3$, we see that both curves have a unique singular point at $[0 : 0 : 1]$.

We can prove that $C \setminus [0 : 0 : 1] \cong \mathbb{A}^1$ (for (a)) or $\cong \mathbb{G}_m$ (for case (b)) respectively. In case (a), notice that $C \setminus [0 : 0 : 1] = C \setminus \mathbb{V}(y)$, so $C \setminus [0 : 0 : 1]$ is the same as the affine curve in the chart $y \neq 0$, which has equation $z = x^3$. This affine curve is isomorphic to $\mathbb{A}^1$ by the isomorphism $f : \mathbb{A}^1_x \to \mathbb{A}^1_{x,z}$ given by $x \mapsto (x, x^3)$.

In case (b), the intersection of C with the equation $x = y$ is precisely $[0 : 0 : 1]$. Thus we consider $C \setminus p = C \setminus \mathbb{V}(x - y) \subset \mathbb{P}^2 \setminus \mathbb{V}(x - y)$, and we use coordinates $Y = \frac{y+x}{y-x}$ and $Z = \frac{z}{y-x}$ for the affine plane $\mathbb{P}^2 \setminus \mathbb{V}(x - y)$.

Rearranging the equation in $\mathbb{P}^2$, we get $x^3 = (y-x)(y+x)z$. Thus on the affine chart $\mathbb{P}^2 \setminus \mathbb{V}(x-y)$, we get $(Y - 1)^3/8 = YZ$. Calculating the coordinate ring of this affine variety (which is iso to $C \setminus p$), we get $\mathrm{Spec}(k[Y, Z]/\langle (Y - 1)^3/8 - YZ \rangle)$. Now we observe that $C \setminus p \subset \mathbb{A}^2_{Y,Z} \setminus \mathbb{V}(Y)$, this is because $y + x = 0$ then $[x : y : z] = [0 : 0 : 1] = p$. Therefore $\mathrm{Spec}(k[Y, Z]/\langle (Y - 1)^3/8 - YZ \rangle) \cong \mathrm{Spec}(k[Y, (Y - 1)^3/8Y]) = \mathrm{Spec}(k[Y^{\pm 1}]) \cong \mathbb{G}_m$, and a parametrisation is given by $\mathbb{G}_m \to C \smallsetminus p$, $Y \mapsto (Y, (Y - 1)^3/8Y)$.

Excision sequence.

Denote $Z = \{p\} \subset C$ as the unique singular point on C. Recall that $A_0(\mathbb{A}^1) = 0$, so excision sequence says that $A_0(\mathbb{G}_m) = 0$. Now consider the excision sequence $\mathbb{Z} \cong A_0(Z) \to A_0(C) \to A_0(C \setminus Z) \to 0$. Because we have proven that $C \setminus Z$ is isomorphic to $\mathbb{A}^1$ (case (a)) and $\mathbb{G}_m$ (case (b)), the excision sequence implies that $A_0(Z) \to A_0(C) \to 0$ is exact.

Consider the degree map $\deg : A_0(C) \to A_0(\mathrm{Spec}\mathbb{C}) = \mathbb{Z}$. The composition $A_0(Z) \to A_0(C) \xrightarrow{\deg} \mathbb{Z}$ is equal to $A_0(Z) \xrightarrow{\cong} A_0(\mathrm{Spec}\mathbb{C})$, which is an isomorphism. Hence $\deg : A_0(C) \to \mathbb{Z}$ is an isomorphism.

On the other hand, $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ factors through $\mathrm{Pic}(C) \to A_0(C) \to \mathbb{Z}$, so the kernel of $\mathrm{Pic}(C) \to A_0(C)$ is identified with $\mathrm{Pic}^0(C)$.

Working out $\mathrm{Pic}^0(C)$.

[Continuing the Abel–Jacobi argument] (a) and (b): We explain why the Abel–Jacobi argument in the last exercise applies to the two examples.

Recall that we proved that there is a unique singular point on both curves. Now we observe that the point $[0:1:0]$ is an inflection point on both curves. Recall that the tangent line at a point $\eta \in C$ is given by the (well-defined) equation $\mathbb{V}(\frac{\partial F}{\partial x}(\eta)x + \frac{\partial F}{\partial y}(\eta)y + \frac{\partial F}{\partial z}(\eta)z) = 0$. Calculating the partial derivatives at $\eta = [0:1:0]$, the tangent lines for both curves have equations $z = 0$, which has intersection multiplicity 3 with the respective curves. We now do the multiplicity calculation: this is a local question so we work with the affine chart $y \neq 0$, namely $\mathbb{A}^2_{x,z}$, in which the equations of the two curves become $z = x^3$ (for (a)) and $z = x^3 + x^2 z$ (for (b)). Now we calculate $\ell\left(\left(\frac{k[x,z]}{(z-x^3)}\right)_{(x,z)}/(z)\right)$ and $\ell\left(\left(\frac{k[x,z]}{(z-(x^3+x^2z))}\right)_{(x,z)}/(z)\right)$ which are both equal to $\ell\left((k[x]/(x^3))_{(x)}\right) = 3$. This shows that the point $[0:1:0]$ is indeed an inflection point. Therefore, the argument in the previous exercise applies to show that $\mathrm{Pic}^0(C)$ is in bijection to $C \smallsetminus [0:0:1]$.

Furthermore, the group law we have constructed in Exercise 4 satisfies that the sum of $f(t) = [t:1:t^3]$ and $f(u) = [u:1:u^3]$ (via the group operation) is equal to $[t+u:1:(t+u)^3]$ (again, we may work in the affine chart $y \neq 0$, then the line through $f(t)$ and $f(u)$ must intersect with the curve at a point with first coordinate $-(u+t)$ by considering sum of the roots, so it must be $f(-(u+t))$; by the same reasoning, the line between $f(-(u+t))$ and $p_0 = [0:1:0]$ must intersect with the curve at a point with first coordinate $(u+t)$, so the point itself is $f(u+t)$, which proves the claim).

Using the parametrisation for case (b) given earlier, we can explicitly calculate and check that the product of $f(Y_1)$ and $f(Y_2)$ is equal to $f(Y_1Y_2)$: this is because solving for the equation that $(Y_1, (Y_1-1)^3/8Y_1), (Y_2, (Y_2-1)^3/8Y_2), (Y_3, (Y_3-1)^3/8Y_3)$ being collinear leads to $Y_1Y_2Y_3 = 1$.

Sheaf cohomology argument. We now give an argument using sheaf cohomology, which does not need to use the existence of an inflection point on C [the fact that the existence of inflection points is true in general]. Recall that $\mathrm{Pic}(C)$ is the cohomology $H^1(C, \mathscr{O}^*)$. Consider the normalisation $\nu : \tilde{C} \to C$ and the sequence

$$0 \to \mathscr{O}_C^* \to \nu_* \mathscr{O}_{\tilde{C}}^* \to \mathcal{F} \to 0.$$

Checking stalks, we see that $\mathcal{F}$ is supported on the singular point $p \in C$. Applying sheaf cohomology long exact sequence, we get a short exact sequence:

$$0 \to H^0(C, \mathcal{F}) \to \mathrm{Pic}(C) \to \mathrm{Pic}(\tilde{C}) \to 0.$$

We notice $H^0(C, \mathscr{O}^*) \to H^0(\tilde{C}, \mathscr{O}^*)$ is the map between constant functions $\mathbb{G}_m \to \mathbb{G}_m$ which is an isomorphism, and $H^1(C, \mathcal{F}) = 0$ because $\mathcal{F}$ has zero-dim support.

We notice that the degree maps $\mathrm{Pic}(C) \to \mathbb{Z}$, $\mathrm{Pic}(\tilde{C}) \to \mathbb{Z}$ fits in the commutative diagram $\mathrm{Pic}(C) \to \mathrm{Pic}(\tilde{C}) \to \mathbb{Z}$. This implies that the map $\mathrm{Pic}(C) \to \mathrm{Pic}(\tilde{C})$ restricts to maps $\mathrm{Pic}^0(C) \to$

$\mathrm{Pic}^0(\tilde{C})$ so we get

$$0 \to H^0(C, \mathcal{F}) \to \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\tilde{C}) \to 0.$$

This way we present $\mathrm{Pic}^0(C)$ as the extension of the abelian variety $\mathrm{Pic}^0(\tilde{C})$ by the commutative group over $\mathbb{C}$ given by $H^0(C, \mathcal{F})$. In both (a) and (b), the normalisation is $\tilde{C} = \mathbb{P}^1$, so we know (from Part III AG) that $\mathrm{Pic}^0(\tilde{C})$ is trivial, so in fact $H^0(C, \mathcal{F}) \cong \mathrm{Pic}^0(C)$. Recall from Exercise 4 that we have $\mathrm{Pic}^0(\mathrm{C}) \cong H^0(C, \mathcal{F})$ as commutative group schemes over k. In the following, we will explicitly compute them in the two examples (a) and (b), which are isomorphic to $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively.

- (b): Since $\mathcal{F}$ is supported at the singular point, it suffices to work with local coordinates $p = (0,0) \in \mathbb{A}^2_{x,y}$. The equation can be factorised as $y^2 = x^2(x+1)$, and so restricting further to the open subset $\{x + 1 \neq 0\}$, we may instead work with $x^2 = y^2$, and after a linear change of variables to $xy = 0$. Thus we only need to work with the local ring $(\frac{k[x,y]}{(xy)})_{(x,y)}$ which normalises to $k[x]_{(x)} \times k[y]_{(y)}$ [integral closures commute with localisation]. The map $(\nu_* \mathscr{O}_{\tilde{C}})^*_p (\nu_* \mathscr{O}_{\tilde{C}})^*_p \to k^*$ given by $(f, g) \to f(p)/g(p)$ passes to an isomorphism $(\nu_* \mathscr{O}_{\tilde{C}})^*_p / \mathscr{O}^*_{C,p} \xrightarrow{\cong} k^*$: from normalisation exact sequence, we see that an invertible function on $(\nu_* \mathscr{O}_{\tilde{C}})^*_p$ descends to an invertible function on $\mathscr{O}^*_{C,p}$ if and only if their values on p agree.
- (a): Similarly, we work locally around $p = (0,0) \in \mathbb{A}^2_{x,y}$. The normalisation of $k[x,y]/(y^2 - x^3)$ is $k[x,y]/(y^2 - x^3) \to k[t]$ given by $x \mapsto t^3, y \mapsto t^2$. We take the completion along the t-adic topology $k[[t^2, t^3]]^\star \to k[[t]]^* \to k$ where the latter map is given by $f \mapsto f'(0)/f(0)$: it is a group homomorphism from the multiplicative group to the additive group. It is not hard to show the surjectivity of the latter map. Given $f = \sum_{i=0}^\infty f_i t^i$ where $f(0) \neq 0$, $f \mapsto 0$ if and only if $f_1 = 0$, hence the formal expansion of f has no t coefficient and comes from $k[[t^2, t^3]]$. Therefore, we proved that $0 \to k[[t^2, t^3]]^* \to k[[t]]^* \to k \to 0$ is an exact sequence of abelian groups (first two groups multiplicative, and k additive). This calculation implies that $k[[t]]^*/k[[t^2, t^3]]^* \cong (k, +)$.

  In the following, we identify the above computation on the level of completion with $H^0(\mathcal{F})$. We first observe $\mathcal{F}$ is supported on a single closed point $p \in C$, hence trivial (as a sheaf of abelian groups). Therefore, as an $\mathcal{O}_{C,p}$-module, $\mathcal{F}$ is isomorphic to $H^0(\mathcal{F})$ where the $\mathcal{O}_{C,p}$-module structure is given by the homomorphism $\mathcal{O}_{C,p} \to \mathcal{O}_{C,p}/\mathfrak{m}_p = k$. In the following, we may abuse notation and consider $\mathcal{F}$ as a sheaf on p itself. Therefore, consider the completion $\hat{\mathscr{O}}_{C,p}$ of $\mathscr{O}_{C,p}$ along $\mathfrak{m}_p$. The pullback $\hat{\mathcal{F}}$ of $\mathcal{F}$ along $\mathrm{Spec}(\hat{\mathscr{O}}_{C,p}) \to \mathrm{Spec}(\mathscr{O}_{C,p})$ is the $\hat{\mathscr{O}}_{C,p}$-module given by $H^0(\mathcal{F})$ along the homomorphism $\hat{\mathcal{O}}_{C,p} \to \mathcal{O}_{C,p} \to \mathcal{O}_{C,p}/\mathfrak{m}_p = k$. This means that $H^0(\mathcal{F}) = H^0(\mathrm{Spec}(\hat{\mathscr{O}}_{C,p}), \hat{\mathcal{F}})$. Our calculations before computed $H^0(\mathrm{Spec}(\hat{\mathscr{O}}_{C,p}), \hat{\mathcal{F}}) = k[[t]]^*/k[[t^2, t^3]]^* = (k, +)$.

In conclusion, we have calculated that $\mathrm{Pic}^0(C) \cong H^0(C, \mathcal{F})$ is isomorphic to $(k, +)$ in case (a) and $(k^\times = \mathbb{G}_m, \times)$ in case (b) . We have seen that $\ker(\mathrm{Pic}(C) \to A_0(C))$ is $\mathrm{Pic}^0(C)$ in both cases, so we have worked out the kernels.

(c) The surface X cut out by $\{z^2 = xy\} \subset \mathbb{A}^3$ is irreducible because it is the image of $\mathbb{A}^2 \to X$, $(s, t) \mapsto (s^2, t^2, st)$, and we can prove that the image of an irreducible topological space under a continuous map is irreducible (which is from Part II AG).

To compute $A_1(X)$, consider the line $L = \{x = z = 0\}$ and the excision sequence $A_1(L) \to A_1(X) \to A_1(X \setminus L) \to 0$. $X \setminus L$ admits a projection map $(x, y, z) \mapsto (x, z)$, which gives a $X \setminus L \cong \operatorname{Spec}(k[x^{\pm 1}, z])$, which is a UFD. This means every prime ideal of height 1 is principal, hence rationally equivalent to zero. Thus, $A_1(X \setminus L) = 0$, which implies that $A_1(X) = \langle [L] \rangle$. We prove that $[L] \neq 0$ and $2[L] = 0$, so that $A_1(X) = \mathbb{Z}/2\mathbb{Z}$ as desired.

- $[L] \neq 0$: as we are working with Weil divisors, it suffices to show that $[L]$ is not principal, ie., $\mathfrak{p} = \langle x, z \rangle$ is not principal. This can be done by taking $\mathfrak{p} \hookrightarrow \mathfrak{m} = \langle x, y, z \rangle \to \mathfrak{m}/\mathfrak{m}^2 = k \cdot \{\overline{x}, \overline{y}, \overline{z}\}$, for instance: write an element in $\mathfrak{p}$ as $p = x \cdot g(x, y, z) + z \cdot h(x, y, z)$, and write $g = \sum_{i \geq 0} g_i(x, y, z)$ where each g_i is homogeneous of degree i, then the image of p in the map $\mathfrak{p} \to \mathfrak{m}/\mathfrak{m}^2$ is equal to $g_0 \bar{x} + h_0 \bar{z}$. Therefore, the image of the map is equal to $\langle \bar{x}, \bar{z} \rangle$. On the other hand, if $\mathfrak{p} = \langle f \rangle$ is principal (with $f = \sum_{i \geq 0} f_i(x, y, z)$), then $f_0 = 0$ because $f \subset \langle x, z \rangle$. Therefore, by doing similar calculations as earlier, we can show that the image will be a one-dimensional linear subspace spanned by $\langle \overline{f_1(x, y, z)} \rangle$. (In the calculation, it's important to note that $f_0 = 0$.)
- $2[L] = 0$ because it is the divisor associated to x.

Here, we prove that there does not exist a Cartier divisor D on X such that $[D] = [L]$. Consider the Cartier divisor given by $D' = \{y = 0\} \subset \mathbb{A}^3$. Notice that the support of D' agrees with that of L'. We calculate that $[D'] = \ell_{\mathscr{O}_{X,L}}(\mathscr{O}_{X,L'}/(y)) \cdot [L'] = 2[L]$. The local calclation of the length is given by $\ell\left((\frac{\mathbb{C}[x,y,z]}{(z^2 - xy)})_{(y,z)}/(x) \right) = \ell\left((\mathbb{C}[y, z]/(z^2))_{(z)} \right) = 2$.

On the other hand, we can check that $D' \cdot L = [p] = [(0, 0, 0)] \in A_0(p)$. So if $[D] = [L]$ for some Cartier divisor D, then $p = D' \cdot L = D' \cdot D = 2D \cdot [L']$ which is a contradiction upon taking degrees.

Exercise 6. (a) There is the following short exact sequence of sheaves cutting out the hypersurface $X \cap H \subset X$: $0 \to \mathscr{O}_X(-m) \to \mathscr{O}_X \to \mathscr{O}_{X \cap H} \to 0$. We tensor it by the invertible sheaf $\mathscr{O}_X(t)$ (which is in particular flat) to get the short exact sequence $0 \to \mathscr{O}_X(t - m) \to \mathscr{O}_X(t) \to \mathscr{O}_{X \cap H}(t) \to 0$. Now consider the long exact sequence associated to this short exact sequence.

It is true in general[1] that for any projective scheme $Y \subset \mathbb{P}^n$ with ample line bundle $\mathscr{O}_Y(1) := \mathscr{O}_{\mathbb{P}^n}(1)|_Y$ and any line bundle L on Y, we have that $\exists N : \forall t > N, i > 0 : H^i(Y, \mathscr{O}_Y(t)) = 0$.

Therefore, the long exact sequence simplifies to

$$0 \to H^0(X, \mathcal{O}(t - m)) \to H^0(X, \mathcal{O}(t)) \to H^0(X \cap H, \mathcal{O}(t)) \to 0$$

implies that $P_{X \cap H}(t) = P_X(t) - P_X(t - m)$. Expanding the two leading terms in $P_X(t) - P_X(t - m)$ implies that $d_{k-1}(X \cap H) = m d_k(X)$.

(b) We perform induction on k, with the case $k = 0$ being clear since X is a point.

From the definition of $c_1(-) \cap -$: $c_1(\mathcal{O}(1)) \cap [X] = [X \cap H]$ for H a hyperplane not containing X, so that $X \cap H$ is a Cartier divisor in X. Iterating this construction, then (c) implies that $d_k(X) = d_{k-1}(X \cap H)$. By inductive hypothesis (applied on $X \cap H$ which has dimension $\dim k - 1$),

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[1] You are not expected to prove this fact in this course.

we have

$$d_k(X) = d_{k-1}(X \cap H) = \deg(\mathcal{O}(1)^{k-1} \cap (X \cap H)) = \deg(\mathcal{O}(1)^{k-1} \cap (X \cap \mathcal{O}(1))) = \deg(\mathcal{O}(1)^k \cap X).$$

Solutions to Sheet II

Exercise 1. (c) *Lemma.* Let E be a rank r vector bundle, and L a line bundle with short exact sequence $0 \to L \to E \to E' \to 0$, then there is a short exact sequence $0 \to \bigwedge^{p-1} E' \otimes L \to \bigwedge^p E \to \bigwedge^p E' \to 0$.

Proof. We explain why there are the maps claimed in the exact sequence. Firstly, $E \to E'$ induces $\bigwedge^p E \to \bigwedge^p E'$. On the other hand, consider $E^{\otimes p-1} \otimes L \to E^{\otimes p} \to \bigwedge^p E$; we can check explicitly that $\ker(E^{\otimes p-1} L \to \bigwedge^{p-1} E) \otimes L \subset \ker(E^{\otimes p-1} \otimes \to \bigwedge^p E)$, hence the map descends to $\bigwedge^{p-1} E \otimes L \to \bigwedge^p E$.

To check exactness it suffices to work on an affine open cover $\{U_i\}_{i \in I}$ which all three bundles are trivialised. Hence, we may assume that $E = E' \oplus L$. Applying wedge powers (performing linear algebra)

$$\bigwedge^p E = \bigwedge^p (E' \oplus L) = \left(\bigwedge^{p-1} E' \otimes \bigwedge^1 L \right) \oplus \bigwedge^p E',$$

so the above sequence simplifies to

$$0 \to \bigwedge^{p-1} E' \otimes L \to \left(\bigwedge^{p-1} E' \otimes \bigwedge^1 L \right) \oplus \bigwedge^p E' \to \bigwedge^p E' \to 0,$$

which is exact.

Proof of the formula. We perform induction on the rank r, and the rank 1 follows from definition. By the splitting principle, it suffices to prove the formula for vector bundles E such that $0 \to L \to E \to E' \to 0$ for some line bundle L. Let α_0 be the Chern root of L and $\alpha_1, \ldots, \alpha_{r-1}$ be the Chern roots of E'. From inductive hypothesis on rank, we have that

$$c_t(\bigwedge^{p-1} E') = \prod_{(0<)i_1 < \cdots < i_{p-1}} (1 + (a_{i_1} + \cdots + a_{i_{p-1}})t).$$

The tensor product formula for Chern classes states that for any rank r vector bundle $\mathcal{E}$ and line bundle E, we have

$$c_t(\mathcal{E} \otimes L) = \sum_{i=0}^{r} t^i c_t(L)^{r-i} c_i(\mathcal{E}).$$

Now Whitney formula says

$$c_t(E) = c_t(L) c_t(E') = (1 + \alpha_0 t) \prod_{i=1}^{r-1} (1 + \alpha_i t),$$

hence $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ are the Chern roots of E.

The short exact sequence $0 \to \bigwedge^{p-1} E' \otimes L \to \bigwedge^p E \to \bigwedge^p E' \to 0$ implies that

$$c_t(\bigwedge^p E) = c_t(\bigwedge^{p-1} E' \otimes L) \cdot c_t(\bigwedge^p E').$$

From inductive hypothesis,

$$c_t\left(\bigwedge^p E' \right) = \prod_{(0<)i_1 < \cdots < i_p} (1 + (a_{i_1} + \cdots + a_{i_{p-1}})t).$$

Now multiplying the two formulas on the right hand side, we get

$$\prod_{0 < i_1 < \cdots < i_{p-1}} (1 + (a_0 + a_{i_1} + \cdots + a_{i_{p-1}})t) \prod_{j_1 < \cdots < j_p, j_1 > 0} (1 + (a_{j_1} + \cdots + a_{i_{j_p}})t),$$

which is equal to $\prod_{(0 \leq) k_1 < \cdots < k_p} (1 + (a_{k_1} + \cdots + a_{k_p})t)$,

Exercise 2. Let $X = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$, then $\mathbb{P}^1 \to X$ given by $[a : b] \mapsto [a^2 : ab : b^2]$ is an isomorphism, with inverse given by gluing together $[x : y : z] \mapsto [x : y]$ and $[x : y : z] \mapsto [y : z]$, which agree on overlaps. Therefore, under the isomorphism $X = \mathbb{P}^1$ (itself) has degree one, but $X \subset \mathbb{P}^2$ has degree 2, because $\deg(X \subset \mathbb{P}^2) = \int_{\mathbb{P}^2} c_1(\mathscr{O}_{\mathbb{P}^2}(1)) \cap [X] = \int_{\mathbb{P}^2} c_1(\mathscr{O}_{\mathbb{P}^2}(1)) \cap c_1(\mathscr{O}_{\mathbb{P}^2}(2)) = 2$.

Exercise 3.

(1) Let $Z = \mathbb{A}^3_{x,y,z}$, $X = \mathbb{V}(x, y)$ and let $Y = \mathbb{V}(xy)$, then the pullback is $Z \subset Y$, which is the closed embedding in $\operatorname{Spec}(k[x, y, z]/(xy))$ given by the ideal $(\bar{x}, \bar{y})$ which is equal to the image of (x, y) under the quotient map $k[x, y, z] \to k[x, y, z]/(xy)$.

We claim that the closed embedding $Z \subset Y$ is not regular. Since $Z \subset Y$ has codimension one, it suffices to show that $Z \subset Y$ is not a Cartier divisor, or the ideal $(\bar{x}, \bar{y})$ is not principal.

Consider the k-linear projection $\pi : k[x, y] \to k \cdot \{x, y\}$ by taking the homogeneous degree one polynomial; it descends to $\pi : k[x, y]/(xy) \to k \cdot \{x, y\}$. Let $(\bar{f}) \subset (\bar{x}, \bar{y})$ and consider $\bar{g} \in (\bar{f})$, and pick lifts of $\bar{g}, \bar{f}$ to $g, f \in k[x, y]$ so that there exists $r \in k[x, y]$ such that $g = rf$. Since $\bar{f} \in (\bar{x}, \bar{y})$, any lift f (which only differs up to (xy)) has zero constant term, namely $f = \alpha x + \beta y + f_2$ for $f_2 \in (x, y)^2$; write $r = r_0 + \alpha' x + \beta' y + r_2$ where $r_0 \in k$ and $r_2 \in (x, y)^2$, we have $g = rf = r_0(\alpha x + \beta y) + g_2$ where $g_2 \in (x, y)^2$, hence the image of g under the projection map $k[x, y] \to k \cdot \{x, y\}$ is $r_0(\alpha x + \beta y)$ which is in the span of $\pi(f)$. Hence the image of projection from (f) to $k \cdot \{x, y\}$ gives a one-dim linear subspace of the image of $(\bar{x}, \bar{y})$ (which is the whole of $k \cdot \{x, y\}$), hence $(\bar{f}) \subsetneq (\bar{x}, \bar{y})$, so the ideal cannot be principal.

(2) Since the question is Zariski local on $z \in Z$, we may work with affine open charts of $Z \subset Y \subset X$, which correspond to $\operatorname{Spec}(R/J) \subset \operatorname{Spec}(R/I) \subset \operatorname{Spec}(R)$ for ideals $J \subset I$ in R.

(i) $\Rightarrow$ (ii): To prove this, it suffices to prove that if i, j are regular embeddings locally, then so is $j \circ i$. In the affine chart, this is equivalent to proving that: given $I = (x_1, \ldots, x_m)$ generated by a regular sequence $x_i \in R$ and let $y_1, \ldots, y_n \in R$ be such that $\bar{y}_1, \ldots, \bar{y}_n \in R/I$ form a regular sequence (which is the ideal $J/I \subset R/I$), then $x_1, \ldots, x_m, y_1, \ldots, y_n$ form a regular sequence in R. This is immediate from the definitions.

(ii) $\Rightarrow$ (iii): It is helpful to note that the sequence in question is the conormal exact sequence $0 \to i^*\mathcal{N}^\vee_{Y/X} \to \mathcal{N}^\vee_{Z/X} \to \mathcal{N}^\vee_{Z/Y} \to 0$ localized at $\mathscr{O}_{X,z}$. Since we have shown that $\mathcal{N}^\vee_{Z/Y}$ is locally free in lecture 8, it suffices to check that the sequence is exact, after which the local freeness of $\mathcal{N}^\vee_{Z/Y} \otimes \mathscr{O}_{X,z}$ implies that the sequence is locally split.

Right exactness is clear since $J^2 \subset I + J^2$ implies that $J/J^2 \to J/(I + J^2)$ is surjective. For exactness at J/J^2 term: we can see that $I \subset I + J^2$ implies that the composition $I \to J/J^2 \to J/(I + J^2)$ is zero; all $\bar{j} \in \ker(J/J^2 \to J/(I + J^2))$ are represented by some $j \in J$ such that $j \equiv i \mod J^2$ for some $i \in I$, so $\bar{j}$ is the image of $\bar{i} \in I/IJ$. This checks exactness at J/J^2.

It suffices to check that $I/IJ \to J/J^2$ is injective. We prove this by checking that $IJ = I \cap J^2$. It's clear to see that $IJ \subset I \cap J^2$. To prove that $I \cap J^2 \subset IJ$, we write $J/I = (\bar{g}_1, \ldots, \bar{g}_m) \subset R/I$ for some $g_i \in J$; by assumption $\bar{g}_1, \ldots, \bar{g}_m$ form a regular sequence. Now consider the exact sequence of Koszul complexes

$$0 \to K_\bullet(R, g_1, \ldots, g_m) \otimes I \to K_\bullet(R, g_1, \ldots, g_m) \to K_\bullet(A/I, \bar{g}_1, \ldots \bar{g}_m) \to 0.$$

Spelling out the terms in the Koszul complexes, we have

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^2 R^m \otimes I \arrow[r] \arrow[d] & \bigwedge^2 R^m \arrow[r, "\gamma"] \arrow[d, "\alpha"] & \bigwedge^2 (R/I)^m \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & I^m \arrow[r] \arrow[d, "\varphi_1"] & R^m \arrow[r, "\beta"] \arrow[d, "\varphi_2"] & (R/I)^m \arrow[r] \arrow[d, "\varphi_3"] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Snake lemma applied to the bottom two (non-zero) rows gives an exact sequence

$$\dots \to \ker(\varphi_2) \xrightarrow{\beta} \ker(\varphi_3) \xrightarrow{\alpha} \operatorname{coker}(\varphi_1) \to \operatorname{coker}(\varphi_2) \to \dots$$

Claim. $\alpha = 0$.

Proof. Since $(\bar{g}_1, \dots \bar{g}_m)$ forms a regular sequence, $H_1(K_\bullet(R/I, \bar{g}_1, \dots \bar{g}_m)) = 0$, hence $\ker(\varphi_3) = \operatorname{im}(\psi)$. Because γ is surjective, $\operatorname{im}(\psi) = \operatorname{im}(\psi \circ \gamma) = \operatorname{im}(\beta \circ \alpha) \subset \operatorname{im}(\beta)$. Therefore, $\ker(\varphi_3) \subset \operatorname{im}(\beta)$, which means that the term $\ker(\varphi_2) \xrightarrow{\beta} \ker(\varphi_3)$ is surjective, therefore $\alpha = 0$. ☐

Vanishing of α implies that $\operatorname{coker}(\varphi_1) \subset \operatorname{coker}(\varphi_2)$. By construction, $\operatorname{coker}(\varphi_1) = I/IJ$, and $\operatorname{coker}(\varphi_1) = A/(g_1, \dots, g_m)$, so $A/(g_1, \dots, g_m) \subset I/IJ$. In particular, for $\alpha \in I \cap J^2$, the class $\bar{\alpha} \in A/(g_1, \dots, g_m)$ is zero, so the class $\bar{\alpha} \in I/IJ$ is zero, hence $\alpha \in IJ$, which implies that $I \cap J^2 \subset IJ$ as desired.

(iii) ⇒ (i): We recall that because $j \circ i$ is a regular embedding, J/J^2 is free as an $\mathscr{O}_{X,z}/J$-module. By assumption the short exact sequence is split exact, we can pick $f_1, \dots, f_n, f_{n+1}, \dots, f_{n+m} \in J$ such that $f_1, \dots, f_n \in I$ and for all $z \in Z$, we have $f_1, \dots, f_n, f_{n+1}, \dots, f_{n+m} \in J_z/\mathfrak{m}_z J_z$ forms a basis as a $k(z)$-vector space.

Claim. $f_1, \dots, f_{n+m}$ is a regular sequence in R.

Proof. By assumption, J/J^2 is locally free of rank $n+m$, so let $h_1, \dots, h_n, h_{n+1}, \dots, h_{n+m}$ be a regular sequence for J, then again $h_1, \dots, h_n, h_{n+1}, \dots, h_{n+m}$ forms a $k(z)$-basis of $J_z/\mathfrak{m}_z J_z$. Write $f_i = \sum_j a_{ij} h_i$ for some $A = (a_{ij}) \in \operatorname{Mat}((n+m) \times (n+m), R)$. Note that because both $\{h_i\}$ and $\{f_j\}$ are bases of $J_z/\mathfrak{m}_z J_z$, the matrix $A \otimes k(z)$ is invertible. Thus, by shrinking to a suitable affine open neighbourhood of z, we may assume that $A \in \operatorname{Mat}((n+m) \times (n+m), R)$ is invertible. This implies that the Koszul complexes $K_\bullet(A, f_1, \dots, f_n, f_{n+1}, \dots, f_{n+m})$ and $K_\bullet(A, h_1, \dots, h_n, h_{n+1}, \dots, h_{n+m})$ are isomorphic, hence $f_1, \dots, f_{n+m}$ is a regular sequence for J in a neighbourhood around z. ☐

This implies also that $f_1, \dots, f_n$ forms a regular sequence for I and $\bar{f}_{n+1}, \dots, \bar{f}_{n+m}$ forms a regular sequence for J/I both in an open neighbourhood of z.

(3) To prove this, we localise at $z \in Z$ and consider the map of local rings $\mathscr{O}_{X,z} \to \mathscr{O}_{Z,z}$; both rings are regular local rings because X, Z are regular at z. It suffices to show that the kernel I of the map (namely the ideal that cut out $Z \subset X$ near z) is generated by a regular sequence. We find a set of generators of I, which turns out to be a regular sequence, by examining $(I + \mathfrak{m}_{X,z}^2)/\mathfrak{m}_{X,z}^2$ as follows.

Suppose $\dim \mathscr{O}_{X,z} = d, \dim \mathcal{O}_{Z,z} = d - c$. We observe that the image of $\mathfrak{m}_{X,z}$, which is $\mathfrak{m}_{X,z}/I$ is equal to the maximal ideal in $\mathscr{O}_{Z,z} = \mathscr{O}_{X,z}/I$. Let $k = \mathscr{O}_{X,z}/\mathfrak{m}_{X,z}$ be the residue field. Observe that there is a short exact sequence of k-vector spaces

$$0 \to (I + \mathfrak{m}_{X,z}^2)/\mathfrak{m}_{X,z}^2 \to \mathfrak{m}_{X,z}/\mathfrak{m}_{X,z}^2 \to (\mathfrak{m}_{X,z}/I)/(\mathfrak{m}_{X,z}/I)^2 \to 0,$$

which implies that

$$\dim_k((I + \mathfrak{m}_{X,z}^2)/\mathfrak{m}_{X,z}^2) = \dim_k(\mathfrak{m}_{X,z}/\mathfrak{m}_{X,z}^2) - \dim_k((\mathfrak{m}_{X,z}/I)/(\mathfrak{m}_{X,z}/I)^2) = c,$$

hence we can find $x_1, \dots, x_c \in I$ such that their images in $I + \mathfrak{m}_{X,z}^2/\mathfrak{m}_{X,z}^2$ form a k-basis and extend to $x_{c+1}, \dots, x_d$ as a k-basis of $\mathfrak{m}_{X,z}/\mathfrak{m}_{X,z}^2$: by Nakayama's lemma $x_1, \dots, x_d$ forms a minimal set of generators of $\mathfrak{m}_{X,z}$. Because $x_1, \dots, x_c$ has k-linearly independent images in $I + \mathfrak{m}_{X,z}^2/\mathfrak{m}_{X,z}^2$, the quotient ring $\mathscr{O}_{X,z}/(x_1, \dots, x_c)$ is a regular local ring of dimension $d - c$, and $(x_1, \dots, x_c) \subset I$ gives a surjection $\phi : \mathscr{O}_{X,z}/(x_1, \dots x_c) \to \mathscr{O}_{X,z}/I$ between two regular local rings of the same dimension. We claim that this map must be injective, hence an isomorphism: let $\bar{y} \in \ker(\phi)$, then because $\mathscr{O}_{X,z}/(x_1, \dots x_c)$ is a regular local ring, $\bar{y}$ is not a zero divisor. Hence $\dim \mathscr{O}_{X,z}/(x_1, \dots, x_c, y) \leq \dim \mathscr{O}_{X,z}/(x_1, \dots, x_c) - 1 = \dim \mathscr{O}_{X,z}/I - 1$, contradiction. Thus $I = (x_1, \dots, x_c)$, and $x_1, \dots, x_c$ forms a regular sequence (we can check that quotienting by each x_i cuts down $\mathscr{O}_{X,z}/(x_1, \dots, x_{i-1})$ by one from the same reasoning as earlier).

Exercise 4.

(1) We first recall the familiar short exact sequences

$$0 \to \mathscr{O}_{\mathbb{P}^m} \to \mathscr{O}_{\mathbb{P}^m}(1)^{\oplus m+1} \to \mathscr{T}_{\mathbb{P}^m} \to 0,$$

where $\mathscr{T}_{\mathbb{P}^m}$ denotes the tangent bundle of $\mathbb{P}^m$, and

$$0 \to \mathscr{T}_X \to \mathscr{T}_{\mathbb{P}^m}|_X \to N_{X/\mathbb{P}^m} \to 0.$$

Dualising both sequences, we get inclusions of vector bundles on X:

$$N^\vee \subset (\mathscr{T}_{\mathbb{P}^m}|_X)^\vee \subset \mathscr{O}_X(-1)^{\oplus m+1}.$$

Taking the projectivisations, there is the inclusion $\mathbb{P}N^\vee \subset \mathbb{P}(\mathscr{O}_X(-1)^{\oplus m+1}) \cong X \times \check{\mathbb{P}}^m$.

$\mathbb{P}N^\vee$ by definition parametrises sub-line bundles $0 \to L \to N^\vee$ of $N^\vee$, which upon dualising corresponds to quotient line bundles $N \to L^\vee \to 0$ of N. Because $N = (\mathscr{T}_{\mathbb{P}^m}|_X)/\mathscr{T}_X$, quotient line bundles of N are equivalent to codimension one subbundles $H \subset (\mathscr{T}_{\mathbb{P}^m}|_X)$ that contains $\mathscr{T}_X$, which recovers the definition of $\tilde{X} \to X$.

(2) We calculate the pullback of $\mathscr{O}_{\hat{\mathbb{P}}^m}(1)$ along the projection map $\mathbb{P}(\mathscr{O}_X(-1)^{\oplus m+1}) \cong X \times \hat{\mathbb{P}}^m \to \hat{\mathbb{P}}^m$.

Note that under the iso $X \times \mathbb{P}(k^{\oplus r+1}) = \mathbb{P}\mathscr{O}_X^{\oplus r+1}$, the pullback of $\mathscr{O}_{\mathbb{P}^r}(-1)$ along the projection $X \times \mathbb{P}(k^{\oplus r+1}) \to \mathbb{P}^r$ is equal to $\mathscr{O}_{\mathscr{O}_X^{\oplus r+1}}(-1)$.

The projection map $\mathbb{P}\mathscr{O}_X(-1)^{\oplus m+1} \to \mathbb{P}^m$ factors through $\mathbb{P}\mathscr{O}_X^{\oplus m+1} \to \mathbb{P}^m$ along the isomorphism $\varphi : \mathbb{P}\mathscr{O}_X(-1)^{\oplus m+1} \to \mathbb{P}\mathscr{O}_X^{\oplus m+1} = \mathbb{P}(\mathscr{O}_X^{\oplus m+1}(-1) \otimes \mathscr{O}_X(1))$, thus the pullback of $\mathscr{O}_{\mathbb{P}^m}(-1)$ is equal to

$$\varphi^* \mathscr{O}_{\mathscr{O}_X^{\oplus r+1}}(-1) = \mathscr{O}_{\mathscr{O}_X^{\oplus r+1}(-1)}(-1) \otimes \pi^* \mathscr{O}_X(1)$$

This formula is given in Fulton, B.5.5. Dualizing on both sides, we get

$$\varphi^* \mathscr{O}_{\mathscr{O}_X^{\oplus r+1}}(1) = \mathscr{O}_{\mathscr{O}_X^{\oplus r+1}(-1)}(1) \otimes \pi^* \mathscr{O}_X(-1).$$

Upon restricting to $N^\vee \subset \mathscr{O}_X(-1)^{\oplus m+1}$, we get that the pullback of $\mathscr{O}_{\mathbb{P}^{\hat{>}}}(1)$ is equal to $\mathscr{O}_{N^\vee}(1) \otimes \pi^*\mathscr{O}_X(-1)$.

(3) The second formula follows from the first one by definition of proper pushforward, so it suffices to prove the first one.

$$\deg f_*[\tilde{X}] = \int_{\hat{\mathbb{P}}^m} c_1(\mathscr{O}_{\hat{\mathbb{P}}^m}(1))^{m-1} \cap f_*[\tilde{X}] \text{ (definition)}$$

$$= \int_{\tilde{X}} c_1(f^*\mathscr{O}_{\hat{\mathbb{P}}^m}(1))^{m-1} \cap [\tilde{X}] \text{ (projection formula on } \tilde{X} \to \hat{\mathbb{P}}^m)$$

$$= \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^i \int_X c_1(\mathscr{O}_X(1))^i \cdot \pi_*(c_1(\mathscr{O}_{N^\vee}(1))^{m-1-i} \cap [\tilde{X}])$$

(formula for $f^*\mathscr{O}_{\hat{\mathbb{P}}^m}(1)$ and projection formula on $\tilde{X} \to X$)

$$= \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^i \int_X h^i \cdot (-1)^{n-i} s_{n-i}(N) \text{ (definition of the Segre class } \star))$$

$$= (-1)^n \int_X (1+h)^{m-1} \cdot (-1)^{m-1} s(N)$$

$$= (-1)^n \int_X (1+h)^{m-1} \cdot (-1)^{m-1} c(T_X)/(1+h)^{m+1} \text{ (segre classes are inverses of Chern classes)}$$

$$= (-1)^n \int_X c(T_X)/(1+h)^2.$$

Justification for $\star$: by definition

$$s_{n-i}(N^\vee) \cap [X] = \pi_*(c_1(\mathscr{O}_{N^\vee}(1))^{(m-n-1)+(n-i)} \cap [\tilde{X}]) = \pi_*(c_1(\mathscr{O}_{N^\vee}(1))^{m-i-1} \cap [\tilde{X}]).$$

Now we relate $s(N)$ to $s(N^\vee)$. Recall that $c_i(N^\vee) = (-1)^i c_i(N)$, so $c_t(N) = c_{-t}(N^\vee)$. Inverting both sides, $s_t(N) = s_{-t}(N^\vee)$, so that $s_i(N) = (-1)^i s_i(N^\vee)$. Combining the two facts, $\pi_*(c_1(\mathscr{O}_{N^\vee}(1))^{m-i-1} \cap [\tilde{X}]) = (-1)^{n-i} s_{n-1}(N)$ as desired.

(4) We consider the embedding $X \hookrightarrow \mathbb{P}^2 \hookrightarrow \mathbb{P}^m$. The map $i : X \hookrightarrow \mathbb{P}^2$ satisfies that $\deg(i^*\mathscr{O}_{\mathbb{P}^2}(1)) = d$ since X has degree d, and the Veronese embedding $\nu : \mathbb{P}^2 \hookrightarrow \mathbb{P}^m$ satisfies that $\deg(\nu^*\mathscr{O}_{\mathbb{P}^m}(1)) = r$, since a hyperplane on $\mathbb{P}^m$ pulls back to a degree r hypersurface on $\mathbb{P}^2$. Composing the two,

$$\deg((\nu \circ i)^*\mathscr{O}_{\mathbb{P}^m}(1)) = dr,$$

which computes $h = c_1(\mathscr{O}_X(1))$ in the above formula.

Implementing the formula proven above, $\deg f_*[\tilde{X}] = -\int_X (1 + c_1(T_X))/(1+h)^2 = -\deg(T_X - 2\mathscr{O}_X(1))$. The degree of the tangent bundle agrees with $\chi(X) = 2 - 2g(X) = 3d - d^2$ by degree-genus formula. Hence $\deg f_*[\tilde{X}] = d(d + 2r - 3)$.

Since the fibers of $\widetilde{X} \to X$ are projective spaces, $\widetilde{X}$ is irreducible, and thus $X^\vee$ is also irreducible. To conclude, we check that $f : \widetilde{X} \to X^\vee$ has $\deg(\widetilde{X}/X^\vee) = 1$.

Claim. To prove $\deg(\widetilde{X}/X^\vee) = 1$ it suffices to find a closed point $H \in X^\vee$ such that the fiber $f^{-1}(H)$ consists of a single reduced point.

Proof. Since $\widetilde{X}, X^\vee$ are both irreducible, $\deg(\widetilde{X}/X^\vee) = 1$ if and only if $R(X^\vee) \to R(\widetilde{X})$ is a degree one field extension. Recall from the lectures that there exists dense open subsets $U_1 \subset \widetilde{X}$ and $U_2 \subset X^\vee$ such that $U_1 = f^{-1}(U_2)$ and $f|_{U_1} : U_1 \to U_2$ is flat. It suffices to check that $U_1 \to U_2$ satisfies that $R(U_1) \to R(U_2)$ is a degree one field extension, which is again equivalent to $\deg(U_1/U_2) = 1$. From the lectures, we know that because $f = f|_{U_1} : U_1 \to U_2$ is finite and flat,

$f^* \circ f_* : A_0(U_1) \to A_0(U_1)$ is equal to multiplying by $\deg(U_1/U_2) = 1$. On the other hand, if such an H exists, we can compute explicitly that $f^* \circ f_*[H] = [f^{-1}(f(H))] = [H]$, so $\deg(U_1/U_2) = 1$ as desired. $\square$

Now we proceed to find such a $H \in X^\vee$. By Bertini's theorem, there exists $\Gamma \subseteq \mathbb{P}^m$ be a hyperplane meeting X in exactly rd reduced points $p_1, \ldots, p_{rd}$. If we can find H containing $T_{p_1}X$ but not $T_{p_i}X$ for $i > 1$, then $f^{-1}(H) = \{p_1\}$ as desired.

To see the existence of such an H, we perform a projective linear transformations to coordinates $Y_0, \ldots, Y_m$ on $\mathbb{P}^m$ such that $p_1 = [1 : 0 : \cdots : 0]$ and $T_{p_1}X$ is given by the line $\mathbb{V}(Y_2, \ldots, Y_m)$. If H is defined by

$$a_0 Y_0 + \cdots + a_m Y_m = 0,$$

we are requiring $a_0 = a_1 = 0$ and that $a_2 Y_2 + \cdots + a_m Y_m$ does not vanish at finitely many other points, namely $p_2, \ldots, p_{rd}$. The existence of such coefficients a_i is clear.

SOLUTIONS TO SHEET III

Question 1. The degree is equal to $\binom{r+s}{r}$.

Write $N = (r+1)(s+1) - 1$. By definition, the degree is equal to $\int_{\mathbb{P}^N} c_1(\mathscr{O}_{\mathbb{P}^N}(1))^{r+s} \cap [\sigma(\mathbb{P}^r \times \mathbb{P}^s)]$. Applying projection formula, it is equal to $\int_{\mathbb{P}^r \times \mathbb{P}^s} \sigma^*(c_1(\mathscr{O}_{\mathbb{P}^N}(1)))^{r+s} \cap [\mathbb{P}^r \times \mathbb{P}^s]$.

Write $[Y_0 : \cdots : Y_N]$ as the coorindates on $\mathbb{P}^N$, then we can explicitly compute that the Cartier divisor $\mathbb{V}(Y_0)$ pulls back to $\mathbb{V}(x_0 y_0)$, and

$$[\mathbb{V}(x_0 y_0)] = [\mathbb{V}(x_0)] + [\mathbb{V}(y_0)] = [H_1 \times \mathbb{P}^s] + [\mathbb{P}^r \times H_2] = \pi_1^*[H_1] + \pi_2^*[H_2],$$

where $H_1 \in A_{r-1}(\mathbb{P}^r)$ and $H_2 \in A_{s-1}(\mathbb{P}^s)$ are hyperplane classes. On the other hand, $[H_1] = c_1(\mathscr{O}_{\mathbb{P}^r}(1)) \cap [\mathbb{P}^r]$ and $[H_2] = c_1(\mathscr{O}_{\mathbb{P}^s}(1)) \cap [\mathbb{P}^s]$, so we know that $c_1(\sigma^* c_1(\mathscr{O}_{\mathbb{P}^N}(1))) = \pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1)) + \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1))$.

Hence we want to calculate

$$\int_{\mathbb{P}^r \times \mathbb{P}^s} (\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1)) + \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1)))^{r+s} \cap [\mathbb{P}^r \times \mathbb{P}^s] =$$
$$\sum_{i=o}^{r+s} \binom{r+s}{i} \int_{\mathbb{P}^r \times \mathbb{P}^s} \pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^i \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1))^{r+s-i} \cap [\mathbb{P}^r \times \mathbb{P}^s]$$

Let $H_1^{(i)} \in A_i(\mathbb{P}^r)$, $H_2^{(j)} \in A_j(\mathbb{P}^s)$ be i resp j-dimensional linear subspaces in $\mathbb{P}^r$ resp $\mathbb{P}^s$. We note that $\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^a \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1))^b \cap [\mathbb{P}^r \times \mathbb{P}^s] = [H_1^{(r-a)} \times H_2^{(s-b)}]$: to see this, we note that when $a = r$ or $b = s$ the formula holds by applying flat pull-back from π_1 or π_2, so we have

$$\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^a \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1))^b \cap [\mathbb{P}^r \times \mathbb{P}^s] = \pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^a \cap [\mathbb{P}^s \times H_2^{(s-b)}].$$

Now identify $H_2^{(s-b)} \cong \mathbb{P}^{s-b} \hookrightarrow \mathbb{P}^s$ and apply the projection formula on the diagram

$$\begin{tikzcd}
\mathbb{P}^r \times H_2^{(s-b)} \arrow[r] \arrow[d] & \mathbb{P}^r \times \mathbb{P}^s \arrow[d] \\
H_2^{(s-b)} \arrow[r] & \mathbb{P}^s
\end{tikzcd}$$

to get

$$\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^a \cap [\mathbb{P}^r \times H_2^{(s-b)}] = i'_*(\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^a \cap [\mathbb{P}^r \times \mathbb{P}^{(s-b)}]) = i'_*([H_1^{r-a} \times \mathbb{P}^{(s-b)}]) = [H_1^{r-a} \times H_2^{s-b}].$$

Thus

$$\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^a \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1))^b \cap [\mathbb{P}^r \times \mathbb{P}^s] \notin A_0(\mathbb{P}^r \times \mathbb{P}^s)$$

unless $a = r, b = s$, in which case

$$\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^r \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1))^s \cap [\mathbb{P}^r \times \mathbb{P}^s] = [H_1^{(0)} \times H_2^{(0)}],$$

which has degree one. Hence the degree is given by the coefficient of the monomial $\pi_1^* c_1(\mathscr{O}_{\mathbb{P}^r}(1))^r \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^s}(1))^s$, which is $\binom{r+s}{r}$.

Question 2.

(1) $\deg(V \subset \mathbb{P}^n) = \prod_{i=1}^m d_i$. This is known when $m - 1$. The inductive step is that from sheet 1 we know given a hypersurface H_d of degree d, $\deg(X \cap H_d) = d \deg(X)$.

(2) Let $\mathscr{I}$ be the homogeneous ideal that cuts out V, then the generators $F_1, \ldots, F_m$ form a map $\bigoplus_{i=1}^m \mathscr{O}_{\mathbb{P}^n}(-d_i) \to \mathscr{O}_{\mathbb{P}^n}$. We take the associated Koszul complex:

$$\cdots \to \bigwedge^2 \left(\bigoplus_i \mathscr{O}_{\mathbb{P}^n}(-d_i) \right) \xrightarrow{F=(F_1,\ldots,F_m)} \bigoplus_{i=1}^m \mathscr{O}_{\mathbb{P}^n}(-d_i) \to \mathscr{O}_{\mathbb{P}^n} \to 0.$$

By assumption the map $\bigoplus_{i=1}^m \mathscr{O}(-d_i) \to \mathscr{O}_{\mathbb{P}^n}$ surjects onto the subsheaf $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^n}$. Hence the following chain complex is exact:

$$\cdots \to \bigwedge^2 \left(\bigoplus_i \mathscr{O}_{\mathbb{P}^n}(-d_i) \right) \xrightarrow{F=(F_1,\ldots,F_m)} \bigoplus_{i=1}^m \mathscr{O}_{\mathbb{P}^n}(-d_i) \to \mathscr{I} \to 0$$

because $F_1, \ldots, F_m$ form a regular sequence.

[More generally, if $X \subset Y$ is the zero locus of a vector bundle section s of $\mathcal{E}$ on Y, the the Koszul complex reads

$$\cdots \to \bigwedge^i \mathcal{E}^\vee \to \cdots \to \mathcal{E}^\vee \to \mathscr{O}_Y \to \mathscr{O}_X \to 0.$$

Here we take $Y = \mathbb{P}^n, X = V$, and $\mathcal{E} = \bigoplus_{i=1}^m \mathscr{O}_{\mathbb{P}^n}(d_i)$. In the end, we take $\mathscr{I} = \ker(\mathscr{O}_Y \to \mathscr{O}_X)$ to get the exact sequence claimed above.]

Now apply $\otimes_{\mathscr{O}_{\mathbb{P}^n}} \mathscr{O}_X$ (which is $\otimes_{\mathscr{O}_{\mathbb{P}^n}} \mathscr{O}_{\mathbb{P}}^n / \mathscr{I}$) to get

$$\cdots \to \bigwedge^2 \left(\bigoplus_i \mathscr{O}_X(-d_i) \right) \xrightarrow{\bar{F}} \bigoplus_{i=1}^m \mathscr{O}_X(-d_i) \to \mathscr{I}/\mathscr{I}^2 \to 0$$

which is right exact. We note that $\bar{F}$ is zero because F_i vanish in $\mathscr{O}_X$, hence $\bigoplus_{i=1}^m \mathscr{O}_X(-d_i) \cong \mathscr{I}/\mathscr{I}^2$. Dualizing, we get $N_{V/\mathbb{P}^n} := (\mathscr{I}/\mathscr{I}^2)^\vee = \bigoplus_{i=1}^m \mathscr{O}_X(d_i)$.

Question 3.

(1) By definition $Z_*(X)$ only have classes $\leq d$ hence so are $A_*(X)$. Also $Z_d(X)$ is generated by $[X_1], \ldots, [X_m]$, so they generate $A_d(X)$, hence every class in $A_d(X)$ can be written in that form.

(2) Let $q : \mathbb{P}(C \oplus \mathbf{1}) \to X$, the definition of Segre class gives that $(e_X Y)_{X_i} \cdot [X_i] = q_*(c_1(\mathscr{O}(1))^{n-d} \cap [\mathbb{P}(C_{X_i} Y \oplus \mathbf{1})])$. Recall that $i : \mathbb{P}(C_{X_i} Y) \subset \mathbb{P}(C_{X_i} Y \oplus \mathbf{1})$ is a hyperplane section. Hence the above is equal to $(q \circ i)_*(c_1(\mathscr{O}(1))^{n-d-1} \cap [\mathbb{P}(C_{X_i} Y)]) = p_*(c_1(\mathscr{O}(1))^{n-d-1} \cap [\mathbb{P}(C_{X_i} Y)])$.

(3) Without loss of generality we may assume $Y_1, \ldots, Y_r$ are all the irreducible components of Y, then $s(X, Y) = \sum_{i=1}^r m_i s(X \cap Y_i, Y_i)$. Taking the $[V]$ coefficients on both sides give the desired equality.

(4) Abusing notation, also denote $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ as f. Then

$$f_* s(f^{-1}(V), Y') = \deg(Y'/Y) s(V, Y).$$

Taking the terms of $\dim V$ (or V''), we have $\sum_{V''} f_*(e_{V''} f^{-1}(V))[V''] = \deg(Y'/Y) e_V Y \cdot [V]$. The desired formula now follows from the definition that $e_{V''} f^{-1}(V) = e_{V''}(f)$.

(5)*

  (a) This follows from the fact that the n-th graded piece of $S_{Y,p}[z]$ is equal (as k-vector spaces) to the direct sum $\bigoplus_{i=0}^n \mathfrak{m}^i/\mathfrak{m}^{i+1}$.

  (b) In the last example sheet, we have seen that the Hilbert function of $(\operatorname{Proj}(S_{Y,p}[z]), \mathscr{O}(1))$ is eventually polynomial, hence the same holds for the Hilbert–Samuel function of $\mathscr{O}_{Y,p}$.

  We note that $(\operatorname{Proj}(S_{Y,p}[z]), \mathscr{O}(1)) = (\mathbb{P}(C_p Y \oplus \mathbf{1}), \mathscr{O}(1))$, which hence has pure dimension, of say ℓ. Thus, the degree of $(\operatorname{Proj}(S_{Y,p}[z]), \mathscr{O}(1))$ is given by $\int_{\operatorname{Proj}(S_{Y,p}[z])} c_1(\mathscr{O}(1))^\ell \cap$

$[\mathrm{Proj}(S_{Y,p}[z])]$ where $\ell = \dim_p Y$. Since the degree of $(\mathrm{Proj}(S_{Y,p}[z]), \mathscr{O}(1))$ coming from the Hilbert polynomial agrees with m_pY that is extracted from the Hilbert–Samuel function, the above integral gives a formula for m_pY.

(c) To conclude, we note that $(\mathrm{Proj}(S_{Y,p}[z]), \mathscr{O}(1)) = (\mathbb{P}(C_pY \oplus \mathbf{1}), \mathscr{O}(1))$, and that on $A_*(p) = A_0(p) = \mathbb{Z}$, by definition of the Segre class we have $e_pY = c_1(\mathscr{O}(1))^\ell \cap [\mathbb{P}(C_pY \oplus \mathbf{1})] = m_pY$ from above.

Question 4.

(1) Let $\mathscr{J}$ be the ideal sheaf cutting out J. The partial derivatives define sections $\mathscr{O}_X^{\oplus n+2} \to \mathscr{O}_X(d-1)$; twisting, we get $\mathscr{O}_X(1-d)^{\oplus n+2} \to \mathscr{O}_X$, and the image of this map is precisely $\mathscr{J}$; therefore, there is a surjection $\mathscr{O}_X(1-d)^{\oplus n+2} \to \mathscr{J}$ Taking symmetric algebras, we have a surjection $\mathrm{Sym}(\mathscr{O}_X(1-d)^{\oplus n+2}) \to \bigoplus_{n\geq 0} \mathscr{J}^{\oplus n}$, which induces a closed embedding $\tilde{X} \to \mathbb{P}_X(\mathscr{O}_X(d-1)^{\oplus n+2}) \cong \mathbb{P}^{n+1} \times X$ (as varieties), and the map $\tilde{X} \to \mathbb{P}^{n+1}$ is given by post-composing with the projection map to the second factor. Let $p : \mathbb{P}_X(\mathscr{O}_X(d-1)^{\oplus n+2}) \to \mathbb{P}^{n+1}$ be the projection, then from calculations analogous to sheet 2, we have $p^*\mathscr{O}_{\mathbb{P}^{n+1}}(1) \cong \pi_X^*\mathscr{O}(d-1) \otimes \mathscr{O}_{\mathscr{O}_X(d-1)^{\oplus n+2}}(1)$. Restricting to the closed subscheme $\tilde{X}$ gets us $\pi_X^*\mathscr{O}(d-1) \otimes \mathscr{O}_{\tilde{X}}(1)$.

(2) Now we calculate the degree

$$\begin{aligned}
\deg f_*[\tilde{X}] &= \int_{\mathbb{P}^{n+1}} c_1(\mathscr{O}_{\mathbb{P}^{n+1}}(1))^n \cap f_*[\tilde{X}] \text{ (definition)}\\
&= \int_{\tilde{X}} c_1(f^*\mathscr{O}_{\mathbb{P}^{n+1}}(1))^n \cap [\tilde{X}] \text{ (projection formula on } \tilde{X} \to \hat{\mathbb{P}}^m)\\
&= \sum_{i=0}^{n} \binom{n}{i} \int_X c_1(\mathscr{O}_X(d-1))^i \cdot \pi_*(c_1(\mathscr{O}_{\tilde{X}}(1))^{n-i} \cap [\tilde{X}])\\
&\quad \text{(formula for } f^*\mathscr{O}_{\hat{\mathbb{P}}^{n+1}}(1) \text{ and projection formula on } \tilde{X} \to X)\\
&= \sum_{i=0}^{n} \binom{n}{i} \int_X (d-1)^i c_1(\mathscr{O}_X(1))^i \cdot \pi_*(c_1(\mathscr{O}_{\tilde{X}}(1))^{n-i} \cap [\tilde{X}])\\
&\quad \text{(Using the fact that } c_1(\mathscr{O}_X(d-1)) = (d-1)c_1(\mathscr{O}_X(1)).)
\end{aligned}$$

We split into whether $n = i$ or not, when $n = i$ the term is equal to $\int_X (d-1)^n c_1(\mathscr{O}_X(1))^n \cap [X] = (d-1)^n \deg(X) = d(d-1)^n$. When $n > i$, we use the formulas that

$$c_1(\mathscr{O}_{\tilde{X}}(-1)) \cap [\tilde{X}] = [\mathbb{P}(C_JX)],$$

$$c_1(\mathscr{O}_{\tilde{X}}(1)) \cap [\mathbb{P}(C_JX \oplus \mathbf{1})] = [\mathbb{P}(C_JX)]$$

to compute that

$$c_1(\mathscr{O}_{\tilde{X}}(1))^{n-i} \cap [\tilde{X}] = -c_1(\mathscr{O}_{\tilde{X}}(1))^{n-i-1} \cap [\mathbb{P}(C_JX)] = -c_1(\mathscr{O}_{\tilde{X}}(1))^{n-i} \cap [\mathbb{P}(C_JX \oplus \mathbf{1})].$$

Note that by definition of Segre class, $\pi_*(c_1(\mathscr{O}_{\tilde{X}}(1))^{n-i} \cap [\mathbb{P}(C_JX \oplus \mathbf{1})])$ is equal to $s_i(J, X)$, namely the i-dimensional component of $s(J, X)$. In summary, the above sum is equal to

$$d(d-1)^n - \sum_{i=0}^{n-1} \int_X (d-1)^i c_1(\mathscr{O}_X(1))^i \cap s_i(J, X) = d(d-1)^n - \sum_{i=0}^{n-1} (d-1)^i \deg(s_i(J, X)).$$

(3) In this case $n = 1$, and $s(J, X) = s_0(J, X) = \sum_{p \in X} (e_JX)_p$ by definition of $(e_JX)_p$, hence the above formula simplifies to $\deg f_*[\tilde{X}] = d(d-1) - \sum_{p \in X} (e_JX)_p$.

Because the normalisation map $\nu : X' \to X$ is proper, $\nu_* s(\nu^{-1}(J), X') = s(J,, X)$. Hence taking coefficients at each point, we get $(e_J X)_p = \sum_{q \in \nu^{-1}(p)} \deg(\nu_*(e_{J'} X)_q) = \sum_{q \in \nu^{-1}(p)} (e_{J'} X)_q$; the second equality holds since $X' \to X$ is birational, so $\deg(X'/X) = 1$. Since X' is non-singular, the closed subscheme $J' = \nu^{-1}(J)$ is either empty (in which case there's nothing to prove) or a Cartier divisor. Then $s(J', X') = [J'] \in A_*(|J'|)$ is equal to $\sum_{q \in J'} \mathrm{ord}_{J'}(q)[q]$ by definition, so $\mathrm{ord}_{J'}(q) = (e_{J'} X)_q$, hence $(e_J X)_p = \sum_{q \in \nu^{-1}(p)} \mathrm{ord}_{J'}(q)$.

(4) For $X = \mathbb{V}(y^2 z - x^3)$, $J = \mathbb{V}(x^2, y^2, yz)$ and $|J| = \{[0 : 0 : 1]\}$. It suffices to use the affine chart $(0, 0) \in X^\circ := \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2$, so that $J = (y, x^2)$ and the normalisation map is given by $\mathbb{A}^1 \to X^\circ$ given by $x \mapsto t^2, y \mapsto t^3$. Hence $J' = \nu^{-1}(J) = (t^3)$, $|\nu^{-1}(J)| = \{0 \in \mathbb{A}^1\}$, and $\mathrm{ord}_{0 \in \mathbb{A}^1}(J') = 3$. Applying the formula ($d = 3$), we get that $\deg f_*[\tilde{X}] = 3$.

SOLUTIONS TO SHEET IV

Question 1. We follow the lecture notes from lecture 24. From the description of $A_*(X)$ as an additive group covered in earlier lectures, every $\beta \in A^pX = A_{n+r-1-p}(X)$ is uniquely written as

$$\beta = \sum_{i=1}^{r-1} \xi^i \cap q^* x_i$$

where $x_i \in A_{n-p+i}Y = A^{p-i}Y$. Therefore, we have a surjection $\Psi : A^*Y[\xi] \to A^*X$ given by $\sum_i \xi^i x_i \mapsto \sum_i \xi^i x_i$. Since $A_*(X)$ is isomorphic as an abelian group to $\bigoplus_{i=0}^{r-1} \xi^i \cap A_*(X)$, the kernel of Ψ is generated by a single element $\xi^r + a_1\xi^{r-1} + \cdots + a_r$ where $a_i \in A^iY$.

It suffices to prove that $a_i = c_i(E)$. To see this, we consider the short exact sequence of tautological bundles

$$0 \to \mathscr{O}_E(-1) \to p^*E \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ has rank $r-1$. Then $c_r(\mathcal{Q}) = 0$ for rank reason. On the other hand, $c_r(\mathcal{Q})$ is the degree r term of $c(\mathcal{Q}) = p^*c(E)/c(\mathscr{O}_E(-1))$, which has degree r term as $c_r(E) + c_{r-1}(E)\xi + \cdots + c_1(E)\xi + \xi^r$, hence the formula vanishes in $A^*(X)$, and this implies that $a_i = c_i(E)$.

Question 2. Let $Z_i = X \backslash U_i$, then $\iota^{(i)} : Z_i \subset X$ are closed subschemes. The excision sequence $A_*(Z_i) \xrightarrow{\iota_*^{(i)}} A_*(X) \to A_*(U_i) \to 0$ implies that $y_i = \iota_*^{(i)} z_i$ for $z_i \in A_*(Z_i)$. Therefore, $y_1 \cdots y_r = \Delta^*(\boldsymbol{\iota}(z_1 \times \cdots \times z_r))$ as given by the following commutative diagram.

$$\begin{tikzcd}
A_*Z_1 \times \cdots A_*Z_r \arrow[r, "\times"] \arrow[d, "(\iota_*^{(i)})_{i=1}^r"'] & A_*(Z_1 \times \cdots \times Z_r) \arrow[d, "\boldsymbol{\iota}"] & \\
A_*Y \times \cdots A_*Y \arrow[r, "\times"'] & A_*Y^r \arrow[r, "\Delta^*"'] & A_*Y
\end{tikzcd}$$

Because the scheme-theoretic intersection $Z_1 \cap \cdots \cap Z_r$ is the fibre product

$$\begin{tikzcd}
Z_1 \cap \cdots \cap Z_r \arrow[r] \arrow[d] & \prod_{i=1}^r Z_i \arrow[d, "\boldsymbol{\iota}"] \\
Y \arrow[r, "\Delta"'] & Y^r
\end{tikzcd}$$

the composition $y_1 \cdots y_r = \Delta^*(\boldsymbol{\iota}(z_1 \times \cdots \times z_r))$ is supported on $|Z_1 \cap \cdots \cap Z_r|$, hence it vanishes upon restricting to $U_1 \cup \cdots \cup U_r$. In particular, when $U_1 \cup \cdots \cup U_r = Y$, the class itself vanishes.

Question 3.

(1) We apply Question 1 on $Y = \mathbb{P}^n$ and $E = \mathscr{O}_Y^{\oplus m+1}$, then $X = \mathbb{P}E$ is the trivial projective bundle over Y, hence isomorphic to $\mathbb{P}^n \times \mathbb{P}^m$.

Let $\pi_1 : X \to \mathbb{P}^n$ and $\pi_2 : X \to \mathbb{P}^m$ be the projections. By definition of tautological line bundle, in this case $\mathscr{O}_E(1) = \pi_2^*\mathscr{O}_{\mathbb{P}^m}(1)$, hence $\xi = c_1(\mathscr{O}_E(1)) = \pi_2^* c_1(\mathscr{O}_{\mathbb{P}^m}(1))$.

Recall that $A^*(\mathbb{P}^n) \cong \mathbb{Z}[c_1(\mathscr{O}_{\mathbb{P}^m}(1)]/(c_1(\mathscr{O}_{\mathbb{P}^m}(1)^{n+1})$. Setting $s = \pi_1^* c_1(\mathscr{O}_{\mathbb{P}^n}(1))$ and $t = \xi$, the projective bundle formula says that

$$A^*(\mathbb{P}^n \times \mathbb{P}^m) \cong A^*(\mathbb{P}^n)[\xi]/(\xi^{m+1} + c_1(E)\xi + \cdots + c_r(E)) = \mathbb{Z}[s,t]/(s^{n+1}, t^{m+1} + c_1(E)t + \cdots + c_r(E)).$$

Because E is trivial, for all $i > 0$, $c_i(E) = 0$, so $A^*(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{Z}[s,t]/(s^{n+1}, t^{m+1})$ as desired.

(2) From the description of $A^*(\mathbb{P}^n \times \mathbb{P}^m)$ given above, $A_0(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z} \cdot [\mathrm{pt}] = \mathbb{Z} \cdot s^n t^m$. Thus the left hand side is equal to the $s^n t^m$ term of the polynomial $\prod_{i=1}^{n+m}(a_i s + b_i t)$. We expand

$$\prod_{i=1}^{n+m}(a_i s + b_i t) = \sum_{\sigma \subset \{1,\dots,n+m\}} \left(\prod_{j\in\sigma} a_j t\right)\left(\prod_{k\in\sigma^c} b_k s\right) = \sum_{\sigma \subset \{1,\dots,n+m\}} \left(\prod_{j\in\sigma} a_j\right)\left(\prod_{k\in\sigma^c} b_k\right) t^{|\sigma|} s^{|\sigma^c|}$$

and see that the $s^n t^m$ term is specified by the formula on the right hand side.

Question 4. We fix an isomorphism $(X \times Y)^2 \cong X^2 \times Y^2$.

Claim. The following diagram is commutative

$$\begin{tikzcd}
A_*(X^2) \times A_*(Y^2) \arrow[r, "\Delta_X^* \times \Delta_Y^*"] \arrow[d, "\times"'] & A_*(X) \times A_*(Y) \arrow[d, "\times"] \\
A_*(X^2 \times Y^2) \arrow[r, "\Delta^*_{X\times Y}"'] & A_*(X \times Y)
\end{tikzcd}$$

Proof of claim. Let $V_1 \subset X^2, V_2 \subset Y^2$ be subvarieties, and let $W_1 \subset X, W_2 \subset Y$ be the fibre products

$$W_1 := (\Delta_X) \times_{X^2} V_1, W_2 := (\Delta_Y) \times_{Y^2} V_2.$$

Then we have isomorphisms

$$(\Delta_{X\times Y}) \times_{X^2 \times Y^2} (V_1 \times V_2) \cong W_1 \times W_2,$$

$$C_{W_1 \times W_2}(X \times Y) \cong C_{W_1} X \boxtimes C_{W_2} Y.$$

Hence the Gysin pullback satisfies that

$$(\Delta_X.V_1) \times (\Delta_Y.V_2) = (s^*_{W_1}[C_{W_1}V_1]) \times (s^*_{W_2}[C_{W_2}V_2]) = s^*_{W_1 \times W_2}[C_{W_1 \times W_2} V_1 \times V_2] = (\Delta_{X\times Y}).(V_1 \times V_2).$$

Now pushing forward $[V_1] \times [V_2] \in A_*(V_1) \times A_*(V_2)$ along $A_*(V_1) \times V_*(V_2) \to A_*(X) \times A_*(Y)$ proves that the above diagram is commutative.

Given the claim, we prove that the above map is a ring homomorphism. Since we have seen that it is a group homomorphism, it suffices to prove multiplicativity. We use the notation $\star_X$ to denote multiplciation on $A^*(X)$ and similar for $Y, X \times Y$.

Let $W_{X,1}, W_{X,2} \subset X$ and $W_{Y,1}, W_{Y,2} \subset Y$ be subvarieties, then

$$\begin{aligned}
(W_{X,1} \star_X W_{X,2}) \times (W_{Y,1} \star_Y W_{Y,2}) &= \Delta_X^*(W_{X,1} \times W_{X,2}) \times \Delta_Y^*(W_{Y,1} \times W_{Y,2}) \text{ (definition of } \star_X, \star_Y) \\
&= \Delta^*_{X\times Y}(W_{X,1} \times W_{X,2} \times W_{Y,1} \times W_{Y,2}) \text{ (commutative diagram)} \\
&= (W_{X,1} \times W_{Y,1}) \star_{X\times Y} (W_{X,2} \times W_{Y,2}) \text{ (definition of } \star_{X\times Y})
\end{aligned}$$

So the map is multiplicative as desired.

The second two parts of the question follow from the fact that the surjectivity and isomorphism (resp.) have been proven on the level of abelian groups in previous example sheets.

Question 5.

(1) This is the same as proving the ring homomorphism $\mathbb{Z}[c_i(\mathcal{Q}) \mid i = 1, \dots, n-d] \to A^*(\mathrm{Gr}_d(E))$ given by

$$f(c_1, \cdots, c_{n-d}) \mapsto f(c_1(\mathcal{Q}), \cdots, c_{n-d}(\mathcal{Q})) \cap [\mathrm{Gr}_d(E)]$$

is surjective. To see this, recall that the classes $[\Omega(\underline{a})]$ form a basis of $A_*(\mathrm{Gr}_d(E))$ and each such class is given by $\Delta_\lambda \cap [\mathrm{Gr}_d(E)]$ where Δ_λ is a polynomial in the Chern classes $c_i(\mathcal{Q})$.

(2) Because of the short exact sequence

$$0 \to \mathcal{S} \to \mathscr{O}_{\mathrm{Gr}_d(E)}^{\oplus n} \to \mathcal{Q} \to 0,$$

there is $c(\mathcal{S}) = \frac{1}{c(\mathcal{Q})} = \sum_{k\geq 0}(-\sum_{i\geq 1} c_i(\mathcal{Q}))^k$. Because $\mathcal{S}$ is of rank d, $c_i(\mathcal{S}) = 0$ for $i = d{+}1, \ldots, n$, so for the same range of i, the homogeneous degree-i parts of $\sum_{k\geq 0}(-\sum_{i\geq 1} c_i(\mathcal{Q}))^k$ vanish.

In the case of $n = 4, d = 2$, the vector bundle $\mathcal{Q}$ has rank 2, so we expand (c_i denotes $c_1(\mathcal{Q})$)

$$\begin{aligned}\frac{1}{1+c_1+c_2} &= 1 - (c_1+c_2) + (c_1+c_2)^2 - (c_1+c_2)^3 + (c_1+c_2)^4 + \cdots \\ &= 1 - c_1 + (c_1^2 - c_2) + (-c_1^3 + 2c_1c_2) + (c_1^4 - 3c_1^2c_2 + c_2^2) + \cdots\end{aligned}$$

Picking out the homogeneous degree 3 and 4 terms, the relations are $c_1(\mathcal{Q})^3 - 2c_1(\mathcal{Q})c_2(\mathcal{Q})$ and $c_1(\mathcal{Q})^3 - 3c_1(\mathcal{Q})^2c_2(\mathcal{Q}) + c_2(\mathcal{Q})^2$.

(3) Let $d = 2, n = 4$, so from the above we know that $A^*(\mathrm{Gr}_d(E))$ is generated by $c_1(\mathcal{Q}), c_2(\mathcal{Q})$ in this case. Consider monomials of the form $\{c_1(\mathcal{Q})^i c_2(\mathcal{Q})^j\}$, the above two relations implies that a $\mathbb{Z}$-basis of $\mathbb{Z}[c_1, c_2]/I$ is given by $\{c_1^i c_2^j \mid (i,j) \in \{0,1,2\} \times \{0,1\}\}$, which has total rank 6.

On the other hand, the number of $\underline{a}$ appearing as $\Omega(\underline{a})$ is the same the number of size-d subsets of $\{1, \ldots, n\}$ (which are then ordered to give $0 < a_1 < \cdots < a_d \leq n$), which is $\binom{n}{d}$. Taking $n = 4, d = 2$, we get a rank 6 abelian group, so the map $\mathbb{Z}[c_1, c_2]/I \to A^*(\mathrm{Gr}_d(E))$ is an isomorphism in this case.