Part III Algebraic Geometry 2023 **Dhruv Ranganathan**

Main references for course

[Har] R. Hartshorne, Algebraic Geometry, Springer (1977). – *this is the main reference for the course.*

[V] R. Vakil, The Rising Sea: Foundations of Algebraic Geometry – *this can be used as a replacement, though some of the formal set up is different.* The latest version available at:

http://math.stanford.edu/~vakil/216blog/

[GW] Görtz and Wedhorn, Algebraic Geometry I, Schemes with examples and exercises, Vieweg+Teubner, 2010.

[EH] D. Eisenbud and J. Harris, The Geometry of Schemes, Springer (2001) – *this is great for examples and intuition*

Plan of course

§0 Brief review of classical Algebraic Geometry and motivation for scheme theory
§1 Sheaves on topological spaces
§2 Definition of schemes, basic properties and morphisms
§3 Locally free and coherent modules
§4 Sheaf cohomology

§0. Preliminaries on classical Algebraic Geometry

In this section, we make explicit basic concepts and results that provide context for the course; this material appeared in Part II Algebraic Geometry for Cambridge students. The course notes are available here:

https://www.dpmms.cam.ac.uk/~dr508/AGIINotes.pdf

The first section on Sheaf Theory will take several lectures and will only incidentally mention any algebraic geometry, so you will have time to look at §0.

A little classical algebraic geometry.
Throughout this discussion, we take the base field k to be **algebraically closed**.

Affine varieties: An *affine variety* $V \subseteq \mathbf{A}^n(k)$ (where, once one has chosen coordinates, $\mathbf{A}^n(k) = k^n$) is given by the vanishing of polynomials $f_1, \ldots, f_r \in k[X_1, \ldots, X_n]$.
If $I = \langle f_1, \ldots, f_r \rangle \triangleleft k[X_1, \ldots, X_n]$ is any ideal, we set

$$V = V(I) := \{z \in \mathbf{A}^n \ : \ f(z) = 0 \ \forall f \in I\}.$$

Projective varieties: First set $\mathbf{P}^n(k) := (k^{n+1} \setminus \{\mathbf{0}\})/k^*$ with *homogeneous coordinates* $(x_0 : x_1 : \ldots : x_n)$. A *projective variety* $V \subseteq \mathbf{P}^n$ is given by the vanishing of homogeneous polynomials $F_1, \ldots, F_r \in k[X_0, X_1, \ldots, X_n]$. If I is the ideal generated by the F_i (a *homogeneous* ideal, i.e. if $F \in I$, then so are all its homogeneous parts), we set

$$V = V(I) := \{z \in \mathbf{P}^n \ : \ F(z) = 0 \ \forall \text{ homogeneous } F \in I\}.$$

Coordinate ring of an affine variety.

If $V = V(I) \subseteq \mathbf{A}^n$, set

$$I(V) := \{f \in k[X_1, \ldots, X_n] \ : \ f(x) = 0 \ \forall x \in V\}.$$

Observe: $V(I(V)) = V$ (tautology) and $I(V(I)) \supseteq \sqrt{I}$ (obvious). Recall that the radical $\sqrt{I}$ of the ideal I is defined by $f \in \sqrt{I} \iff \exists m > 0$ s.t. $f^m \in I$.

Hilbert's Nullstellensatz (note $k = \bar{k}$): $I(V(I)) = \sqrt{I}$. ([R] §3, [AM] pp 82-3).

Coordinate ring: $k[V] := k[X_1, \ldots, X_n]/I(V)$. This may be regarded as the ring of polynomial functions on V, and it is a finitely generated reduced k-algebra. Recall that a k-algebra is a commutative ring containing k as a subring; it is finitely generated if it is the quotient of a polynomial ring over k, and *reduced* if $a^m = 0 \Rightarrow a = 0$.

Given an affine subvariety $W \subseteq V$, have $I(W) \supseteq I(V)$ defining an ideal of $k[V]$, also denoted $I(W) \triangleleft k[V]$.

Corollary of 0-satz: If $\mathbf{m}$ is a maximal ideal of $k[V]$, then $\mathbf{m} = \mathbf{m}_P$ for some $P \in V$, where $\mathbf{m}_P$ is the maximal ideal $\{f \in k[V] \ : \ f(P) = 0\}$.

Proof. 0-satz implies $I(V(\mathbf{m})) = \sqrt{\mathbf{m}} = \mathbf{m} \neq k[V]$. So $V(\mathbf{m}) \neq \emptyset$, since otherwise $I(V(\mathbf{m})) = k[V]$. Choose $P \in V(\mathbf{m})$; then $\mathbf{m}_p \supseteq \mathbf{m}$. Since $\mathbf{m}$ maximal, this implies $\mathbf{m}_P = \mathbf{m}$.

Observe that $\{P\} = V(\mathbf{m}_P) = V(\mathbf{m})$, and so there exists a natural bijection

$$\{\text{points of affine variety } V\} \longleftrightarrow \{\text{maximal ideals of } k[V]\} \qquad (\dagger)$$

Definition. A variety W is *irreducible* if there do not exist proper subvarieties W_1, W_2 of W with $W = W_1 \cup W_2$.

Lemma 0.1. A subvariety W of an affine variety V is irreducible $\iff$ $\mathcal{P} = I(W)$ is prime, i.e. $\iff$ $k[W]$ is an ID (integral domain).

Proof. ($\Rightarrow$) If $I(W)$ not prime, there exist $f, g \notin I(W)$ such that $fg \in I(W)$. Set $W_1 := V(f) \cap W$ and $W_2 := V(g) \cap W$; then W_1, W_2 are proper subvarieties with $W = W_1 \cup W_2$, i.e. W not irreducible.

($\Leftarrow$) If W_1, W_2 are proper subvarieties with $W = W_1 \cup W_2$, choose $f \in I(W_1) \backslash I(W)$ and $g \in I(W_2) \backslash I(W)$; then $fg \in I(W)$, i.e. $I(W)$ not prime.

For a projective variety $V \subseteq \mathbf{P}^n$, we let $I(V) \triangleleft k[X_0, X_1, \ldots, X_n]$ be the *homogeneous ideal* of V, by definition generated by the homogeneous polynomials vanishing on V.

Exercise. Show that a projective variety V is irreducible $\iff$ $I(V)$ is prime.
(($\Leftarrow$) as in (0.1), ($\Rightarrow$) by considering homogeneous parts of polynomials.) Generalizing (†), for V an affine variety, we have a bijection given by $W \mapsto I(W)$,

$$\{\text{irreducible subvarieties } W \text{ of an affine variety } V\} \longleftrightarrow \{\text{prime ideals of } k[V]\}.$$

Proof. Given a prime ideal $\mathcal{P} \triangleleft k[V]$, the Nullstellensatz implies $I(V(\mathcal{P})) = \sqrt{\mathcal{P}} = \mathcal{P}$ in $k[V]$, so there is an inverse map.

Projective Nullstellensatz. Suppose I is a homogeneous ideal in $k[X_0, X_1, \ldots, X_n]$ and $V = V(I) \subseteq \mathbf{P}^n$. The Projective Nullstellensatz ([R] p82) says:
If $\sqrt{I} \neq \langle X_0, X_1, \ldots, X_n \rangle$ (the *irrelevant* ideal), then $I(V) = \sqrt{I}$.

Proof. An easy deduction from the Affine Nullstellensatz, noting that I also defines an affine variety in $\mathbf{A}^{n+1}$, the *affine cone* on the projective variety $V \subseteq \mathbf{P}^n$. **Decomposition of variety into irreducible components.**

For V an affine or projective variety, there is a decomposition $V = V_1 \cup \ldots \cup V_N$ with the V_i irreducible subvarieties and the decomposition is essentially unique.

Proof. Suppose V is affine (similar argument for V projective): If there does not exist such a finite decomposition in the above form, then there exists a strictly decreasing sequence of subvarieties

$$V = V_0 \supset V_1 \supset V_2 \supset \ldots.$$

(If $V = W \cup W'$, then at least one of W, W' has no such decomposition and let this be V_1; continue in same way using Countable Axiom of Choice to obtain sequence.)

Hence in $k[V]$, $0 = I(V_0) \subseteq I(V_1) \subseteq \ldots$. Hilbert's Basis Theorem implies that there exists N such that $I(V_{N+r}) = I(V_N)$ for all $r \geq 0$. Hence $V_{N+r} = V(I(V_{N+r})) = V(I(V_N)) = V_N$ for all $r \geq 0$, a contradiction.

An easy "topological" argument ([R] Exercise 3.8, [W]) with the Zariski topology (see below) shows that the decomposition is essentially unique.

Zariski topology. Let V be a variety (affine or projective), then the *Zariski topology* is the topology on V whose closed sets are the subvarieties. This is the underlying topology for this course

We check this is a topology. Wlog take V affine. Clearly V and $\emptyset$ are closed. Observe that for ideals $(I_\alpha)_{\alpha \in A}$ of $k[V]$, we have $V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$ is closed. Finally we claim for ideals I, J of $k[V]$ that $V(IJ) = V(I) \cup V(J)$ $(= V(I \cap J))$ is closed.

Proof. Clearly $V(IJ) \supseteq V(I \cap J) \supseteq V(I) \cup V(J)$. Suppose however there exists a point $P \in V(IJ) \setminus (V(I) \cup V(J))$: we can choose $f \in I$ such that $f(P) \neq 0$ and $g \in J$ such that $g(P) \neq 0$. Then $fg \in IJ$ with non-zero value at P, a contradiction.

Note that V being irreducible as a topological space corresponds to the previous definition. Also, we have a well-defined concept of connectedness.

When V is affine, we have a basis of open sets of the form $D(f)$ for $f \in k[V]$, where $D(f) := \{x \in V \ : \ f(x) \neq 0\}$; any *open* set is of the form $V \setminus V(f_1, \ldots, f_r) = \bigcup_{i=1}^r D(f_i)$. If $V = \mathbf{A}^1$, get *cofinite* topology; in fact Zariski topology is only Hausdorff for a finite set of points. For V projective, we have a basis of open sets of the form $D(F) = V \setminus V(F)$, for F a homogeneous polynomial.

Exercise. The Zariski topology is compact in the usual sense (called precompact in some terminology since it is not Hausdorff), i.e. any open cover of V has a finite subcover.

Function fields of irreducible varieties

If V is an *irreducible* affine variety, then the *field of rational functions* or the *function field* $k(V) :=$ fof $k[V]$. Here $k[V]$ is an integral domain and fof denotes the field of fractions. In fact, we define the *dimension* of V by $\dim V := \operatorname{tr\,deg}_k k(V)$.

For $V \subseteq \mathbf{P}^n$ an irreducible projective variety, we define

$$k(V) := \{F/G \ : \ F, G \text{ homogeneous polynomials of the same degree, } G \notin I(V)\}/\sim$$

where the zero polynomial has any degree and where $F_1/G_1 \sim F_2/G_2 \iff F_1G_2 - F_2G_1 \in I(V)$. Need V irreducible here, i.e. $I(V)$ prime, to show that $\sim$ is transitive, and hence an equivalence relation.

If $V \subseteq \mathbf{P}^n$ an irreducible projective variety and U a non-empty affine piece of V (say $U = V \cap \{X_0 \neq 0\}$), then U is an affine variety, $U \subseteq \mathbf{A}^n$ with affine coordinates $x_i = X_i/X_0$ for $i = 1, \ldots, n$, the equations for U coming from those for V by "putting $X_0 = 1$". It is an easy check now that U is irreducible and $k(V) \cong k(U)$, the isomorphism being given by "putting $X_0 = 1$".

We say that $h \in k(V)$ is *regular* at $P \in V$ if it can be written as a quotient f/g with $f, g \in k[V], g(P) \neq 0$ (affine case), or F/G with F, G homogeneous polynomials of the same degree, $G(P) \neq 0$ (projective case).

Define $\mathcal{O}_{V,P} := \{h \in k(V) \ : \ h \text{ regular at } P\}$, the *local ring of V at P,* with maximal ideal $\mathbf{m}_{V,P} := \{h \in \mathcal{O}_{V,P} \ : \ h(P) = 0\}$, the kernel of the *evaluation map* $\mathcal{O}_{V,P} \to k$ given by evaluation at P. $\mathcal{O}_{V,P}$ is a *local ring,* i.e. $\mathbf{m}_{V,P}$ is the unique maximal ideal. Since $\mathcal{O}_{V,P} \setminus \mathbf{m}_{V,P}$ consists of units of $\mathcal{O}_{V,P}$ and any proper ideal consists of non-units, any proper ideal is contained in $\mathbf{m}_{V,P}$, and hence $\mathbf{m}_{V,P}$ is the unique maximal ideal.

Morphisms of affine varieties

For $V \subseteq \mathbf{A}^n$, $W \subseteq \mathbf{A}^m$, a morphism $\phi : V \to W$ is a map given by elements $\phi_1, \dots, \phi_m \in k[V]$. This yields a k-algebra homomorphism $\phi^* : k[W] \to k[V]$ (where $\phi^*(f) = f \circ \phi$; so if y_j a coordinate function on W induced from polynomial Y_j, we have $\phi^*(y_j) = \phi_j$). Conversely, given a k-algebra homomorphism $\alpha : k[W] \to k[V]$, we define a morphism $\alpha^* = \psi : V \to W$ given by elements $\alpha(y_1), \dots, \alpha(y_m) \in k[V]$. Note that $\psi(P)$ is in W, since for all $g \in I(W)$, $g(\psi(P)) = g(\alpha(y_1), \dots, \alpha(y_m))(P) = (\alpha(g(y_1, \dots, y_m)))(P) = 0$ since $g(y_1, \dots, y_m) = 0$ in $k[W]$.

Observe: For $\phi : V \to W$, we have $\phi^{**} = \phi$; for $\alpha : k[W] \to k[V]$, we have $\alpha^{**} = \alpha$. For $\psi : U \to V$ also a morphism of affine varieties, we have $\phi\psi$ a morphism $U \to W$ with $(\phi\psi)^* = \psi^*\phi^*$. For $\beta : k[V] \to k[U]$ a morphism of k-algebras, we have $(\beta\alpha)^* = \alpha^*\beta^*$.

We deduce that affine varieties V, W are *isomorphic* (i.e. there is an invertible morphism between them) $V \cong W \iff k[W] \cong k[V]$ as k-algebras. Recall: the k-algebras which occur as coordinate rings are the finitely generated reduced k-algebras. So formally, there is an equivalence of categories between the category of affine varieties over k and their morphisms, and the opposite of the category of finitely generated reduced k-algebras and their morphisms, i.e. there is a contravariant equivalence between the category of affine varieties and the category of finitely generated reduced k-algebras.

Thus affine algebraic geometry over k is a branch of commutative algebra. Commutative Algebra may be interpreted as affine algebraic geometry once one has generalized varieties to *schemes*.

For (irreducible) *affine* varieties, we can reconstruct the variety (up to isomorphism) from its ring of everywhere regular rational functions by (0.2) below; for irreducible projective varieties, the only everywhere regular rational functions are the constants (see Corollary 2 to Proposition 2.2).

Lemma 0.2. For V an irreducible affine variety,

$$\{f \in k(V) \ : \ f \text{ regular everywhere}\} = k[V].$$

Proof. Exercise.