Logarithmic Gromov–Witten cycles from toric varieties

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Let X be a toric variety equipped with its toric logarithmic structure. In Molcho–R, we proved that after fixing the discrete data, the cycle of logarithmic maps from curves to the rubber target geometry X with respect to its full toric boundary lies in the tautological ring of the moduli space of curves [8, Theorem A]. In this note, I sketch a strengthening of this result, for arbitrary Gromov–Witten cycles from X itself, rather than only it's rubber geometry. The strategy is to observe that the moduli space of rubber stable maps admits evaluation maps to a rubber evaluation space, and is compatible with the evaluation and virtual structures in the non-rubber geometries. These "rubber evaluations" appear to be natural in the logarithmic context, and are closely related to Maulik–Pandharipande's rubber calculus.

1 Evaluations and Gromov–Witten cycles

We begin with a precise statement of the claimed result. Let X be a toric variety. We fix discrete data Γ: the genus g, the number of marked points n, the contact orders of these marked points. Since X is toric, the contact orders determine the curve class. There is a moduli space of logarithmic stable maps $\mathsf{K}_\Gamma(\mathsf{X})$ parameterizing maps with these data. There is a universal diagram:

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathsf{X} \\ \mathsf{K}_\Gamma(\mathsf{X}) & \end{tikzcd}$$

There is a forgetful morphism

$$\pi : \mathsf{K}_\Gamma(\mathsf{X}) \to \overline{\mathcal{M}}_{g,n}.$$

1.0.1 Evaluation maps

For each marked point p_i, there is an evaluation map

$$ev_i : \mathsf{K}_\Gamma(\mathsf{X}) \to W_i$$

where W_i is a stratum of X. To identify it explicitly, note that the contact order at p_i is encoded precisely the data of a point u_i in the fan Σ of the toric variety X. The point u_i lies in the relative interior of a cone σ_i, which is dual to the closed stratum W_i. The logarithmic structure on W_i is taken to be the stratum structure, i.e. we view W_i as a toric variety and equip it with its standard logarithmic structure[1]. By combining these maps, there is a consolidated evaluation

$$ev : \mathsf{K}_\Gamma(\mathsf{X}) \to \mathsf{Ev}(\mathsf{X}).$$

Writing began at an altitude of 11500 m while flying over Saudi Arabia.

[1]The point to pay attention to is that we are not equipping W_i with the pullback logarithmic structure, which includes the data of the normal bundle in of W_i in X as a generic logarithmic direction. Gross and Siebert prefer this latter structure for their formalism, but then have to enhance the logarithmic structure on their moduli spaces to obtain an evaluation. The two differ only by cotangent classes from the target, and therefore only by descendants.

The target is obviously a toric variety. The simplest evaluation classes are obtained by pulling back evaluation classes and pushing forward:

$$\pi_\star ev^\star(\gamma) \in \mathsf{CH}^\star(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}).$$

1.0.2 Enhanced evaluation classes

There are more sophisticated classes. These first arose from discussions with Davesh and will appear in the second logarithmic DT theory paper. Let

$$\mathsf{Ev}^\dagger(\mathsf{X}) \to \mathsf{Ev}(\mathsf{X})$$

be any equivariant modification. Let γ be any cohomology class on this modification. By taking a fine and saturated logarithmic base change, we obtain a modification of the moduli space which maps to this blowup:

$$\mathsf{ev}^\dagger : \mathsf{K}_\Gamma(\mathsf{X})^\dagger \to \mathsf{Ev}^\dagger(\mathsf{X}).$$

The source is equipped with a virtual fundamental class by the construction in [9, Section 3]. Indeed, it is shown there that there is a well-defined virtual class in the inverse limit of Chow homology groups over all modifications, taken under the proper pushforward maps. By composition, there is still a map

$$\pi^\dagger : \mathsf{K}_\Gamma(\mathsf{X})^\dagger \to \overline{\mathcal{M}}_{g,n},$$

which I will continue to denote by π in a mild abuse of notation.

Definition 1.0.1. A *primary logarithmic Gromov–Witten cycle* in $\overline{\mathcal{M}}_{g,n}$ is any class of the form

$$\pi^\dagger_\star \mathsf{ev}^{\dagger,\star}(\gamma) \in \mathsf{CH}^\star(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$$

for any Chow cohomology class γ in any blowup $\mathsf{Ev}^\dagger(\mathsf{X})$.

1.0.3 Main result

The main claim in this note is the following.

Theorem 1.0.2. *Primary logarithmic Gromov–Witten cycles from toric targets lie in the tautological ring of the moduli space of curves.*

One could go further, and notice that there are actually logarithmic Gromov–Witten cycles in the *logarithmic* Chow ring of $\overline{\mathcal{M}}_{g,n}$, but this adds another layer of notational complexity, and nothing particularly interesting. One could also include descendants. Of course, the ancestor cotangent classes can be added in later on $\overline{\mathcal{M}}_{g,n}$. The full descendant theory will differ from the ancestors only by boundary, which can likely be handled using the inductive procedure used in [8].

Although it is not immediately apparent, this is a strengthening of the main theorem in Molcho–R [8, Theorem A]. In order to see this, first note that it is harmless to add an additional marked point with 0 contact order to the discrete data, since the pullback maps on Chow homology will be injective after doing this. The toric contact cycle in loc. cit. can then be obtained by pulling back a point class in X under the evaluation class of this map.

2 Proof of the main result

2.1 Plan

The strategy of the proof is as follows. The first step will be to pass from the evaluation space $\mathsf{Ev}(\mathsf{X})$ to a torus quotient by the dense torus of X, which acts factorwise on the evaluation space; this will function as a rubber

Written while over the Mediterranean. Ground speed is 828 kph, with the outside temperature -60 C.

evaluation space, much in the same way that there is a rubber mapping space. Second, we observe that evaluation map to this quotient can be factored through the space of rubber maps to X. Third, given a class in the quotient, we can lift to it a class in the cohomology of the Artin fan. The evaluation map to this Artin fan is given by purely tropical data. Fourth, and finally, we realize that evaluation class as coming from the ring of piecewise polynomials in the moduli space of curves itself. The result then reduces to what is already known about the toric contact cycle.

2.2 A compatibility statement

We continue to fix discrete data. There are two moduli spaces we are interested in:

$$\mathsf{K}_\Gamma(\mathsf{X}) \quad \text{and} \quad \mathsf{TC}(\Gamma),$$

where $\mathsf{TC}(\Gamma)$ is the moduli space of rubber stable maps as constructed in [8], building on [6]. The precise choice of $\mathsf{TC}(\Gamma)$ is not critical. When X is $\mathbb{P}^1$ we take it to be a moduli space of curves equipped with a piecewise linear function, whose associated line bundle is trivial [3, 6]. When X is general, we blowup X until it maps to a product of projective lines, and then define $\mathsf{TC}(\Gamma)$ by fibre product, with the same analysis in [8]. We have a map

$$\mathsf{K}_\Gamma(\mathsf{X}) \to \mathsf{TC}(\Gamma).$$

It is logarithmically smooth, and in fact, a family of toric varieties. The logarithmic smoothness can be seen as follows. First, we can replace X with the logarithmic multiplicative group $\mathbb{G}_{\mathsf{log}}^n$ with

$$\mathsf{X} \to \mathbb{G}_{\mathsf{log}}^n$$

being proper and logarithmically étale map. A map from a curve to $\mathbb{G}_{\mathsf{log}}$ is the data of a section $\mathsf{M}_{\mathsf{C}}^{\mathsf{gp}}$. We obtain a section $\overline{\mathsf{M}}_{\mathsf{C}}^{\mathsf{gp}}$ whose associated line bundle is trivial. We obtain a canonical map

$$\mathsf{K}_\Gamma(\mathbb{G}_{\mathsf{log}}^n) \to \mathsf{TC}(\Gamma).$$

The map

$$\mathsf{K}_\Gamma(\mathsf{X}) \to \mathsf{K}_\Gamma(\mathbb{G}_{\mathsf{log}}^n)$$

is a subdivision, and the map

$$\mathsf{K}_\Gamma(\mathbb{G}_{\mathsf{log}}^n) \to \mathsf{TC}(\Gamma)$$

is torsor under the group $\mathbb{G}_{\mathsf{log}}^n$. It follows that

$$\mu : \mathsf{K}_\Gamma(\mathsf{X}) \to \mathsf{TC}(\Gamma).$$

is a toric variety bundle. By performing weak semistable reduction, after replacing source and target with subdivisions, we can assume that this map is flat.

Proposition 2.2.1. *We have the following equality of Chow homology classes:*

$$[\mathsf{K}_\Gamma(\mathsf{X})]^{\mathsf{vir}} = \mu^\star[\mathsf{TC}(\Gamma)]^{\mathsf{vir}}.$$

Proof. Follows from unwinding the obstruction theories; omitted. □

2.3 Rubber evaluation spaces and torus quotients

Let T be the dense torus in X. The evaluation space $\mathsf{Ev}(\mathsf{X})$. The torus T acts on each stratum on X, since each stratum is a compactification of an orbit of T. The action on each stratum need not be effective. If in addition, the action of T on the full evaluation space $\mathsf{Ev}(\mathsf{X})$ is not effective, then all Gromov–Witten classes vanish.

Proposition 2.3.1. *Suppose that the* T *action on* $\mathsf{Ev}(\mathsf{X})$ *is not effective. Then all primary logarithmic Gromov–Witten cycles vanish.*

Proof. Let $\mathsf{H} \subset \mathsf{T}$ be a positive dimensional subgroup that acts trivial on $\mathsf{Ev}(\mathsf{X})$. We may assume that X is smooth by toric resolution, and therefore that the evaluation cycle γ is represented by a cycle. It follows that the cycle $\mathsf{ev}^{-1}(\gamma)$ has an effective H action. The pushforward to $\overline{\mathcal{M}}_{g,n}$ now vanishes since the forgetful map has positive dimensional fibers on this cycle. $\square$

Note that it is only the primary classes that vanish. One can of course pass to strata in the moduli space $\mathsf{K}_\Gamma(\mathsf{X})$ which do pushforward to something nonzero. These will certainly also be tautological, but the methods are different to the ones we are interested in here.

The next step in our plan is to replace the evaluation space $\mathsf{Ev}(\mathsf{X})$ with a quotient by T. Up to replacing $\mathsf{Ev}(\mathsf{X})$ with a subdivision, we have a map

$$\mathsf{Ev}(\mathsf{X}) \to \overline{\mathsf{Ev}}(\mathsf{X}),$$

with $\overline{\mathsf{Ev}}(\mathsf{X})$ a toric compactification of the quotient torus of the dense torus $\mathsf{Ev}(\mathsf{X})$ by T. We may also assume that this map is flat, either by semistable reduction, or by applying the Chow quotient construction [1, 5]. I will call this latter space the rubber evaluation space.

The key point here is that I want to replace $\mathsf{K}_\Gamma(\mathsf{X})$ with rubber, so one should build a rubber version of the evaluation space as well, which is exactly what we've just done.

2.4 Rubber evaluation maps

In what follows, to avoid an overuse of subdivisions, I will work with implicitly by replacing moduli spaces with appropriate subdivisions. I now assume that T acts on the evaluation space $\mathsf{Ev}(\mathsf{X})$ via a subtorus of the dense torus.

Proposition 2.4.1. *The following diagram is cartesian in all categories:*

$$\begin{tikzcd} \mathsf{K}_\Gamma(\mathsf{X}) \arrow[r] \arrow[d] & \mathsf{Ev}(\mathsf{X}) \arrow[d] \\ \mathsf{TC}(\Gamma) \arrow[r, "\overline{\mathsf{ev}}"'] & \overline{\mathsf{Ev}}(\mathsf{X}). \end{tikzcd}$$

The vertical maps are flat and the virtual classes on the left are compatible.

2.4.1 Completing the proof

The proof now follows from chasing diagrams. As a corollary of the above proposition, we reduce the calculation to the toric contact cycle. Indeed, the virtual class on $\mathsf{K}_\Gamma(\mathsf{X})$ is pulled back from $\mathsf{TC}(\Gamma)$, and the map to the moduli space of curves factors through $\mathsf{TC}(\Gamma)$. Therefore, by compatibility of pull/push and the projection formula, we reduce to the following. Let γ be a Chow cohomology class on $\mathsf{Ev}(\mathsf{X})$ and $\overline{\gamma}$ its pushforward in $\overline{\mathsf{Ev}}(\mathsf{X})$. It follows from the compatibility diagram above that the Gromov–Witten cycle associated to γ can be calculated by capping $\mathsf{TV}(\Gamma)$ with the $\overline{\mathsf{ev}}$ pullback of the class $\overline{\gamma}$ and pushing down in $\overline{\mathcal{M}}_{g,n}$.

Now, let $\mathsf{A}(\overline{\mathsf{Ev}}(\mathsf{X}))$ be the Artin fan of the toric variety $\overline{\mathsf{Ev}}(\mathsf{X})$. Let α be an equivariant lift of $\overline{\gamma}$ to a class on the stack $\mathsf{A}(\overline{\mathsf{Ev}}(\mathsf{X}))$. By composition, we have a map

$$\mathsf{TC}(\Gamma) \to \mathsf{A}(\overline{\mathsf{Ev}}(\mathsf{X}))$$

and we need only pullback α. In order to understand this latter map, we observe that since the target is an Artin fan and the map is logarithmic, there is a factorization

$$\mathsf{TC}(\Gamma) \to \mathsf{A}(\mathsf{TC}(\Gamma)) \to \mathsf{A}(\overline{\mathsf{Ev}}(\mathsf{X}))$$

The proof was completed as the flight finally crossed the 40000 ft mark, which took a long time. It is likely easier now that we have burned off a significant amount of fuel. We flew over Naples at the start of the proof and are over Terracina as I complete the proof.

through the Artin fan of $\mathsf{TC}(\Gamma)$. The latter is precisely the Artin fan associated to the tropical space $\mathsf{TC}(\Gamma)^{\mathrm{trop}}$ constructed in [8]. This is equivalent data to the combinatorial map of cone complexes

$$\mathsf{TC}(\Gamma)^{\mathrm{trop}} \to \Sigma(\overline{\mathsf{Ev}}(\mathsf{X})).$$

Now we recall from [8, Section 4] that $\mathsf{TC}(\Gamma)^{\mathrm{trop}}$ is a subcomplex of a subdivision of $\mathcal{M}_{g,n}^{\mathrm{trop}}$. By arbitrarily extending this subcomplex to a complete subdivision of $\mathcal{M}_{g,n}^{\mathrm{trop}}$ and using the basic compatibilities, we reduce to the problem to calculating a class of interest to one of the form

$$[\mathsf{TC}(\Gamma)] \cap p \in \mathsf{CH}^\star(\overline{\mathcal{M}}_{g,n}^{\dagger}; \mathbb{Q})$$

where p is a class on the Artin fan of a subdivision of $\overline{\mathcal{M}}_{g,n}$. The toric contact cycle lies in the logarithmic tautological ring by the results of [8], and the classes from the Artin fan are contained in this ring by definition; the definition of the logarithmic tautological ring is the one used in [7]. Since both classes lie in the logarithmic tautological ring of $\overline{\mathcal{M}}_{g,n}$, so does their product and we conclude. The proof is complete.

□

3 Concluding remarks

Various other strengthenings of the result are possible. I have left out the discussion of descendants; they differ from ancestors by passing to strata in $\mathsf{K}_\Gamma(\mathsf{X})$ before pushing down. The main new idea that is necessary here is the boundary splitting. The first steps in handling this are taken in [2].

The simplest structural generalization is to allow X to be a toric variety bundle over a base manifold B, equipped with the vertical toric boundary as the logarithmic structure. The result is then that one obtains tautological classes in the space of maps to B. There is no difficulty in extending the result to this case, though some comparison will be required to invoke [4]; alternative, the method of [8] will have to be redone over a base. Both routes should present no difficulties, but I have not written out all the details.

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