A GUI FOR LOGARITHMIC INTERSECTIONS

(Edited for handwriting after)

Goal: Understand intersection problems of the following flavour.

$$\overline{K} \xrightarrow{\pi} \overline{M}$$

SMOOTH WITH SNC BOUNDARY

Logarithmic moduli with a virtual class

$\boxed{\text{Want to understand the “RIGHT” intersections of classes } \pi_* [\overline{K}]^{vir}}$

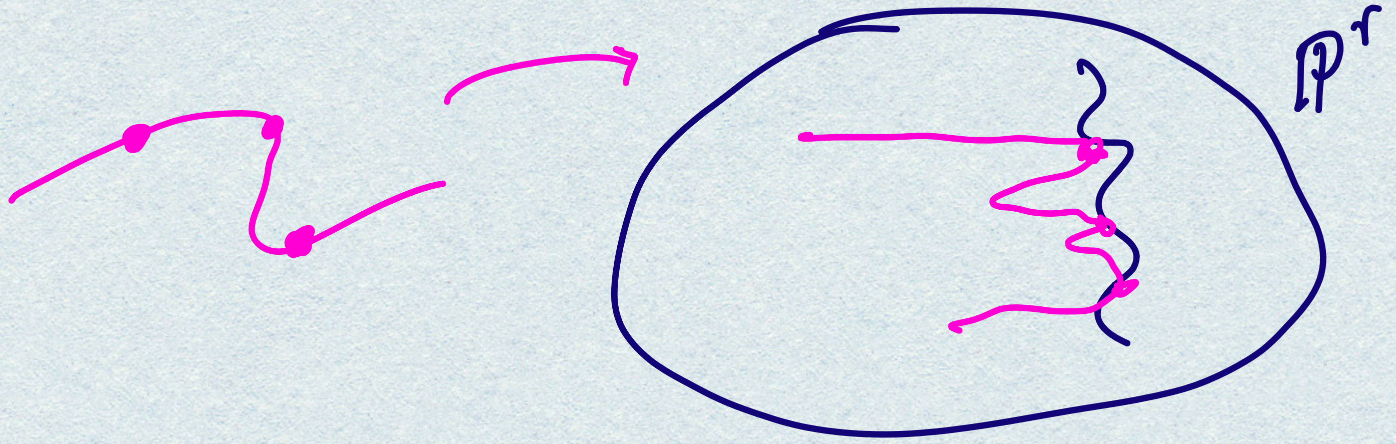
EXAMPLES: Two; there are more...

Ex 1: $\overline{M} = \overline{M}_{0,n}(\mathbb{P}^r, d)$;

Relative GW cycles

$$\overline{K} := \overline{M}_{0,n}(\mathbb{P}^r | H, \mu)\,[\mu \vdash d]$$

CURVES WITH TANGENCY μ ALONG H



<u>Ex 2</u>: $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}$; $\overline{\mathcal{K}}$ = SPACE OF MAPS TO $\boxed{\text{RUBBER}}$ $\mathbb{P}^1$ WITH CONTACT ORDER $\chi \in \mathbb{Z}^n_0$

DOUBLE RAMIFICATION CYCLE

$\boxed{\text{COMPACTIFY.}}$

$$DR_g(\chi) = \left\{ C \xrightarrow{\pi} \mathbb{P}^1 \;\middle|\; \begin{array}{l} C\text{: marked; } \pi\text{: stable,} \\ \text{contact } \chi\text{; } f \sim g \text{ upto scaling} \end{array} \right\}$$

whatever floats your boat

① Rubber ② Log maps to $\mathbb{P}^1$ & rigidify ③ Tropical divisors

④ Abel-Jacobi resolution

$\boxed{\text{QUESTION}}$ Can we understand the cycle

$$\pi_*[\overline{\mathcal{K}}]^{vir} \in A^*(\overline{\mathcal{M}}; \mathbb{Q})$$

THERE ARE GOOD ANSWERS:

- For rel. GW cycles in $g=0$ [Vakil, Gathmann] RECURSIVE.

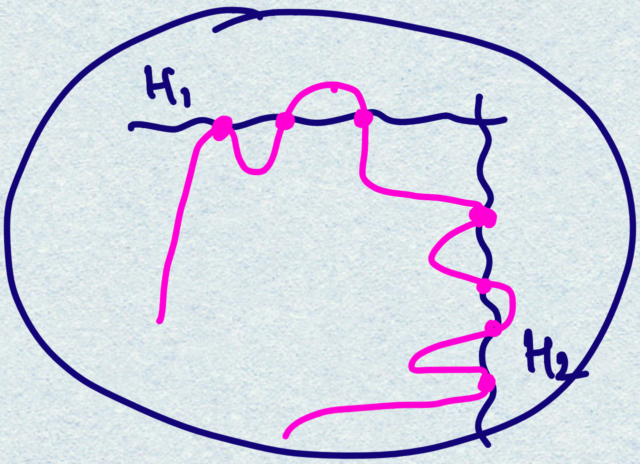
- DR great answer [Pixton, JPPZ]

END OF THIS BIT

NEW SUPPLY OF CLASSES {GEOMETRY OF S.N.C PAIRS}

Let H_1, H_2 be two hyperplanes in $\mathbb{P}^r$:

study curves $C \longrightarrow \mathbb{P}^r$ where $\mu_1, \mu_2 \vdash d$ are fixed contact orders and give us



To compactify, we can do something cheap: take closure $\left\{ \mathcal{M}_{0,n}(\mathbb{P}^r|H_1,H_2,\mu_1,\mu_2) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \right\}$

If you don't like logarithmic structures yet, this is enough for a rich example

QUESTION Can we calculate this class of the closure?

SIMILAR: DDR = "Double ramification cycle"

The surface version of DR; not a closure but basically reasonable to think about it this way.

... SO WHAT'S THE DDR?

LOGARITHMIC COMPACTIFICATION OF A SIMPLE PROBLEM:

Fix $x \in \mathbb{Z}_0^n$; $y \in \mathbb{Z}_0^m$. Then

"OPEN"

$$\mathrm{DDR}_g^{\circ}(x,y) =$$

$$\left\{ (C, p_1 \ldots p_n, q_1, \ldots, q_m) \xrightarrow{f} \mathbb{P}^1 \times \mathbb{P}^1 \;\middle|\; \begin{array}{l} g(C) = g \text{; projections} \\ \text{ramify as } x \text{ \& } y \text{ as} \\ \text{before} \\ \text{up to } \mathbb{G}_m^2\text{-scaling} \end{array} \right\}$$

$\boxed{\text{Compactify?}}$ $\quad \mathrm{DDR}_g^{\circ}(x,y) \hookrightarrow \mathrm{DDR}_g(x,y)$

Options:
1. Log GW theory to double rubber $\mathbb{P}^1 \times \mathbb{P}^1$
2. Log maps to $\mathbb{P}^1 \times \mathbb{P}^1$ + rigidify
3. ... Holmes-Pixton-Schmitt.

any pick

I want a formula for

$$[\mathrm{DDR}_g(x,y)]^{vir} \in A^*(\overline{\mathcal{M}}_{g,n+m}; \mathbb{Q})$$

{ASIDE ON DEFINITION OF DDR}

DDR via LOG GWT :

$$\mathcal{M}^{\log}_{g,x,y}(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{G}_m^2} \longrightarrow \overline{\mathcal{M}}_{g,n+m}$$

Maulik-Pandharipande trick.

Alternate: Don't do rubber. Add a point w/ trivial contact, send it to $(1,1) \in \mathbb{P}^1 \times \mathbb{P}^1$, pullback and pushforward

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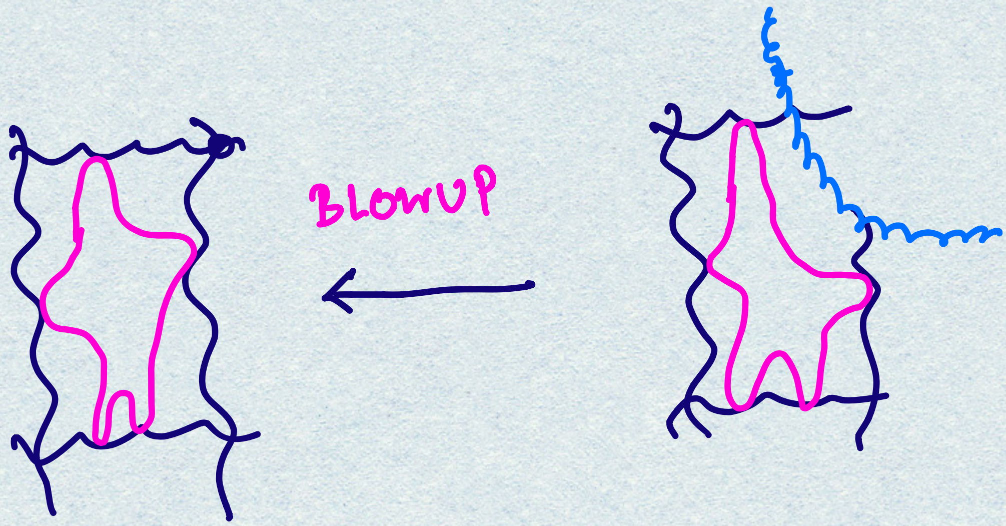
But: why isn't this just a product of the old answers in the RING $A^*(\overline{\mathcal{M}}; \mathbb{Q})$?

→ Turns out, this is an inescapable feature of the logarithmic theory.

But a <u>sidenote</u> first: why is the logarithmic theory not the "<u>RIGHT</u>" theory?

Major thing you want is invariance under strata <u>blowups</u>:



Expect the DDR to be invariant under <u>blowups</u>

<u>Intersections</u>:

$$\begin{tikzcd} \underbrace{DR_g(X)}_{\text{LOG STRUCTURE}} \arrow[r] \arrow[d, dash] \arrow[dr, phantom, "\square"] & \underbrace{\overline{\mathcal{M}}_{g,n}}_{\text{SMOOTH NC PAIR.}} \\ ? \arrow[r, dash] & \underbrace{DR_g(Y)}_{\text{MORE LOG STR}} \arrow[u] \end{tikzcd}$$

There are two options for the log ?

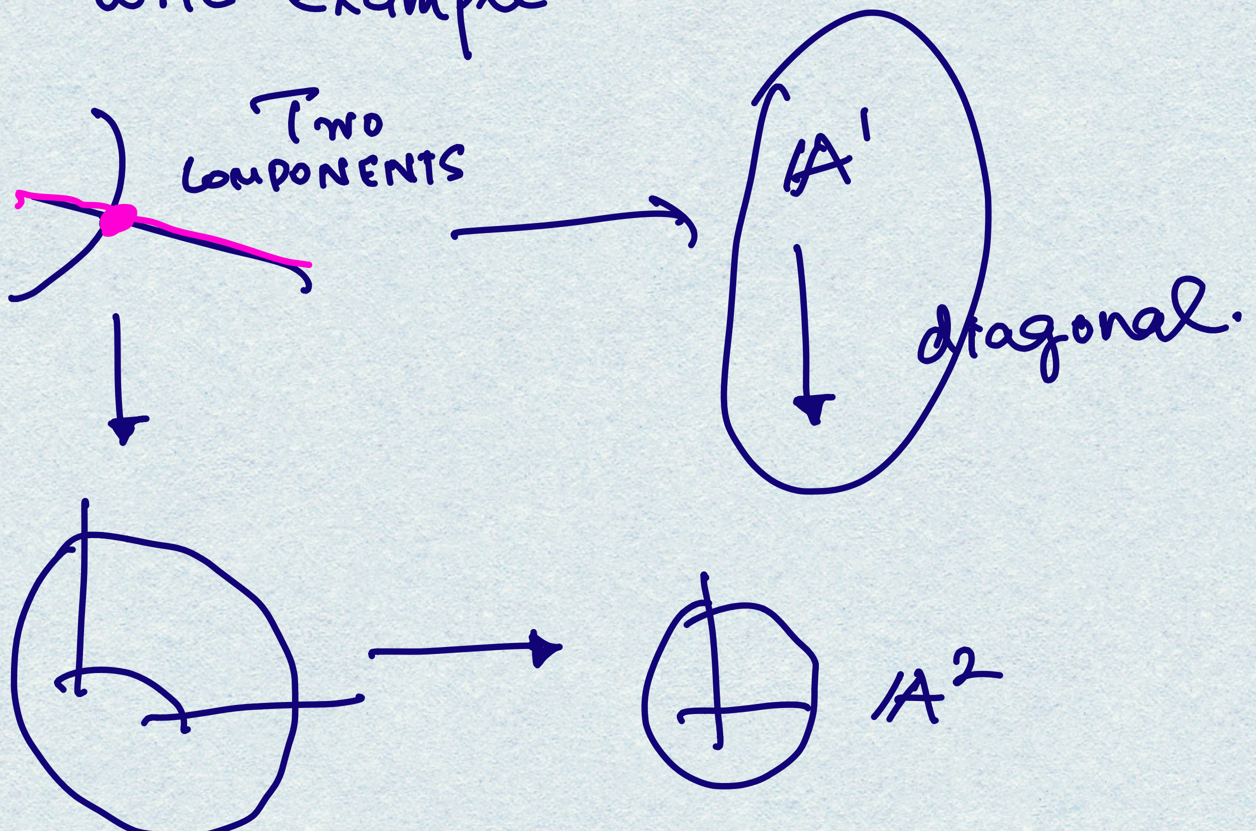
- schematic fiber product
- leads to naive products

"WRONG"

- The fine & saturated fiber product.

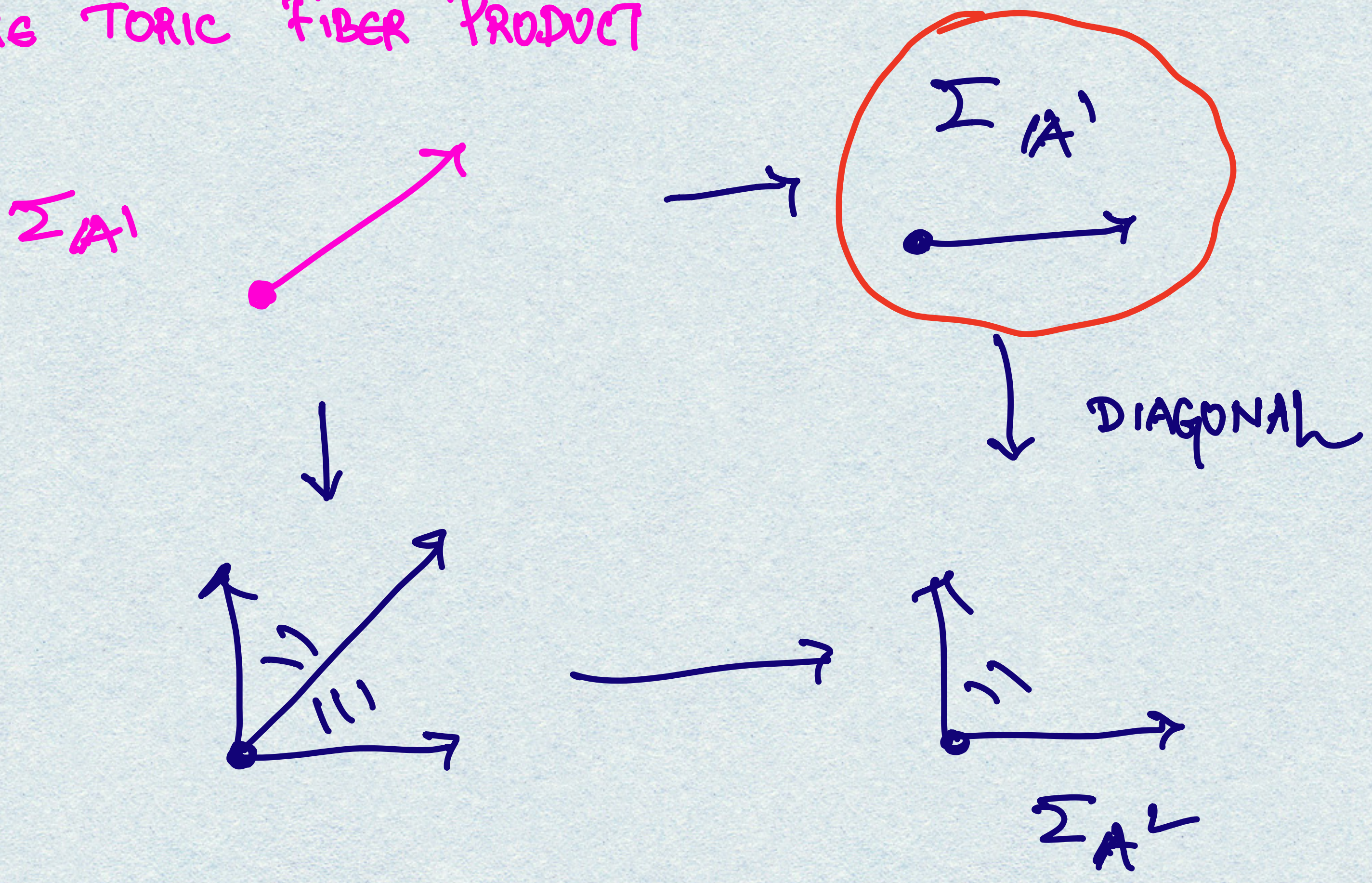
"RIGHT"

This is visible (and fully comprehensible) in <u>ONE</u> toric example.



so schematic fiber products are not toric.

THE TORIC FIBER PRODUCT



LESSON: The toric guy is the strict transform of the other guy.

In the non-toric logarithmic case the phenomenon is visible via Artin fans, effectively reducing to the toric/fan case [Ulirsch, friends].

Concretely:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \arrow[r, squiggly] & \text{SNC pair} \end{tikzcd}$$

log schemes

— If $X \to Z$ is flat, reduced fiber then the fiber products agree [Tom: Z generically trivial log structure? Yes probably].

Basically, this suggests the following false statement

$\mathrm{DR}_g(X) \longrightarrow \overline{\mathcal{M}}_{g,n}$ we blowup such that the strict transform of $\mathrm{DR}_g(X)$ is flat w/ reduced fibers over $\overline{\mathcal{M}}_{g,n}$.

▲ Interpret properly: strict transform, flatness, etc. interpreted as {fs log pullback, integral, saturated}

... but in practice, blowup the tropical moduli spaces to give this output at the toric/Artin fan level.

Where to compute the $DR_g(x)$?

Subdivide $\overline{\mathcal{M}}_{g,n}^{+} \longrightarrow \overline{\mathcal{M}}_{g,n}$ such that the $DR_g(x)$ is transverse to the boundary

i.e. stabilizes under strict transform.

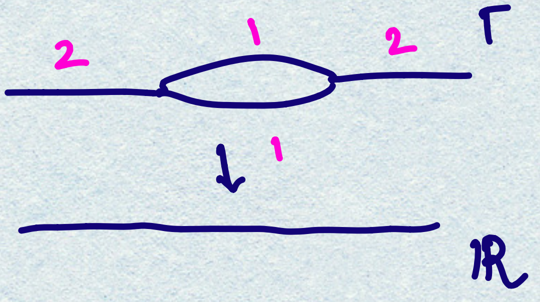
$\longrightarrow$ $DR_g(x) \longrightarrow \overline{\mathcal{M}}_{g,n}$ intersects strata of $\overline{\mathcal{M}}_{g,n}$ that you think it shouldn't from a naive dimension count of the strata.

This is easy from the tropical point of view.

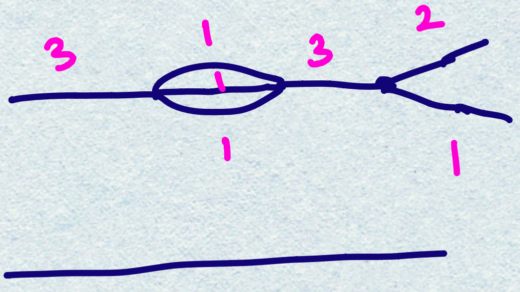
Cone (stack) complex

$$DR_g(x) \xrightarrow{\text{trop}} \left\{ \begin{array}{l} \Gamma : \text{tropical curve} \\ \Gamma \to \mathbb{R} : \text{balanced PL map with} \\ \qquad \text{asymptotics given by } x \end{array} \right\}$$

EXAMPLES (didn't include in talk) slopes



$x = (2, -2)$



$x = (3, -2, -1)$.

Obvious map: $DR_g^{\text{trop}}(x) \longrightarrow \mathcal{M}_{g,n}^{\text{trop}}$

{BASICALLY CONE COMPLEXES}

{MAP IS AN INJECTION}

1. Subdivide such that $DR_g^{\text{trop}}(x)$ is a union of faces of $\mathcal{M}_{g,n}^{\text{trop}}$

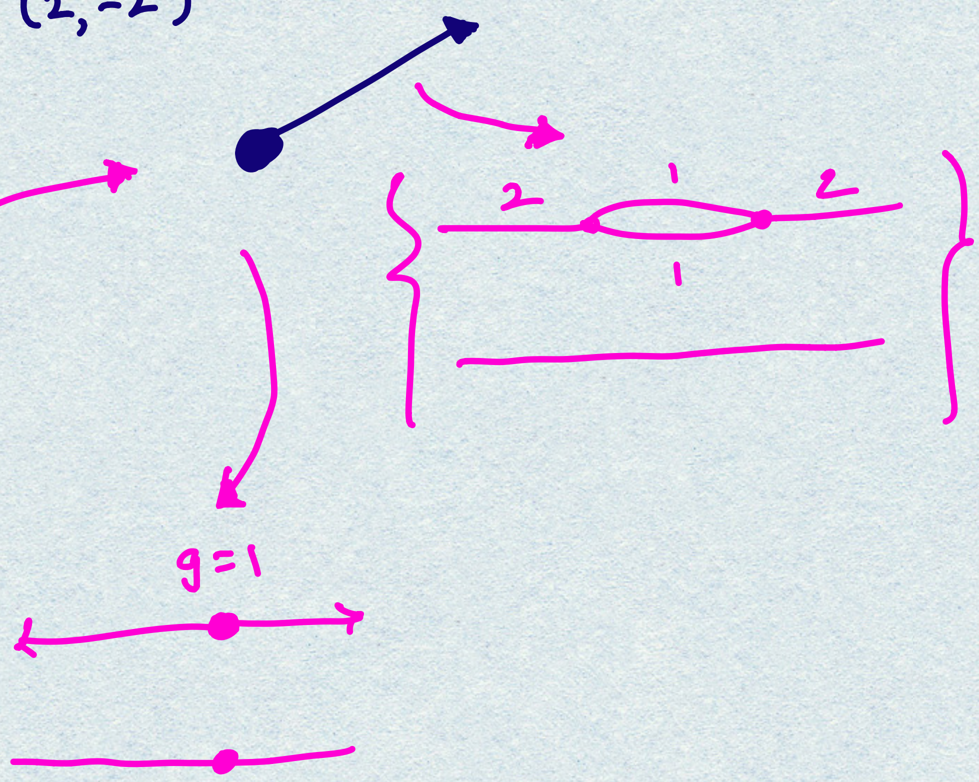
2. Compute strict transform

THM: The strict transforms DO SATISFY PRODUCT FORMULAS.

Ex: $g=1 \quad x=(2,-2)$

$DR_1^{trop}(2,-2)$

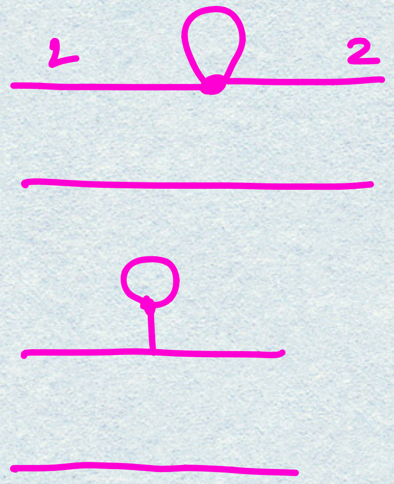
tropical moduli contains this ray



IRRELEVANT CONES in $DR_1(2|-2)$:

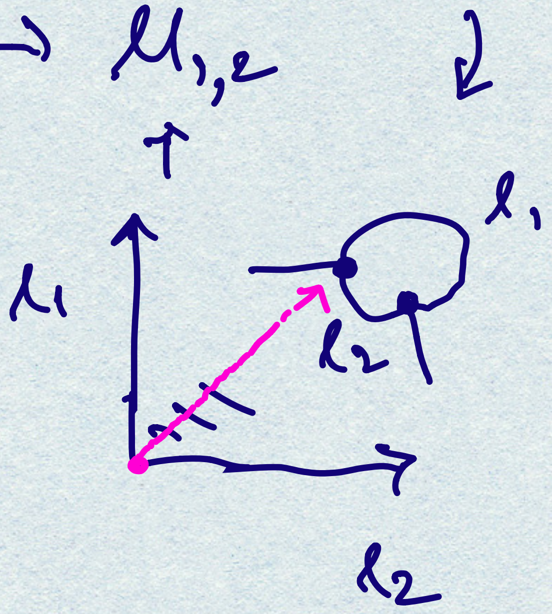
1.

2.



don't enter into calculation.

$$DR_1^{trop}(2|-2) \longrightarrow \overline{\mathcal{M}}_{1,2}$$



Basically is that diagonal toric example from before.

Take $\overline{\mathcal{M}}^{+}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ given by a subdivision

such that the map

$$DR_g^{trop}(x) \longrightarrow \mathcal{M}_{g,n}^{trop}$$

has image a union of <u>cones</u> of $\mathcal{M}^{+,trop}_{g,n}$

How to compute:

$$\begin{tikzcd} \widetilde{DR}_g(\pi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{NOT fiber product!}"] & DR_g(\pi) \arrow[d] \\ \overline{\mathcal{M}}_{g,n}^{+} \arrow[r, "\text{blowup}"'] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

Now we want to compute.

$$[P_i x] - [\widetilde{DR}_g(x)] \in A^*(\overline{\mathcal{M}}^{+}_{g,n})$$

"correction terms".

Outside compact type, corrections are plentiful.

This difference "can be computed" Fulton formula.

P_{1E} → Segre classes

Strata in $\overline{\mathcal{M}}_{g,n}$

Exceptional strata in $\overline{\mathcal{M}}_{g,n}^{+}$

of pullbacks to DR of strata in $\mathcal{M}_{g,n}$.

Most complicated, but can control (Aluffi).

Easy outcome: $\mathrm{DDR}_g(x,y) \in R^*(\overline{\mathcal{M}}_{g,n})$.

For an explicit computation:

$$\mathrm{DR}_g(x) \ \& \ \mathrm{DR}_g(y)$$

compute strict transforms & intersect in $\overline{\mathcal{M}}_{g,n}^{+}$.

$$\Big\downarrow \pi_*$$

$$\overline{\mathcal{M}}_{g,n}$$

EXAMPLE: Try $g=1$ $x=y=(2|-2)$.

Want to calculate $DDR_1(x,y) \in A^2(\overline{\mathcal{M}}_{1,4})$

① TRANSVERSALIZE

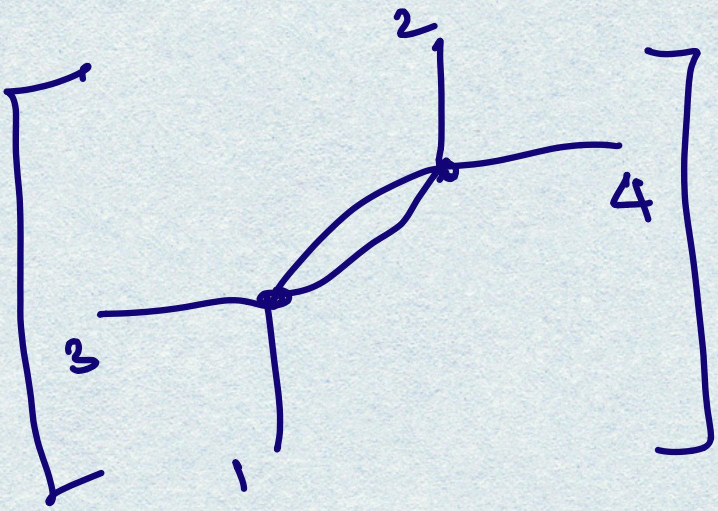
$$\begin{array}{c} DR_1(2,-2) \\ \downarrow \\ \overline{\mathcal{M}}_{1,2} \end{array}$$

easy to get $\widetilde{DR}_1(2,2)$ on $\overline{\mathcal{M}}^+_{1,2}$

② Pullback to $\overline{\mathcal{M}}^+_{1,4}$

take product & push to $\overline{\mathcal{M}}_{g,n}$.

③ There is a nonzero correction in $A^2(\overline{\mathcal{M}}_{1,4};\mathbb{Q})$.



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