Algebraic Geometry

Part III

Autumn 2023

Lecture Notes.

§0: Preliminary Remarks

0.1 : Goals & Non-Goals

- The course is a STARTER KIT
- Mastery of scheme theory is NOT a goal
- scheme theory represents a SPECTACULAR revolution in pure mathematics; I will try to guide you towards an understanding of why.
- Example sheets are crucial!

0.2 The plan:

- Basics of sheaves on topological spaces
- Definitions of schemes & morphisms
- Properties of schemes & morphisms
- Rapid Introduction to sheaf cohomology.

0.3: Prerequisites:

- Basic undergraduate maths [algebra, topology, etc]
- Commutative algebra [co-requisite or willingness to read].

0.4 Resources:

- Dhruv's course page [notes]
- Texts: Hartshorne; Vakil ; Intuition: Eisenbud-Harris
- Commutative Algebra: Atiyah-MacDonald & PART III
- Web: MathOverflow & MathStackExchange

(YouTube) AGITTOC "pseudo lectures" by Ravi

- Example sheets/classes ☆: SAGES reading group

0.6 Why scheme theory? [Not examinable]

- Many motivations: more rigorous foundations, interactions with number theory (Weil conjectures) but also MODULI THEORY.

- Moduli already of interest in early 20^{th} century (Italian school).

MODULI

$\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\} / \text{scaling}$; varieties are zero loci of a set of homog. polynomials

Study varieties of a given "type" simultaneously.

Simple Examples:

- The set of lines in $\mathbb{P}^2$: $\{aX + bY + cZ = 0\}$ parametrized by $\{(a,b,c)\} \setminus \{0\} / \text{scaling}$

$$\{\text{Lines in } \mathbb{P}^2\} = \mathbb{P}^2.$$

- Set of degree d hypersurfaces in $\mathbb{P}^n$ is $\mathbb{P}^{\binom{n+d-1}{d}-1}$ using same logic.

Except not: some degree d poly's f factorize as $f = f_1^2 \cdot f_2$, f_1 non-const.

for example is $(X+Y+Z)^2 = 0$ really a "conic"?

So maybe $U_d \subseteq \mathbb{P}^{\binom{n+d-1}{d}-1}$ a subset of those $[f]$ st $\mathbb{V}(f)$ is degree d is enough

But now limits of varieties do not exist!

In scheme theory, $\mathbb{V}(f)$ & $\mathbb{V}(f^2)$ are very different objects!

Very generally scheme theory finds these "limiting" objects. For example, there is a "space": $\mathrm{Var}(\mathbb{P}^n) \subseteq \mathrm{Hilb}(\mathbb{P}^n) \leftrightarrow \{\text{schemes in } \mathbb{P}^n\}$

$\updownarrow$

$\{\text{Varieties in } \mathbb{P}^n\}$

Contains all "limit points".

This "compactness" is a sign of a more complete theory

The Weil Conjectures

Let $f \in \mathbb{Z}[\underline{X}]$ be a homogeneous polynomial

Two Worlds (Weil 1949)

1. $X = \mathbb{V}(f) \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$ a $\boxed{\text{projective hypersurface.}}$

assume X is $\boxed{\text{smooth}}$:

no point on X where $\frac{\partial f}{\partial x_i}(p) = 0$ for all i

X is a compact topological space in Euclidean top.

Numbers: $b_0(X), b_1(X), \ldots, b_{2n}$ $\boxed{\text{Betti Numbers}}$

In the $\mathbb{C}$-Euclidean topology X can be triangulated, and $\sum (-1)^i b_i(X) = \chi_{top}(X)$.

2. Fix prime number p with X smooth over $\overline{\mathbb{F}}_p$

Define $N_m := \# X(\mathbb{F}_{p^m})$

Now package this:

$$\zeta(X,t) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} t^m\right)$$

the Weil Zeta Function

UNBELIEVABLE THEOREM ! [Grothendieck]

1. $\zeta(X,t)$ is a ratio of polynomials:

$$= \boxed{\frac{P_0(t)\, P_2(t) \ldots P_{2n}(t)}{P_1(t)\, P_3(t) \ldots P_{2n-1}(t)}}$$

2. The degree of $P_i(t)$ is equal to the Betti number b_i.

The topology of X over $\mathbb{C}$ is connected to the number of points on X over $\mathbb{F}_q$!

Again about families: really, the polynomial f defines a scheme that sees both $\mathbb{C}$ & $\mathbb{F}_p$ properties.

§1 : Beyond algebraic varieties

1.1 Summary of varieties (affine case)

k = algebraically closed field $\qquad \mathbb{A}^n_k := k^n$ as a set

Affine varieties are subsets of $\mathbb{A}^n_k$ of the form

$$\mathbb{V}(S) \qquad S \subseteq k[x_1, \dots, x_n]$$

common zero set

Note: $\mathbb{V}(S) = \mathbb{V}(I(S))$.
$= \mathbb{V}(\sqrt{I})$

Morphisms: Given varieties V in $\mathbb{A}^n$ & W in $\mathbb{A}^m$ a morphism is given by

$$\varphi : V \longrightarrow W \subseteq \mathbb{A}^m_k$$
$$\varphi = (f_1, \dots, f_m)$$

where f_i are restrictions of polynomials in $\{x_1, \dots, x_n\}$

Isomorphisms are those with 2-sided inverse

Basic correspondence

$$\left\{ \begin{matrix} \text{Affine varieties } V \\ \text{over } k \end{matrix} \right\} \Big/ \text{isom.}$$
$$\Big\updownarrow$$
$$\left\{ \begin{matrix} \text{fin. generated } k\text{-algebras } A \\ \text{without nilpotent elements} \end{matrix} \right\}$$

How? Given V a representative of isom. class write $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$, for radical ideal I and send $V \longmapsto k[\underline{x}]/I$.

Conversely, write $A \cong k[\underline{x}]/I$ take $V = \mathbb{V}(I)$. CHECK: Independence of choice!

The algebra is called the COORDINATE RING. $k[V]$

Functoriality of Basic Correspondence:

$$\begin{array}{c} \text{Morphisms}\,(V, W) \\ \updownarrow \\ \text{Ring Homomorphisms}_k\,(k[W], k[V]) \end{array}$$

identity on k.

Topology: $V = \mathbb{V}(g) \subseteq \mathbb{A}^n_k$

Zariski Topology:

closed sets $= \mathbb{V}(S)$ for $S \subseteq k[V]$

$= \mathbb{V}($ideal gen. by $S)$.

Closed sets are where functions vanish.

(Exercise: check this!)

Nullstellensatz: Fix V with coord. ring $k[V]$

Given $p \in V$ we evaluate functions at p.

$\boxed{ev_p : k[V] \twoheadrightarrow k}$; $m_p := \text{kernel}(ev_p)$

and conversely, by Hilbert's Nullstellensatz

$\{\text{points of } V\} \longleftrightarrow \{\text{maximal ideals of } k[V]\}$

POINTS OF V CORRESPOND TO MAXIMAL IDEALS in $k[V]$

1.2 LIMITATIONS:

Question 1.2.1: What is an abstract variety?

Should be something that is "LOCALLY" an affine variety.

Example 1.2.2 (non-algebraically closed fields)

$$I = (x^2+y^2+1) \subseteq \mathbb{R}[x,y]$$

then $$\mathbb{V}(I) = \emptyset \subseteq \mathbb{R}^2$$

But I is prime, therefore radical.

Nullstellensatz fails since I is NOT unit.

Question 1.2.3: On what topological space

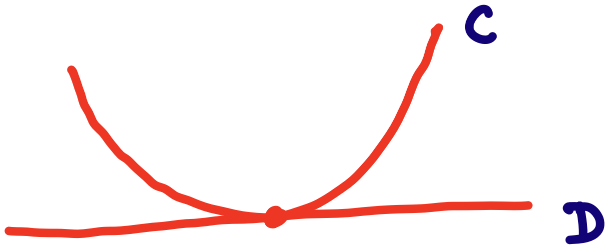
X is $\mathbb{R}[x,y]/(x^2+y^2+1)$ NATURALLY the space of functions?

Question 1.2.4 (Similar) On what topological space is $\mathbb{R}[x]$ the ring of functions? And the rings $\mathbb{Z}$, $\mathbb{Z}[x]$?

Example 1.2.5 (Why restrict to radical ideals?)

Take $C = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2_k$

$D = \mathbb{V}(y)$



$$C \cap D = \mathbb{V}(y, y - x^2) = \boxed{\mathbb{V}(x^2, y) = \mathbb{V}(x, y)}$$

1 POINT

Now if $D_\delta = \mathbb{V}(y + \delta)$ $\delta \in \mathbb{C}$

$$C \cap D_\delta = \{\pm\sqrt{\delta}\}$$ 2 POINTS

> Intersections of varieties don't want to be varieties.

Remark 1.2.6 (Moduli) If $X \xrightarrow{\pi} B$ is a morphism of varieties how is geometry of $\pi^{-1}(a)$ & $\pi^{-1}(b)$ for $a, b \in B$ related? How do you parameterize varieties

§ 1.3 SPECTRUM OF A RING

Let A be a commutative ring with identity. We will define a topological space on which A is the ring of functions.

<u>Definition 1.3.1</u> The Zariski spectrum of A is $\operatorname{Spec} A = \{ \mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal} \}$

- Given a ring hom $\varphi : A \longrightarrow B$ we get a map

$$\varphi^{-1} : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

- Given $f \in A$, we can evaluate at $\mathfrak{p} \in \operatorname{Spec} A$

by $\bar{f} \in A/\mathfrak{p} \subseteq FF(A/\mathfrak{p})$

[for any point $\mathfrak{p} \in \operatorname{Spec} A$]

Functions are field-valued, but the field changes from point to point and $ev_{\mathfrak{p}}$ is not always surjective

Example 1.3.2 $A = \mathbb{Z}$ then $\operatorname{Spec} \mathbb{Z}$ is the set of prime mubers plus 0.

Pick a $\boxed{\text{FUNCTION}}$ e.g $132 \in \mathbb{Z}$

$\boxed{\text{EVALUATE}}$ $132(p) := 132 \bmod p$

The codomain of the function changes from point-to-point

Example 1.3.3 $A = \mathbb{R}[x]$ then

$\operatorname{Spec} \mathbb{R}[x] = \mathbb{C} / \underbrace{\text{complex conjugation}}_{\text{Galois group!}}$ & (0)

$=$ Upper half plane in $\mathbb{R}^2$ & (0)

Exercise 1.3.4 Draw $\operatorname{Spec} A$ for $A = \mathbb{Z}[x]$ and $A = k[x]$ for k arbitrary field.

Why not maximal ideals? Functoriality!

§1.4 TOPOLOGY ON SPEC(A)

Zariski topology $=$ Zero sets of functions

Fix $f \in A$ & $\mathfrak{p} \in \operatorname{Spec}(A)$; then

$$\mathbb{V}(f) = \{ \mathfrak{p} \in \operatorname{Spec}(A) : \bar{f} = 0 \bmod \mathfrak{p} \text{ i.e. } f \in \mathfrak{p} \}$$

Points where f vanishes.

Similarly for $I \subseteq A$ an ideal

$$\mathbb{V}(I) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p} \text{ for all } f \in I \}$$

i.e. $I \subseteq \mathfrak{p}$

PROPOSITION 1.4.1: The sets $\mathbb{V}(I) \subseteq \operatorname{Spec} A$ for all ideals $I \subseteq A$ form the closed sets of a topology — the Zariski topology.

Proof: Easy facts: $\emptyset$ and $\operatorname{Spec} A$ are closed.

Since $\mathbb{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathbb{V}(I_\alpha)$ arbitrary intersections are closed. [I_1+I_2 is the smallest ideal containing $I_1 \cup I_2$]

REQUIRES SOME THOUGHT:

$$\mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$$

$\supseteq$: clear

Claim: $\mathbb{V}(I_1 \cap I_2) \subseteq \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$

$I_1 I_2 \subseteq I_1 \cap I_2 \subseteq \mathfrak{p}$ with $\mathfrak{p}$ prime then

I_1 or I_2 is contained in $\mathfrak{p}$.

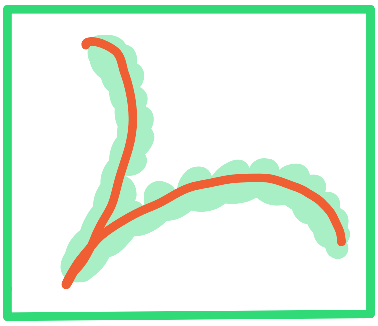
we used primality ! $\square$

Example 1.4.2 (How does this compare with old skool)

Let $k = \bar{k}$ and consider Spec $k[X,Y]$ is k^2 plus one point for each irreducible curve and an extra point corresponding to (0).

what are closures of the weird points ?

ROUGH PICTURE:



$y^2 = x^3$

(0) is dense; (y^2-x^3) closure includes all of $\mathbb{V}(y^2-x^3)$. Similar for $y=x^2$, $x-y$, etc.

POINTS aren't CLOSED!

§1.5 FUNCTIONS ON OPENS

Let $f \in A$. Then define
$U_f = (\operatorname{Spec} A) \setminus \mathbb{V}(f)$

Distinguished open.

LEMMA 1.5.1 The distinguished opens form a basis for the topology on Spec A.

Proof: Exercise (Example Sheet I) $\square$

The localization a ring R at $s \in R$ is $R_s := \dfrac{R[x]}{(xs-1)}$

LEMMA 1.5.2 The subspace U_f is (naturally) homeomorphic to $\operatorname{Spec} A_f$

Proof: Primes in A_f are primes in A that miss f; (why?) Let $j : A \to A_f$

be the canonical map. Bijection is:

- $\mathfrak{q} \subseteq A_f$ a prime take $j^{-1}(\mathfrak{q})$
- $\mathfrak{p} \subseteq A$ a prime avoiding f, take $\mathfrak{p}_f := \mathfrak{p} \cdot A_f$

Claim: this $\mathfrak{p}_f$ is prime.

To see this, observe $A_f/\mathfrak{p}_f = (A/\mathfrak{p})_{\bar{f}}$

But $\bar{f} \neq 0$, so $(A/\mathfrak{p})_{\bar{f}} \subseteq FF(A/\mathfrak{p})$. Thus $A_f/\mathfrak{p}_f$ is a domain & $\mathfrak{p}_f$ is prime. $\square$

Basic Facts: about distinguished open sets

- $U_f \cap U_g = U_{fg}$ (Easy)
- $U_{f^n} = U_f$ for all $n \geq 1$ (Easy)
- The rings A_f and A_{f^n} are isomorphic (if f^n has an inverse x, then f has inverse $x \cdot f^{n-1}$)
- Containment $U_f \subseteq U_g$ if and only if f^n is a multiple of g for some n

Proof: "if" direction is clear. Suppose $U_f \subseteq U_g$ so $\mathbb{V}(f) \supseteq \mathbb{V}(g)$. Recall

$\mathbb{V}(f)$ is the intersection of all primes containing f, but this is $\sqrt{(f)}$

[use that nilradical is intersection of all primes]

So now $\sqrt{(f)} \subseteq \sqrt{(g)}$ as required. $\square$

FORESHADOWING

Fix A a ring; $X = \operatorname{Spec} A$. We have an assignment:

$$\left\{\begin{array}{lcl}\text{Distinguished Open Sets} & \longrightarrow & \text{Rings} \\ U_f & \longmapsto & A_f := \mathcal{O}_X(U_f)\end{array}\right.$$

Meaning: Open Set $\longmapsto$ Functions on that open.

"Functoriality": Suppose $U_{f_1} \subseteq U_{f_2}$. Thus $f_1^n = f_2 \cdot f_3$ and so $U_{f_1^n} = U_{f_2 f_3} = U_{f_2}$.

Get a map: $A_{f_2} \longrightarrow A_{f_3}$

GUIDEPOST for the next steps: Extend to an assignment:

Opens on Spec $A \to$ Rings

General open

$$\mathcal{O}_X(U) = \left\{ \text{families } (f_V)_{V \subseteq U;\, V \text{ distinguished}} \text{ of functions st if } V \subseteq W \subseteq U \text{ with } W \text{ distinguished then } f_W \text{ restricts to } f_V \right\}.$$

Unnecessary but fancy: $\mathcal{O}_X(U) = \mathcal{O}_X\left(\bigcup_\lambda B_\lambda\right) = \varprojlim_\lambda \mathcal{O}_X(B_\lambda)$

OBSERVE: This is naturally a ring. Every open set in Spec A has a ring of functions; if $U \subseteq U'$ then there is a restriction

$$\mathcal{O}_X(U') \longrightarrow \mathcal{O}_X(U).$$

PUNCHLINE: A scheme is SOMETHING OBTAINED BY GLUEING THE DATA STRUCTURES ABOVE.

VECTOR SPACES $\rightsquigarrow$ MANIFOLDS.

§2 SHEAVES:

Sheaves formalize objects that you know and the behaviour we just saw.

2.1 PRESHEAVES

The basic instance:

If X is a topological space:

$$\text{Open Sets} \longrightarrow \text{Ab. Groups.}$$
$$U \longmapsto \{ f : U \to \mathbb{R} \mid f \text{ continuous} \}$$

DEFINITION 2.1.1 A $\boxed{\text{presheaf}}$ of abelian groups on a topological space X is an association

$$\mathcal{F} : \text{Opens in } X \longrightarrow \text{Ab. Groups}$$
$$U \longmapsto \mathcal{F}(U)$$

such that if $U \subseteq V$ there is a homomorphism $res^V_U : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ with $res^U_U = id$ and $res^V_U \circ res^W_V = res^W_U$. for $U \subseteq V \subseteq W$ opens.

Similarly presheaf of sets, rings, etc...

Remark 2.1.2 (Language) A presheaf is therefore a FUNCTOR from the CATEGORY $\mathrm{Open}(X)$ to abelian groups.

Objects : Opens
Morphisms : Inclusions

Morphisms between presheaves

what should it be? Definition by "DUH"

DEFINITION 2.1.3 A morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of presheaves on X is, for each U, a homomorphism

$$\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

commuting with restrictions:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{res}^U_V"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V) \end{tikzcd}$$

$V \subseteq U$ opens

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves is injective/surjective if $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective/surjective for all U

§2.2 SHEAVES: DEFINITIONS & EXAMPLES

What additional properties does the sheaf of continuous functions satisfy?

DEFINITION 2.2.1: A sheaf $\mathcal{F}$ is a presheaf such that:

S1: If $U \subseteq X$ is open and $\{U_i\}$ is an open cover of U then for $s \in \mathcal{F}(U)$ with $s|_{U_i} := \operatorname{res}^{U}_{U_i}(s) = 0$ for all i, then $s = 0$.

S2: If U & $\{U_i\}$ as above, given $s_i \in \mathcal{F}(U_i)$ with $\boxed{s_i = s_j \text{ on } U_i \cap U_j}$ s_i glue

AMUSING DEDUCTION: If $\mathcal{F}$ is a sheaf on X then $\mathcal{F}(\emptyset) = \{e\}$.

A morphism of sheaves is a morphism of the underlying presheaves.

Sheaves on X form a category.

Example 2.2.2 If X is a topological space the sheaf of continuous functions:

$$\mathcal{F}(U) = \{ f : U \to \mathbb{R} ; \text{ continuous} \}$$

is a sheaf.

Non-Example 2.2.3 Let $X = \mathbb{C}$ with euclidean topology. set

$$\mathcal{F}(U) = \{ f : U \to \mathbb{C} : f \text{ bounded } \& \text{ analytic} \}$$

This is not a sheaf; bounded doesn't GLUE.

Non-Example 2.2.4 Fix a group G and set

$$\mathcal{F}(U) = G.$$

If U_1 & U_2 are disjoint, then by sheaf axioms $\mathcal{F}(U_1 \cup U_2)$ is forced to be $G \times G$.

But it should be G.

Example 2.2.5 (the constant sheaf) Fix G and set $\mathcal{F}(U) = \{ f : U \to G \mid f \text{ locally constant} \}$

This is the sheaf that 2.2.4 wants to be.

Example 2.2.6: If V is an affine/projective/quasi projective irreducible variety, set

$$\mathcal{O}_V(U) = \{ f \in k(V) \mid f \text{ is regular at } p \text{ for all } p \in U \}$$

$k(V) = \mathrm{Frac}\, k[V]$; regular means near p, can write $f = r/s$ with $s(p) \neq 0$.

This is called the STRUCTURE SHEAF $\mathcal{O}_X$

Check sheaf axioms [obvious!]

In "VARIETY THEORY" $k(V)$ gets used a LOT.
The sheaf is the same data but with better/more flexible user interface!

§2.3 BASIC CONTRUCTIONS

$\mathcal{F}$ a sheaf on X

Terminology: A section of $\mathcal{F}$ over U is some element
$s \in \mathcal{F}(U)$.

Construction 2.3.1 (stalks) Fix p in X. Then

$$\mathcal{F}_p = \text{stalk at } p$$
$$= \{(s,U) \mid s \in \mathcal{F}(U)\} / \sim$$

with $(s,U) \sim (s',U')$ if there exists nonempty $W \subseteq U' \cap U'$ such that

$$s'|_W = s|_W$$

We call elements of $\mathcal{F}_p$ a germ at p.

Example 2.3.2: Calculate $\mathcal{O}_{\mathbb{A}^1,0}$ — the stalk of the structure sheaf of $\mathbb{A}^1$ at 0.

Using "variety theory" Ex. 2.2.6. Answer: it is rational functions $f(t)/g(t)$ with $g(0) \neq 0$

The following shows the power of the sheaf axioms

PROPOSITION 2.3.3: If $f: \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves on X such that for all p

$$f_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p \text{ is an isomorphism.}$$

Then f is an isomorphism.

Meaning what?

PROOF: We will show that

$$f_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \text{ is an isomorphism}$$

for all U; define f^{-1} via f_U^{-1}.

Exercise: Show that this defines an inverse map of sheaves i.e. compatibility with restriction.

Injectivity: Suppose $s \in \mathcal{F}(U)$ with $f_U(s) = 0$. Then the germ of s is 0 in every stalk $\mathcal{F}_p$ for $p \in U$, by injectivity of f_p.

Unwind definition: there exist opens U_p around every p with $s|_{U_p} = 0$. Cover U by U_p. Use sheaf axioms.

Surjective: Let $t \in \mathcal{G}(U)$; we will build $s \in \mathcal{F}(U)$.

At the level of stalks, we have an iso, so this determines stalks in $\mathcal{F}_p$ for all $p \in X$. Now choose representatives (s_p, U_p) with $s_p \in \mathcal{F}(U_p)$.

By shrinking U_p if necessary, we can assume that $\boxed{f_{U_p}(t|_{U_p}) = s_p}$ by def'n of equiv. relation.

Now, injectivity shows that these glue. So writing $U_{pq} = U_p \cap U_q$: $f_{U_{pq}}(s_p|_{U_{pq}} - s_q|_{U_{pq}})$

$$= t|_{U_{pq}} - t|_{U_{pq}} = 0$$

By sheaf axioms these glue. By the sheaf axioms, the resulting section maps to t.

$\square$

⚡ Take note of the logic: injectivity was needed for proving surjectivity

REMARK 2.3.4 Even easier (Exercises)

(i) $\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p$ is injective by S1.

(ii) Given $\begin{tikzcd} \mathcal{F} \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] & \mathcal{G} \end{tikzcd}$ with $\varphi_p = \psi_p$ for all p

then $\varphi = \psi$.

DEFINITION 2.3.5 (sheafification) If $\mathcal{F}$ is a presheaf on X then a morphism $sh : \mathcal{F} \longrightarrow \mathcal{F}^{sh}$ is a sheafification if $\mathcal{F}^{sh}$ is a sheaf and for any "map" $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ with $\mathcal{G}$ a sheaf there is a unique diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "\varphi"'] & \mathcal{F}^{sh} \arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

that commutes

Remark 2.3.6: (i) Unique if it exists — if $\mathcal{F}^{+}$ were another, we get a diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{sh} \arrow[d, shift right] \\ & \mathcal{F}^{+} \arrow[u, shift right] \end{tikzcd}$$

Now use uniqueness to show isom.

(ii) Presheaf morphisms induce morphisms of sheafification

Trivial exercise

[Stalks of presheaves make sense; $f \in \mathcal{F}_p$ is germ at p]

Proposition/Construction 2.3.7

Sheafification exists. Given a presheaf $\mathcal{F}$ on X

define:

$\mathcal{F}^{sh}(U) = \{ (f_p)_{p \in U} : f_p \in \mathcal{F}_p$ & for every p there exists an open $V_p \subseteq U$ containing p and a section $s \in \mathcal{F}(V_p)$ st $s_q = f_q$ for all $q \in V_p \}$.

PROOF THIS WORKS:

- Restriction maps are clear; clearly a sheaf!
- The map $\mathcal{F} \to \mathcal{F}^{sh}$ is obvious.
- Exercise: verify the universal property. □

Note: $(\mathcal{F}^{sh})^{sh} = \mathcal{F}^{sh}$

Corollary 2.3.8: The stalks of $\mathcal{F}$ & $\mathcal{F}^{sh}$ coincide.

Exercise 2.3.9: Find a nonzero presheaf whose sheafification is zero. [This is actually rather stupid]

§2.4 KERNELS, COKERNELS, ETC

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves.

The presheaf KERNEL/IMAGE/COKERNEL assigns

$$U \longmapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U)) \text{ ; etc.}$$

If φ is a map of sheaves then

Exercise 2.4.1: The presheaf kernel of a map of sheaves is a sheaf.

Beware of the cokernel:

Example 2.4.2: $X = \mathbb{C}$, $\mathcal{O}_X = (\text{holomorphic functions}, +)$

and $\mathcal{O}_X^* = (\text{nowhere } 0 \text{ holomorphic functions}, \times)$. Now define

$$\exp : \mathcal{O}_X \to \mathcal{O}_X^* \; ; \; \mathcal{O}_X(U) \to \mathcal{O}_X(U)^*$$

$\ker(\exp) =$ constant sheaf $2\pi i \mathbb{Z}$.

Cokernel is not a sheaf! Take

$U_1 = \mathbb{C} \setminus [0,\infty)$; $U_2 = \mathbb{C} \setminus [0,-\infty)$

$$U = U_1 \cup U_2 = \mathbb{C} \setminus \{0\}.$$

Take $f = z$ in $\mathcal{O}_X(U)$. This lies in the presheaf cokernel of exp. But on U_i the cokernel is 0 because logarithm exists.

$\square$.

DEFINITION 2.4.3 For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves, the sheaf cokernel/image is the sheafification of the presheaf cokernel/image.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is injective/surjective if $\ker \varphi = 0$ / $\operatorname{im} \varphi = \mathcal{G}$.

Remark 2.4.4 (crucial!) the sequence

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$ is

exact as sheaves; in fact for X a $\mathbb{C}$-manifold

Remark 2.4.5: Do the kernel & cokernel deserve their name? What properties should they satisfy? The kernel of $\varphi: A \longrightarrow B$ is the data

$$\ker \varphi \longrightarrow A \quad \text{universal for}$$

things becoming 0: for any diagram

$$\begin{tikzcd} & K \arrow[dl, dashed, "\exists!"'] \arrow[d] \arrow[dr, "0"] & \\ \ker\varphi \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

sending K to 0, there is a unique filled in diagram.

Proximate Notions 2.4.6

(i) Subsheaf: $\mathcal{F} \subseteq \mathcal{G}$ if there are inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ compatible with restrictions

(ii) Quotient sheaf: the sheafification of

$$U \longmapsto \mathcal{G}(U)/\mathcal{F}(U).$$

Warning 2.4.7 If $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ is surjective the maps on any particular open need not be.

FACTS 2.4.7 These follow from the same kind of arguments we've used. I will not provide proofs.

(i) Stalks of kernel and image are kernel and image of the stalk maps.

(ii) Injectivity and surjectivity are stalk local properties BUT f surjective does not mean f_U's are always surjective Exponential Sequence !

§2.5 MOVING BETWEEN SPACES

Given $f : X \to Y$ with $\left.\begin{array}{l}\mathcal{F} \text{ on } X \\ \mathcal{G} \text{ on } Y\end{array}\right\}$ sheaves

DEFINITION 2.5.1 (pushforward) Define the presheaf $f_* \mathcal{F}$ on Y by $U \mapsto \mathcal{F}(\underbrace{f^{-1}(U)}_{\text{OPEN}})$

PROPOSITION 2.5.2 The pushforward is a sheaf.

Proof: Trivial. $\square$

DEFINITION 2.5.1 (inverse image) The inverse image presheaf is defined by

$$(f^{-1}\mathcal{G})^{pre}(V) = \{(s_U, U) : U \text{ open containing } f(V) \ \&\ s_U \in \mathcal{G}(U)\}/\sim$$

for $V \subseteq X$ open

where $\sim$ identifies pairs that agree on a smaller open. The inverse image is

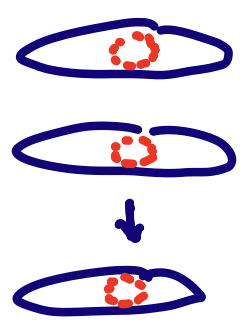
$$f^{-1}\mathcal{G} = (f^{-1}\mathcal{G}^{pre})^{sh}.$$

Example 2.5.2 (why is the sheafification necessary?)

Take $f : X \to Y$ with $X := Y \amalg Y$.

Take $\mathcal{G} = \underline{\mathbb{Z}}$ constant sheaf & $\mathcal{F} := (f^{-1}\mathcal{G})^{pre}$

Then fix $V \subseteq Y$ open and $U = f^{-1}(V)$



$$\mathcal{F}(U) = \mathcal{G}(V) = \mathbb{Z}$$

But $U = V \amalg V \subseteq Y \amalg Y$ so

$$\mathcal{F}^{sh}(U) = \mathcal{G}(V) \times \mathcal{G}(V) = \mathbb{Z} \times \mathbb{Z}$$ by sheaf axioms.

Contradiction!

Example 2.5.3 (Two simplest examples)

For $\mathcal{F}$ a sheaf on X and $\pi : X \to pt$ then $\pi_* \mathcal{F}$ is a sheaf on a point, i.e. an abelian group. which one?

GLOBAL SECTIONS

$$\mathcal{F}(X) =: \Gamma(X, \mathcal{F}) =: H^0(X, \mathcal{F}).$$

For $i : p \hookrightarrow X$ inclusion of a point and $\mathcal{G}$ a "sheaf on p" i.e. an abelian group A, then $i_* \mathcal{G}$ is the sheaf on X st

$$(i_* \mathcal{G})(U) = \begin{cases} 0 & \text{if } p \notin U \\ A & \text{if } p \in U \end{cases}$$

SKYSCRAPER SHEAF at p WITH STALK A

§3 SCHEMES Spec A has a sheaf; we globalize

§3.1 AFFINE SCHEMES

Let A be a ring and $S \subseteq A$ multiplicatively closed. Then

$$S^{-1}A = \{(a,s) : s \in S, a \in A\}/\sim$$

with $(a,s) \sim (a',s') \iff s''(as' - a's) = 0$ in A. for some $s'' \in S$.

Example 3.1.1: (i) Take $S = \{1, f, f^2, \dots\}$

(ii) Take $S = S \setminus \mathfrak{p}$ with $\mathfrak{p}$ a prime.

$A_\mathfrak{p}$ will be the stalk of the structure sheaf at $\mathfrak{p}$. I now take the route of Vakil — not Hartshorne

SHEAF on a BASE:

Sheaf $\mathcal{F}$ on X gives $\mathcal{F}: \{\text{Base opens}\} \to \text{Groups}$ + natural restrictions

Reverse this Given a base $\{B_i\}$ with $\mathcal{F}(B_i)$ assignments & $\mathrm{res}^{B_i}_{B_j}$ satisfying

SB1: if $B = \cup B_i$ with B in the base and $res^B_{B_i}(f) = res^B_{B_j}(g)$ for all f & g then $f \equiv g$.

SB2: If $B = \cup B_i$ with $f_i \in F(B_i)$ and agreeing on overlaps then there exists $f \in F(B)$ with $f|_{B_i} = f_i$. Go look at the discussion at the end of §1.

Call this a **SHEAF on a BASE** $\mathcal{B} = \{B_\alpha\}$

PROPOSITION 3.1.2 A sheaf on a base F with base $\mathcal{B}$ determines a sheaf $\mathcal{F}$ by

$$\mathcal{F}(B_i) = F(B_i)$$

agreeing with restriction maps, where $B_i \in \mathcal{B}$. It is unique up to unique isomorphism.

PROOF: (i) Determine the stalks $\mathcal{F}_p$ via the basis.

(ii) Use "sheafification trick" and define

$$\mathcal{F}(U) = \{ (f_p \in \mathcal{F}_p)_{p \in U} \mid \text{for all } p \in U \text{ there exists } s \in F(B),\ B \text{ a basis open around } p \text{ and with } s_q = f_q \text{ for all } q \in B \}.$$

Sheaf axioms are clear.

(iii) Natural maps $\mathcal{F}(B) \to \mathfrak{F}(B)$ are isomorphism
Check injective & surjective (compatible germs).

COLLECTION 3.1.2$\frac{1}{2}$

- A a ring; Spec A has basic opens

$U_f = \operatorname{Spec} A_f$

- For $f, g \in A$, $U_f \subseteq U_g \iff f^n = g \cdot a$ for some $n \geq 1$ and some $a \in A$.

[Key: $\mathbb{V}(g) \subseteq \mathbb{V}(f)$, and $\sqrt{(g)}$ is intersection of primes containing (g). So $\sqrt{(f)} \subseteq \sqrt{(g)}$

- If $U_f = U_g$ then by above, $A_f \cong A_g$
[Use universal property to get maps]

- If $f^n = g \cdot a$ there is a localization map $A_g \xrightarrow[\text{invert } a]{} A_{g \cdot a} \xrightarrow{\text{iso}} A_{f^n} \xrightarrow{\text{iso}} A_f$

PROPOSITION 3.1.3 Let A be a ring. The assignment

$$U_f = \{p \in \operatorname{Spec} A \mid f \notin p\} \mapsto A_f$$

is a sheaf on the base of distinguished opens in $\operatorname{Spec} A$. If $U_f \subseteq U_g$ then $f^n = a \cdot g$ for some $n \geq 1$, so get restriction maps: $A_g \to A_{f^n} \xrightarrow{\sim} A_f$

Well-defined: see collection $3.1.2\frac{1}{2}$.

PROOF: We check SB1 & SB2 on the base & set $B = \operatorname{Spec} A$ in the verification for simplicity; general case is similar,

Prologue: If $\{U_{f_i}\}_{i \in I}$ cover $\operatorname{Spec} A$, then finitely many cover [Why? $\sum_i (f_i) = (1)$]

SB1: Suppose $\operatorname{Spec} A = \bigcup_{i=1}^{n} U_{f_i}$; $U_{f_i} = \operatorname{Spec} A \setminus V(f_i)$.

Given $s \in A$ with $s|_{U_{f_i}} = 0$ for all i

then $f_i^m s = 0$ for appropriate m. [by defin. of localization]

But $(f_1^m, \dots, f_n^m) = (1) = A$ [b/c U_{f_i} COVER $\operatorname{Spec} A$]

$1 = \left(\sum r_i f_i^{m_i}\right)$. Now clear that

$s \cdot 1 = s = 0$.

SB2: Say $\operatorname{Spec} A = \bigcup_{i \in I} U_{f_i}$. and elements in A_{f_i} agreeing in $A_{f_i f_j}$ — do they glue?

First suppose I is finite.

On $U_{f_i} \rightsquigarrow$ have $\dfrac{a_i}{f_i^{l_i}}$. Write $g_i = f_i^{l_i}$

noting $U_{f_i} = U_{g_i}$. Overlaps: $U_{g_i g_j}$ (why?)

Overlap condition: $(g_i g_j)^{m_{ij}} \cdot (a_i g_j - a_j g_i) = 0$

Rewrite using algebra & fact that $U_f = U_{f^k}$, for $k \geq 1$

Assume $m = m_{ij}$ by taking the largest.

Write $b_i = a_i g_i^m$; $h_i = g_i^{m+1}$.

On each U_{h_i} have b_i / h_i.

Overlap condition:

$$h_j b_i = h_i b_j$$

But U_{h_i} cover Spec A so

$$1 = \sum r_i h_i \qquad r_i \in A.$$

$\underbrace{\text{Now write}}_{\text{Constructive!}}$ $r = \sum r_i b_i$ with r_i, b_i as above

Now verify this restricts correctly — elementary algebra

When I is infinite, pick a finite subcover with $(f_1, \ldots, f_n) = A$ and U_{f_i} a cover. Construct r as above. Need that this satisfies all restrictions

Now given $(f_1, \ldots, f_n, f_\alpha)$ we get a "new" r'. By SB1 $r' = r$

$\square$.

DEFINITION 3.1.4 The structure sheaf on Spec $A = X$ is the sheaf associated to the sheaf on the base

$$U_f \longmapsto A_f. \quad \text{Denoted } \mathcal{O}_{\mathrm{Spec}\, A} = \mathcal{O}_X.$$

Note that the stalks are A_p.

We are now basically there — a scheme is a pair $(X, \mathcal{O}_X)$ with $\mathcal{O}_X$ a sheaf of rings locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Provided we can understand "isomorphisms".

Terminology 3.1.5: · A ringed space $(X, \mathcal{O}_X)$ is a topological space with a sheaf of rings.

· An isomorphism $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism & an isomorphism

$$\mathcal{O}_Y \xrightarrow{\simeq} \pi_* \mathcal{O}_X$$

· An affine scheme $(X, \mathcal{O}_X)$ is a ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

DEFINITION 3.1.6 A scheme is a ringed space $(X, \mathcal{O}_X)$ locally isomorphic to an affine scheme i.e. every point $p \in X$ has a neighborhorhood U_p st $(U_p, \mathcal{O}_X|_{U_p})$ is isomorphic to an affine scheme (dependending on p).

§ 3.2 EXAMPLES OF SCHEMES

EXAMPLES 3.2.1 (INTERESTING RINGS)

- $k[x_1, \ldots, x_n]$ • Quotients by ideals THE GOLD STANDARD

- Monoid rings: A toric monoid P is the positive integer span of finitely many elements $\{v_1, \ldots, v_k\} \subseteq \mathbb{Z}^n$. The MONOID RING over $\mathbb{Z}$ is
$$\mathbb{Z}[P] = \{ \sum a_u x^u \mid a_u \in \mathbb{Z};\ u \in P \}$$
Dummy

$P = \mathbb{N}^2 \subseteq \mathbb{Z}^2$ then $\mathbb{Z}[P] \cong \mathbb{Z}[X, Y]$

$P = \mathbb{Z}^2$ then $\mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[X^{\pm}, Y^{\pm}]$

- Invariant rings: R^G or $k[x_1, \ldots, x_n]^G$ — Quotients of varieties. G a group $\curvearrowright k[X]$

- Artinian rings: for instance $k[t]/t^2$ or more generally k-algebras that are finite dim'l k-vector spaces. Space X contains little to no information here! All $\mathcal{O}_X$

Examples 3.2.2 (Open subschemes) Fix a scheme X, $U \subseteq X$ open (write: $i : U \hookrightarrow X$). Then set $\mathcal{U} = \mathcal{O}_X|_U = i^{-1}\mathcal{O}_X$

Simple case: $U = U_f \subseteq \operatorname{Spec} A$

PROPOSITION 3.2.3 The space $(U, \mathcal{O}_X|_U)$ is a scheme.

Pf: Let $p \in U \subseteq X$; since X a scheme, find $V_p \subseteq X$ open, affine nbd. Now, $V_p \cap U$ is open in affine. But there is a distinguished open in V_p inside $V_p \cap U$ and these are always affine. $\square$

Example 3.2.4: Take $U = \mathbb{A}^{n^2}_k \setminus \{\text{determinant} = 0\}$

GL_n is a scheme & a group.

Definition: Affine n-space$/k$ $= \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$

Example 3.2.5: (a non-affine scheme)

$$\begin{cases} \mathbb{A}^2_k := \operatorname{Spec} k[X,Y] \\ \qquad \& \\ U = \mathbb{A}^2_k \setminus \{(0,0)\}. = \{(x,y) \text{ the ideal}\} \end{cases}$$

CLAIM: $(U, \mathcal{O}_U)$ not affine.

Calculate $\mathcal{O}_U(U)$: Write $U_x = \mathbb{A}^2 \setminus \mathbb{V}(x)$

$U_y = \mathbb{A}^2 \setminus \mathbb{V}(y)$

Note: $U = U_x \cup U_y$ and $U_x \cap U_y = \mathbb{A}^2 \setminus \mathbb{V}(xy)$

$\mathcal{O}_U(U_x) = k[x, x^{-1}, y]$ $\quad$ $\mathcal{O}_U(U_y) = k[x, y, y^{-1}]$

$$\mathcal{O}_U(U_x \cap U_y) = k[x, x^{-1}, y, y^{-1}]$$

Restriction maps are obvious inclusions.

By sheaf axioms: $\mathcal{O}_U(U) = k[x, y]$. Contradiction!

[But why? Use maximal ideal m with $\mathbb{V}(m) = \emptyset$]

CLARIFICATION (Given in Lecture on Oct 27)

Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ global section

Fix $p \in X$ a point. The stalk

$\mathcal{O}_{X,p}$ is a ring with a unique maximal ideal.

[on an affine Spec A, it is A_p]

Say f vanishes at p if restriction of f to $\mathcal{O}_{X,p}$ lies in the maximal ideal.

Thus, for $f \in \Gamma(X, \mathcal{O}_X)$, $\mathbb{V}(f)$ = vanishing locus of f.

Later we will do this more generally, replacing $f \in \Gamma(X, \mathcal{O}_X)$ by sections of line bundles or vector bundles.

§ 3.3 INTERLUDE ON GLUING SHEAVES

X a topological space with a cover $\{U_\alpha\}$ and sheaves $\mathcal{F}_\alpha$ and isomorphisms

$$\phi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$$

inverse image sheaf

satisfying $\boxed{\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}}$ on $U_{\alpha\beta\gamma}$.

COCYCLES

with $\phi_{\alpha\alpha} = id$, $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$.

Then this gives to a sheaf $\mathcal{F}$ on X.

Construction 3.3.1 Given $V \subseteq X$ define $\mathcal{F}(V)$ as tuples (s_α) with $s_\alpha \in \mathcal{F}_\alpha(V \cap U_\alpha)$. with

$$\phi_{\alpha\beta}(s_\alpha|_{V \cap U_{\alpha\beta}}) = s_\beta|_{V \cap U_{\alpha\beta}}.$$

— CONDITION (*)

Proposition 3.3.2: $\mathcal{F}$ is a sheaf and $\mathcal{F}|_{U_\alpha}$ is $\mathcal{F}_\alpha$.

Proof: • $\mathcal{F}$ is a presheaf: given $(s_\alpha) \in \mathcal{F}(V)$, $W \subseteq V$ open, take $(s_\alpha)|_W = \left(\text{res}^{V\cap U_\alpha}_{W\cap U_\alpha}(s_\alpha)\right)_\alpha$

Gives an element of $\mathcal{F}(W)$ because $\phi_{\alpha\beta}$ are isomorphisms of sheaves, so commute with restricti[ons]

Axiom S1: trivial; Axiom S2: fun & trivial!

But not done! Claim $\mathcal{F}_\gamma = \mathcal{F}|_{U_\gamma}$ on U_γ and here we use cocycle condition.

What is the isomorphism from $\mathcal{F}_\gamma \longrightarrow \mathcal{F}|_{U_\gamma}$

Given $V \subseteq U_\gamma$ and $s \in \mathcal{F}_\gamma(V)$

take its image in $\mathcal{F}(V)$ to be

$$\left(\phi_{\gamma\alpha}\left(s|_{V\cap U_\alpha}\right)\right)_\alpha$$

But check this satisfies Condition (*)

$$\phi_{\alpha\beta}\circ\phi_{\gamma\alpha}\left(s|_{V\cap U_\alpha\cap U_\beta}\right) = \phi_{\gamma\beta}\left(s|_{V\cap U_\alpha\cap U_\beta}\right)$$

§ 3.4 More schemes

□

Take schemes $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ with opens $U \subseteq X$ & $V \subseteq Y$ with an isomorphism

$$(U, \mathcal{O}_X|_U) \xrightarrow{\cong} (V, \mathcal{O}_Y|_V)$$ meaning what?

Then we can glue!

$X \amalg Y / U \sim V$ with the glued structure sheaf.

This generalizes cleanly

Example 3.4.1: (Bug-eyed line) Let

$X = \operatorname{Spec} k[t]$ & $Y = \operatorname{Spec} k[u]$ both $\mathbb{A}^1$.

$U = \operatorname{Spec} k[t, t^{-1}]$ & $V = \operatorname{Spec} k[u, u^{-1}]$. both "$\mathbb{A}^1 \setminus pt$"

Glue via $t \longleftrightarrow u$

Compare from topology: $\mathbb{R}_x \amalg \mathbb{R}_y / x \sim y$ for $x = y \neq 0$.

This is the canonical example of Hausdorff failing. But schemes are already not Hausdorff. But still...

This scheme is not affine. Calculate that $\mathcal{O}_X(X) = k[t]$ but there is an extra point!

↳ Make rigorous!

Example 3.4.2: (Projective line)

$X = \operatorname{Spec} k[t]$ & $Y = \operatorname{Spec} k[u]$ both $\mathbb{A}^1$.

$U = \operatorname{Spec} k[t, t^{-1}]$ & $V = \operatorname{Spec} k[u, u^{-1}]$. both "$\mathbb{A}^1 \setminus pt$"

Glue via $t \longleftrightarrow u^{-1}$.

PROPOSITION 3.4.3 $\mathbb{P}^1_k$ has only constants as the global functions; in particular $\mathbb{P}^1$ is not affine.

Proof: The only polynomials in t that are polynomials in $1/t$ are constant.

[Why is this a proof? Use sheaf axioms!].

Example 3.4.4 Take $\mathbb{A}^2_k$ with doubled origin.

Notice intersection of two affines is not.

GLUING SCHEMES (EXAMPLE SHEET 1)

Given
- schemes $X_i \quad i \in I$
- open subschemes $X_{ij} \subseteq X_i$; $X_{ii} = X_i$
- isomorphisms $f_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$; $f_{ii} = \mathrm{id}$

such that $f_{ik}\Big|_{X_{ij} \cap X_{ik}} = f_{jk}\Big|_{X_{ji} \cap X_{jk}} \circ f_{ij}\Big|_{X_{ij} \cap X_{ik}}$

There is a unique scheme X with cover X_i

KEY EXAMPLE: Projective space.

A ring A. Take $X_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$

$$U_{ij} = \mathbb{V}(x_j/x_i)^c \subseteq X_i$$

Isom: $U_{ij} \xrightarrow{\sim} U_{ji}$ identifying $\frac{x_k}{x_i} \mapsto \frac{x_k}{x_j} \cdot \frac{x_j}{x_i}$

Output: a scheme $\mathbb{P}^n_A$ called projective space.

A HEADS UP: An important condition for us will be separated. It will be the analogue of Hausdorff. It will imply that (affine $\cap$ affine) is affine.

§ Appendix THE PROJ CONSTRUCTION — motivation

- A few words of motivation — it is actually hard to produce schemes that are not "open in proj" — i.e. quasi-projective i.e. PART II AG

- "Separated" will be the "AG Hausdorff" condition.
  "Proper" will be the "AG Compact" condition.

Proj constructions will always give us proper ($\Rightarrow$ separated) things.

DEFINITION 3.5.1: A $\mathbb{Z}$-grading on a ring A is a decomposition $A = \bigoplus_{n \in \mathbb{Z}} A_n$ such that $A_k \cdot A_j \subseteq A_{j+k}$ "multiplication respects grading"

But what is the geometry behind graded rings?

PROPOSITION 3.5.2 (for motivation) Let A be a (finitely generated nilpotent free) $k=\bar{k}$-algebra.
Let $V = \mathrm{mSpec}(A)$ i.e. the variety of A.
Then a k^*-action on V given by a morphism

$k^* \times V \longrightarrow V$ is the same thing as

a grading of A by $\mathbb{Z}$.

Variety Theory: Define $\mathbb{P}^n_k = \mathbb{A}^{n+1}_k \setminus \underline{0} / k^*$.

Only homogeneous polynomials make sense:

i.e. $\sum_\alpha a_\alpha \underline{x}^\alpha$ $\quad \alpha \in \mathbb{N}^{n+1}$ with

degree $(\underline{x}^\alpha) = d$. In other words:

$$k[x_0, .., x_n] = \bigoplus_{d \geq 0} S_d \; ; \quad S_d \text{ is } k\text{-span}$$

of degree d monomials.

Observe how both "graded" and k^*-action appear naturally.

This works not just for $\mathbb{P}^n_k$ but for projective varieties:

"irrelevant point"

$$\begin{tikzcd}
\pi^{-1}(V) = \tilde{V} \arrow[r] \arrow[d] & \mathbb{A}^n_k \setminus \underline{0} \arrow[d, "\pi"', "k^*\text{-quotient}"] \\
V \arrow[r, phantom, "\subseteq"] & \mathbb{P}^n_k
\end{tikzcd}$$

$\| \;$ $\overline{V}$ = closure of $\tilde{V}$ in $\mathbb{A}^n$

$V = \mathbb{V}(\text{homogeneous poly's})$

Notice that $\tilde{V}$ is k^*-invariant as is $\overline{V}$! why?

Therefore to get a projective variety:

(i) Take $\overline{V} \subseteq \mathbb{A}^n_{k+1}$ a k^*-invariant variety

(ii) Throw out junk i.e. $\underline{0}$ b/c it is dumb.

(iii) Take a quotient.

§3.5 THE PROJ CONSTRUCTION

We've lifted $\mathbb{A}^n_k$ into scheme theory

Want to do the same for $\mathbb{P}^n_k$

Spec $k[x_1,\dots,x_n]$ gives the "new" scheme theory $\mathbb{A}^n_k$.

Similarly Proj $\overbrace{k[x_0,\dots,x_n]}^{\text{standard grading}}$ will give the "new" $\mathbb{P}^n_k$

DEFINITION 3.5.1: A $\mathbb{Z}$-grading on a ring A is a decomposition $A = \bigoplus_{n\in\mathbb{Z}} A_n$ such that

$A_k\cdot A_j \subseteq A_{j+k}$ "multiplication respects grading"

I will only treat Proj A in a simplified setting. Assume A is $\mathbb{Z}_{\geq 0}$-graded. Moreover, A is generated over A_0 by degree 1 elements.

A_0 : subring of A

$A_+ := \bigoplus_{i>0} A_i$ ideal in A — the "irrelevant ideal".

$I \subseteq A$, is homogeneous if generated by homog. elts.

DEFINITION 3.5.2 : The set Proj $A_\bullet$ is the set of homogeneous primes of $A_\bullet$ not containing A_+.

If $I \subseteq A_\bullet$ is homogeneous:

$$\mathbb{V}(I) := \{ \mathfrak{p} \in \operatorname{Proj} A_\bullet \mid \mathfrak{p} \text{ contains } I \}$$

The Zariski topology has closed sets given by $\mathbb{V}(I)$.

We now cover by affines using degree 1 elts.

Let $f \in A_1$ and $U_f = \operatorname{Proj} A_\bullet \setminus \mathbb{V}(f)$

These form a cover; ring $A[\frac{1}{f}]$ is $\mathbb{Z}$-graded.

PROPOSITION 3.5.2 Bijection between

$$\left\{ \begin{array}{l} \text{Homogeneous primes} \\ \text{of } A \text{ missing } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Primes in the} \\ \text{ring } (A[1/f])_0 \end{array} \right\}$$

(or hom. primes in A_f)

degree 0 in localization

Proof : Construction of bijection.

Homog. primes in A_f are in bijection with hom. primes of A not containg f.

Suppose $q \subseteq (A[1/f])_0$ a prime. Then let $\psi(q)$ be generated by:

$$\bigcup_{d \geq 0} \left\{ a \in A_d \;\middle|\; \frac{a}{f^d} \in q \right\} \subseteq A$$

Given $p \subseteq A$. homogeneous, take

$$\varphi(p) = (A[1/f] \cdot p \cap A[1/f]_0)$$

Now check compositions: $\varphi \circ \psi = id$ [easy]

<u>But $\psi \circ \varphi$ is trickier</u>:

- suppose $p \in U_f \subseteq \operatorname{Proj} A$, if $a \in p \cap A_d$, then $a/f^d \in \varphi(p)$ so $a \in \psi(\varphi(p))$ so $p \subseteq \psi(\varphi(p))$. — one containment

- If $a \in \psi(\varphi(p))$ then $a/f^d \in \varphi(p)$ for some d, so there is $b \in p$, st $b/f^e = a/f^d$.

<u>So for some</u> k, $f^k(f^d b - f^e a) = 0$

But $f^{e+k} \notin p$, so $a \in p$. — reverse containment

Compatible with Zariski topology — Exercise!

Remark 3.5.3: A basis for the Z-topology on $\operatorname{Proj} A_\bullet$ is given by opens $U_f = \mathbb{V}(f)^c$ for $f \in A_+$.

Notice, we have a natural identification

$$U_f = \operatorname{Spec}(A_\bullet[1/f])_0 \text{ by above}$$

State of the union: $\operatorname{Proj} A_\bullet$ is a set of homogeneous prime ideals. It is covered by U_f which are Spec of a ring and have structure sheaves, where by hypothesis, f can be taken to be degree 1.

If $(\operatorname{Proj} A_\bullet)_f = \mathbb{V}(f)^c$, and $f, g \in A_1$, we have

$$(\operatorname{Proj} A_\bullet)_f \cap (\operatorname{Proj} A_\bullet)_g \quad \text{is}$$

$$\operatorname{Spec}\, (A_\bullet[1/f])_0[f/g] = \operatorname{Spec}\,(A_\bullet[f^{-1}, g^{-1}])_0$$

This gives gluing data — cocycle condition is immediate from properties of localization.

Lots of new examples! ⚠ Proj is not functorial!

§4 MORPHISMS

We have now lots of examples of schemes coming from "variety theory". We want MAPS

We've seen these in passing. • eg $U \subseteq X$ open subscheme, • given $A \to B \rightsquigarrow \operatorname{Spec} B \to \operatorname{Spec} A$

§4.1 MORPHISMS OF SCHEMES & LOCALLY RINGED SPACES

Given a scheme $(X, \mathcal{O}_X)$ the stalks $\mathcal{O}_{X,p}$ are LOCAL RINGS. Given a "function" $f \in \mathcal{O}_X(U)$ we can ask whether f vanishes at p (or ask its value in $\mathcal{O}_{X,p}/p\mathcal{O}_{X,p}$.

DEFINITION 4.1.1 A morphism of ringed spaces

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y) \text{ is a pair } (f, f^\#)$$

with $f : X \longrightarrow Y$ continuous

$f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ sheaves of rings on Y.

This is the "obvious" idea. But:

Possible to have $f : X \longrightarrow Y$ with $h(p) = q$

and $U \subseteq Y$ open with $q \in U$, and $h \in \mathcal{O}_Y(U)$ that vanishes at q, but f^*h does not vanish at p.

We simply impose this condition by hand

Given $f: X \to Y$ ringed space map, there is an induced map $f^\#: \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p}$.

Careful! Why does this map exist?

DEFINITION 4.1.2 $(X, \mathcal{O}_X)$ is locally ringed if stalks $\mathcal{O}_{X,p}$ are local [automatic for schemes]. A morphism of locally ringed spaces:

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

with $\underbrace{f^\#(m_p) \subseteq m_{f(p)}}_{\text{"Local homomorphism"}}$ in stalks

DEFINITION 4.1.3: If $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ are schemes, a morphism of schemes is a morphism as locally ringed spaces

[What does it buy us?] If $\varphi: X \to Y$ morphism of schemes, if $s \in \mathcal{O}_{Y,\varphi(p)}$ is invertible then $\varphi^{\#}(s) \in \mathcal{O}_{X,p}$ is too. You can tell where functions vanish by composition.

THEOREM 4.1.9: There is a natural bijection

$$\left\{ \begin{array}{l} \text{Morphisms from} \\ \operatorname{Spec} B \to \operatorname{Spec} A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Ring homomorphisms} \\ A \to B \end{array} \right\}$$

PROLOGUE: A section $s \in \mathcal{F}(U)$ is a coherent collection of elts in stalks $\mathcal{F}_p$ for $p \in U$

PROOF: 1. $A \to B$ induces a scheme map

2. Every scheme map arises this way

=

1. Given $\varphi: A \to B$, $f: \operatorname{Spec} B \to \operatorname{Spec} A$ sends $\mathfrak{p}$ to $\varphi^{-1}(\mathfrak{p})$. Gives map at topological level. Continuity is formal from:

$$f^{-1}(\mathbb{V}(I)) = \mathbb{V}(\varphi(I)).$$

Now we build:

$$f^{\#}: \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}.$$

by specifying what happens at stalk level.

Take: $A_{\varphi^{-1}(p)} \longrightarrow B_p$

$$\frac{a}{s} \longmapsto \frac{\varphi(a)}{\varphi(s)}$$

well-defined: If $s \notin \varphi^{-1}(p)$ then $\varphi(s) \notin p$.

It is automatically local! The maximals are pB_p and $\varphi^{-1}(p) A_{\varphi^{-1}(p)}$.

Now think on opens: Given $U \subseteq \operatorname{Spec} A$ open

take $f^{\#} : \mathcal{O}_{\operatorname{Spec} A}(U) \longrightarrow \mathcal{O}_{\operatorname{Spec} B}(\varphi_{top}^{-1}(U))$

View sections as compatible collections of germs.

$$[p \mapsto s(p)]_{p \in U} \longmapsto [q \mapsto \varphi_q(s(f(q)))]_{q \in f^{-1}(U)}$$

$s(p) \in A_p$ where $\varphi_q : A_{\varphi^{-1}(q)} \longrightarrow B_q$ is map induced by localization.

View $\mathcal{O}_{\operatorname{Spec} A}(U) \subseteq \prod_{p \in U} \mathcal{O}_{\operatorname{Spec} A, p}$ and $\mathcal{O}_{\operatorname{Spec} B}$ similarly

Does it define map of sheaves? If s is locally at p written $\frac{a}{h}$

$f^{\#}(s)$ is written as $\varphi(a)/\varphi(h)$. Thus, $f^{\#}$ is a morphism of sheaves

Therefore $A \to B$ yields a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$.

2. Conversely, take: $(f, f^{\#}) : \operatorname{Spec} B \to \operatorname{Spec} A$.

Using global section maps: $g : \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)$, we get: $g : A \longrightarrow B$. Now, plug g into construction above. Must show we get $(f, f^{\#})$ as given.

Two things: topological map & sheaf map.

Since g is compatible with restriction to stalks:

$$\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\
\mathcal{O}_{\operatorname{Spec} A, f(p)} \arrow[r] & \mathcal{O}_{\operatorname{Spec} B, p}.
\end{tikzcd}$$

commutes. Equivalently:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \\
A_{f(p)} \arrow[r, "f^{\#}"'] & B_p.
\end{tikzcd}$$

also commutes.

By locality $(f_p^{\#})^{-1} \mathfrak{p} B_{\mathfrak{p}} = f(p) A_{f(p)}$

By commutativity $f(p) = g^{-1}(p)$. Thus, topologically we get the right map. They agree on stalks so we're done. $\square$

§ 4.2 A FEW BASIC NOTIONS (HOUSEKEEPING)

Open & closed immersions; closed subschemes.

DEFINITIONS 4.2.1

$f : X \to Y$ is an open immersion if f induces an isomorphism onto an open subscheme of Y i.e. $(U, \mathcal{O}_Y|_U)$; $U \subseteq Y$ open

$g : X \to Y$ is a closed immersion if g^{top} is a homeomorphism ont a closed subset and $g^\# : \mathcal{O}_Y \to g_* \mathcal{O}_X$ is surjective.

Example 4.2.2 : Take $k[t] \longrightarrow k[t]/t^2$ and take Spec. This is closed.

AWKWARD DEFINITION 4.2.3 : A closed subscheme is an equivalence class of closed immersions, where $[X \to Y] \sim [X' \to Y]$ if there is a triangle.

$$\begin{tikzcd} X' \arrow[rr, "\text{iso}"] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

Scheme theoretic points: Let K be any field. A K-valued point of a scheme X is a morphism

$$\mathrm{Spec}\, K \longrightarrow X.$$

We write the set of all such maps as $X(K)$.

Example 4.2.6 Take $X = \mathbb{P}^n_{\mathbb{C}}$. Then $X(\mathbb{C})$ is the $\mathbb{P}^n_{\mathbb{C}}$ you know and love.

Remark 4.2.5: For any ring R we could define R-valued points similarly. Infact, we can do the same for S any scheme! We will therefore obtain

$$F_X : \mathrm{Rings} \longrightarrow \mathrm{Sets}$$
$$R \longmapsto X(R).$$

This "functor of points" is eventually very useful, but I want to stay close to geometry.

Very concrete! Given $p \in X$, there is an affine open U around p. Setting $k(p) = FF(\mathcal{O}_{X,p})$ we get

$$\mathrm{Spec}\, k(p) \longrightarrow U \hookrightarrow X$$

Every point is a scheme theoretic point

§ 4.3 FIBRES & FIBRE PRODUCTS

Motivation: Fibre products are a common generalization of several operations: (0) The right notion of product.

(i) $X_1 \hookrightarrow Y$ & $X_2 \hookrightarrow Y$ closed subschemes

Intersection "$X_1 \cap X_2$" is a fibre product

(ii) Given $X \xrightarrow{f} Y$ a morphism and $y \in Y$, the fibre $f^{-1}(y)$ is a scheme

(iii) The intuitive statement that $\mathbb{P}^n_{\mathbb{C}}$ is obtained from $\mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{Z} \hookrightarrow \mathbb{C}$.

DEFINITION 4.3.1 Let

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

be morphisms of schemes. The **fibre product** is a scheme $X \times_S Y$ with maps

$$\begin{tikzcd} X \times Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

commuting,

such that for any

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

commuting, there is a unique map

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

with all diagrams commuting. If exists, unique upto unique iso

Makes sense in any category. If X, Y, B were just sets then $X \times_B Y \subseteq X \times Y$ as pairs that project to the same point of B.

THEOREM 4.3.2 Fibre products of schemes exist.

PROOF: [Hartshorne Theorem 3.3 — do look it up!]

1. Affine Case: If X, Y, S are affine with rings A, B, R then $\mathrm{Spec}\,(A \otimes_R B)$ satisfies the

universal property. To verify, notice that by same ideas in previous lecture, a map

$Z \longrightarrow \operatorname{Spec} A \otimes_R B$ is a ring map

$A \otimes B \longrightarrow \Gamma(Z, \mathcal{O}_Z)$.

Globalization: Slowly turn the 3 pieces into affines.

2. If $X \times_S Y$ exists and $U \subseteq X$ is open then $U \times_S Y$ exists: take $p_X^{-1}(U)$ with open subscheme structure.

3. If X is covered by $\{X_i\}$ then if $X_i \times_S Y$ exists they can be glued to $X \times_S Y$.

why? The schemes already glue to X, but the maps to Y can also be glued — this is easier than you think — no cocycle conditions !

4. For any X but Y & S affine, $X \times_S Y$ exists. Since X & Y are interchangeable, $X \times_S Y$

exists for affine S.

5. Cover S by affines $\{S_i\}$. Let X_i & Y_i be the p_X & p_Y preimages. $X_i \times_{S_i} Y_i$ exist.

But in fact, $X_i \times_{S_i} Y_i = X_i \times_S Y$ [Think about intersections! You have flexibility!]

6. Now glue again ∇!

$\square$

Notation:

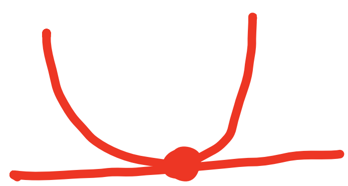
$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

Little box says Z is the fibre product.

Examples 4.3.3: with some honest geometry

(i) $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C}$. [why?]

(ii) Take $C = \operatorname{Spec} \mathbb{C}[x,y]/y-x^2$



$L = \operatorname{Spec} \mathbb{C}[x,y]/(y)$

Then $C \times_{\mathbb{A}^2} L = \operatorname{Spec} \mathbb{C}[x]/x^2 \rightsquigarrow$ "FAT POINT"

(iii)

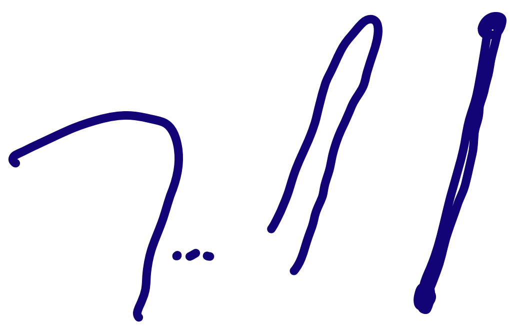
Total space of family

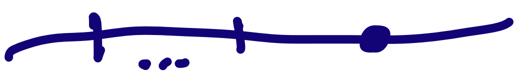
$$\begin{tikzcd} \operatorname{Spec} \dfrac{\mathbb{C}[x,y,t]}{(y^2+tx)} \arrow[r] & \operatorname{Spec} \mathbb{C}[t] \\ \operatorname{Spec} \dfrac{\mathbb{C}[x,y]}{y^2} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \operatorname{Spec} \mathbb{C}[t]/(t) \arrow[u] \end{tikzcd}$$

Family of mostly conics.

closed point 0.

"DOUBLED" X-Axis



More generally: 

(iv) Recall that given $p \in S$ we defined $\kappa(p) = FF(A/p)$ with $\operatorname{Spec} A \hookrightarrow S$ an open neighborhood. Given $X \to S$ the scheme theoretic fiber of $X \to S$ at p is

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \kappa(p) \arrow[r] & S \end{tikzcd}$$

SCHEME OVER $\kappa(p)$

⚠ If S is arithmetic eg $\operatorname{Spec} \mathbb{Z}$, fibres live in different FIELDS !!

(v) In Ex (iii) take $\operatorname{Spec} \mathbb{C}(t) \hookrightarrow \operatorname{Spec} \mathbb{C}[t]$

The generic fibre of π

$$\begin{tikzcd} \operatorname{Spec} \dfrac{\mathbb{C}(t)[x,y]}{(y^2+tx)} \arrow[r] \arrow[d] & \operatorname{Spec} \dfrac{\mathbb{C}[x,y,t]}{(y^2+tx)} \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{C}(t) \arrow[r] & \operatorname{Spec} \mathbb{C}[t] = S \end{tikzcd}$$

Consolidates information that is constant on an open set in the base S

§4.3½ : Example sheet II contains many basic notions — reduced, irreducible, integral, noetherian, finite type. — You should read the sheet at a minimum. (i) We will not need it for now (ii) I will supply a number of examples later

Language 4.3.4 In scheme theory, we often fix a base scheme S and study the collection of schemes $X \to S$. If no such choice is made, we take $\underbrace{S = \operatorname{Spec} \mathbb{Z}}_{\text{Terminal object.}}$ implicitly

In variety theory, $S = \operatorname{Spec} k$ $(k = \bar{k})$.

The product of varieties X & Y is

$$X \underset{\operatorname{Spec} k}{\times} Y.$$

In $\underbrace{Sch/S}_{\text{schemes over } S}$, given X/S & Y/S the product in this category is $X \underset{S}{\times} Y$.

The "USUAL" PRODUCT never comes up [until you start using $\mathbb{C}$ + Euclidean topology].

Let $k = \bar{k}$ be a field. A scheme $X \to \operatorname{Spec} k$ is a variety if there exists a finite cover $\{U_i\}$ of X st $\mathcal{O}_X(U)$ is a finitely generated k-algebra without nilpotents. + Separated

§4.4 SEPARATED MORPHISMS

Given X a scheme X_{top} is essentially never Hausdorff. But bug-eyed-line is worse than $\mathbb{A}^1$ or $\mathbb{P}^1$ – why?

Hausdorff is about separating pairs of points and so can be phrased in $\boxed{\text{topology}}$ as

$$X \text{ is Hausdorff} \iff \Delta_X \subseteq \underbrace{X \times X}_{\text{product topology}} \text{ is closed.}$$

DEFINITION 4.4.1: Given $X \to S$ a scheme map the diagonal $\Delta_{X/S}$ is the morphism below:

$$\begin{tikzcd} X \arrow[dr, "\Delta_{X/S}"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\ & X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

(*) [induced by universal property]

Write Δ insted of $\Delta_{X/S}$ when clear.

Useful: If $U, V \subseteq X$ then $X \times_{X \times_S X} (U \times V) = U \cap V$.

PROPOSITION 4.4.2: Let $X \to S$ be a scheme map. The diagonal is a locally closed immersion,

i.e. $$\begin{tikzcd} X \arrow[r, "\text{CLOSED}"] \arrow[rr, bend right, "\Delta"'] & U \arrow[r, "\text{OPEN}"] & X \times_S X \end{tikzcd}$$

PROOF: We find open in $X \times_S X$ in which X is closed

Say S is covered by $\{V_i\}$ affine opens and X is covered by affines $\{U_{ij}\}$ so U_{ij} covers preimage of V_i

Let $U_{ij} \to V_i$ be the maps induced by

$$\begin{tikzcd}
U_{ij} \arrow[r] & X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V_i \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Now $U_{ij} \times_{V_i} U_{ij}$ is affine & covers the diagonal

Also, $\Delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$ [Use Diagram ⊛ !]

For U_{ij} affine

$$U_{ij} \hookrightarrow U_{ij} \times_{V_i} U_{ij}$$ is $\boxed{\text{clearly}}$

a closed immersion.

If you remember the definitions.

□

PROPOSITION 4.4.3 If $X \to S$ is a map of affine schemes then $\Delta_{X/S}$ is a closed immersion

$[A \otimes_B A \to A$ always surjective$]$

DEFINITION 4.4.4 ← spooky! A morphism $X \to S$ is separated if the diagonal is a closed immersion.

Easy fact: If $X \to Y$ is a locally closed immersion whose image is a closed topological subset, then it is closed [definition chasing]

Examples 4.4.5: (i) For any ring R the morphism $\mathbb{A}^n_R \to \operatorname{Spec} R$ is separated.

(ii) The bug-eyed line $\mathbb{A}^1_k \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1_k$ is NOT separated over $\operatorname{Spec} k$.

(iii) For a ring R $\mathbb{P}^n_R \to \operatorname{Spec} R$ is separated.

(iv) Open & closed embeddings are separated.

(v) Composition of separated maps are too.

(vi) Base extensions of separated morphisms

Exercises — or we will prove later.

PROPOSITION 4.4.6: Let R be any ring. Then $\mathbb{P}^n_R \longrightarrow \operatorname{Spec} R$ is separated, [recalling $\mathbb{P}^n_R := \operatorname{Proj} R[x_0,\dots,x_n]$]

<u>Proof</u>: We want to show that

$$\begin{tikzcd} \mathbb{P}^n_R \arrow[r, "\Delta"'] & \mathbb{P}^n \times_R \mathbb{P}^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}^n \arrow[d] \\ & \mathbb{P}^n \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

is closed. Suffices to show after restricting to an open cover of $\mathbb{P}^n \times_R \mathbb{P}^n$.

[recall from construction of $\mathbb{P}^n_R$]

Set $A_\bullet = R[\underline{x}]$ and $U_i = \operatorname{Spec}(A_\bullet[\frac{1}{x_i}])_0$

The schemes $U_i \times_R U_j$ cover $\mathbb{P}^n \times_R \mathbb{P}^n$

Now: $U_i \times_R U_j = \operatorname{Spec} R[x_1/x_i, \dots, x_n/x_i, y_1/y_j, \dots, y_n/y_j]$

<u>Exercise</u>: restriction of diagonal is exactly

$$U_i \cap U_j \longrightarrow U_i \times_R U_j$$

$$R\left[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}\right]\left[\frac{x_i}{x_j}\right] \longleftarrow R\left[\frac{x_1}{x_i},\dots,\frac{x_n}{x_i},\frac{y_1}{y_j},\dots,\frac{y_n}{y_j}\right]$$

replace y's with x's. Map is clearly surjective, so Δ is closed. $\square$

NICE FACT: Closed in affine is always affine [ExSh III]

PROPOSITION 4.4.7: Let U, V be affine opens of a separated scheme X/S. Suppose S is affine. Then $U \cap V$ is affine.

Exercise: In situation above: $U \cap V = (U \times_S V) \cap \Delta$

Proof: The following diagram is Cartesian

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta_{X/S}"'] & X \times_S X \end{tikzcd}$$

Since $\Delta_{X/S}$ is separated $U \cap V \to U \times_S V$ is closed. $\square$

§4.6 PROPERNESS

Recall from ExSh II $f: X \to Y$ is finite type if Y has a cover by $\{V_\alpha\}$ st $V_\alpha = \operatorname{Spec} A_\alpha$; $f^{-1}(V)$ has a finite cover by $U_{\alpha\beta}$ affine such that $U_{\alpha\beta} = \operatorname{Spec} B_{\alpha\beta}$ and B_i is finitely generated over A_α

Example: $\operatorname{Spec} k[\underline{X}]/I$

DEFINITION 4.6.1: A scheme map $X \xrightarrow{f} Y$ is closed if f is a closed map. It is universally closed if f' is also closed for any base extension:

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

$\boxed{\text{WHY?}}$ Think about closed vs. compact.

A scheme map $f: X \to Y$ is proper if it is separated, finite type, and universally closed.

Example: Closed immersions are proper. $\mathbb{A}^1_k$ not!

OBSERVATION: If $f: X \to Y$ is proper and $Z \to Y$ is a morphism, $X \times_Y Z \to Z$ is proper.

PROPOSITION 4.6.2: Suppose is any commut. ring. Then $\mathbb{P}^n_R \to \operatorname{Spec} R$ is proper.

Proof: Suffices to show $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is universally closed. But by taking covers, suffices to show for all R, the map

$\mathbb{P}^n_R \xrightarrow{\pi} \operatorname{Spec} R$ is closed. Let $Z \subseteq \mathbb{P}^n_R$ be Zariski closed. Need equations for $\pi(Z)$.

Let $\mathfrak{p} \subseteq R$ be prime and $\kappa(\mathfrak{p}) = FF(R/\mathfrak{p})$.

Want to know: For which $\mathfrak{p}$ is the scheme

$$Z_{\mathfrak{p}} = Z \times_{\operatorname{Spec} R} \operatorname{Spec} \kappa(\mathfrak{p})$$ nonempty?

Now, say Z is cut out by $g_1, g_2, \ldots$ homogeneous.

Then $Z_{\mathfrak{p}}$ is nonempty if and only if

$$\sqrt{(g_1, g_2, \ldots)} \not\supseteq (x_0, \ldots, x_n)$$

Notice: Does not really depend on $\mathfrak{p}$.

Equivalently, for all d positive integer:

$$(x_0, \ldots, x_n)^d \not\subseteq (g_1, g_2, \ldots)$$

Equivalently, if $S = R[\underline{x}]$ with usual grading the map:

$$\bigoplus_i S_{d-\deg(g_i)} \longrightarrow S_d$$

$$f_i \longmapsto f_i g_i \quad \text{is not surjective.}$$

This is a polynomial condition with integer coefficients, independent of p.

These precisely cut out $\pi(Z)$ in Spec R, so we have the claim.

$\square$

Proper morphism have similar properties:

- Compositions of proper maps are proper
- Base extensions are proper

These will be proved in one way in ExSh's. I will give another way

Here forward assume all schemes are Noetherian

A discrete valuation ring is a local PID.

Examples 4.4.6: $\mathbb{C}[[t]]$, $\mathcal{O}_{\mathbb{A}^1,0}$, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ p-adic integers.

If A is a DVR $\operatorname{Spec} A$ is a connected doubleton. $\bullet \rightsquigarrow \cdot$ | GERMS OF CURVES

OPEN CLOSED

max'l ideal has a gen. called the uniformizer

There is a valuation: $A \setminus \{0\} \longrightarrow \mathbb{Z}_{\geq 0}$

THEOREM 4.6.2 (VALUATIVE CRITERIA): let $X \xrightarrow{f} Y$ be a scheme map. Then f is separated with $FF(A) = K$ is separated iff for any DVR A

given

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is at most 1 lift

making everything commute. It is universally closed if there is at least 1 lift. It is proper if there is exactly 1 lift and f is of finite type.

COROLLARY 4.6.3

(i) $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper

(ii) $\mathbb{A}^n_A \to \operatorname{Spec} A$ for $n \geq 1$ is NOT proper

(iii) Closed subschemes of $\mathbb{P}^n_A$ are proper over Spec A.

(iv) Closed immersions are proper

(v) Composition of proper morphisms are proper.

(vi) Consider

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ Y \arrow[r] & S \end{tikzcd}$$

. If f is proper then f' is proper.

(vii) Properness of $f : X \to Y$ is local on Y.

THEOREM 4.6.4 : $\mathbb{P}^n_A \to \operatorname{Spec} A$ satisfies existence & uniqueness parts of valuative crit.

Proof from Hartshorne; not lectured in '23.

Proof: By base change, can take $A = \mathbb{Z}$.

We check the valuative criterion: take a DVR R. $T = \operatorname{Spec} R$, & let $U = \operatorname{Spec} K$ where $K = FF(R)$.

Consider the lifting diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

By induction: assume $U \subseteq \mathbb{V}(X_i)^c$ for all i, where $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[X_0, \dots, X_n]$.

Now, x_i/x_j lie in the stalk at the image of U i.e. they are well-defined and let f_{ij} be the pullback in K and

note: $f_{ij} \cdot f_{jk} = f_{ik}$

Now let $d_0, \dots, d_n$ be $v(f_{i0})$. If d_k is the smallest, then $v(f_{ik}) \geq 0$.

Now define: $\mathbb{Z}\left[\frac{x_0}{x_k}, \dots, \frac{x_n}{x_k}\right] \longrightarrow R$

$$x_i/x_k \longmapsto f_{ik}.$$

□

SAMPLE USES OF VALUATIVE CRITERION

i. $\mathbb{A}^1_k \longrightarrow \operatorname{Spec} k$ is not proper:

Write: $\mathbb{A}^1_k = \operatorname{Spec} k[x]$

Take $\operatorname{Spec} k[[t]]$ and $U = \operatorname{Spec} k((t))$.

Let $\begin{cases} U \rightarrow \mathbb{A}^1 \text{ be given by} \\ \frac{1}{t} \leftarrow\!\shortmid x. \end{cases}$ Map does not extend.

ii. Compositions: $X \longrightarrow Y \longrightarrow S$ proper morphism. So write:

$$\begin{tikzcd} & & & X \arrow[d] \\ & & & Y \arrow[d] \\ U \arrow[urrr] \arrow[uurrr] & \subseteq & \operatorname{Spec} A \arrow[r] & S \end{tikzcd}$$

First lift to $\operatorname{Spec} A \rightarrow Y$, then lift the resulting to $\operatorname{Spec} A \rightarrow X$.

§4.6 – a brief interlude on other types of morphisms

$X \xrightarrow{f} Y$ a scheme map.

(i) FINITE: Cover Y by affines $\mathrm{Spec}\, B_i = U_i$ st $V_i = f^{-1}(U_i)$ is open affine $\mathrm{Spec}\, A_i$ and A_i is a fg B_i MODULE.

Examples: Non-constant maps of smooth curves

closed immersions

(ii) FLAT: At every $p \in X$ the map

$$f^{\#} : \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X, p} \text{ makes } \mathcal{O}_{X,p}$$

a flat $\mathcal{O}_{Y, f(p)}$ module [injectivity of $\mathcal{O}_{Y,f(p)}$ modules is preserved] "Everything is flat over a field"

UTILITY: Given

$$\begin{tikzcd} Z_\eta \arrow[r, hook, "\text{closed}"'] & \mathbb{P}^n_{\mathbb{C}((t))} \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{\mathbb{C}[[t]]} \arrow[d] \\ & \mathrm{Spec}\, \mathbb{C}((t)) \arrow[r] & \mathrm{Spec}\, \mathbb{C}[[t]] \end{tikzcd}$$

There exists a unique $Z \hookrightarrow \mathbb{P}^n_{\mathbb{C}[[t]]}$ that is

FLAT OVER $\mathbb{C}[[t]]$

(iii) More sophisticated ring theory gives notions of étale map [covering spaces], unramified maps [immersions in topology], smooth map [submersions]. I have equipped you with enough background to make sure the work to understand such notions is in the affine / ring case.

§5 MODULES OVER $\mathcal{O}_X$

§5.1 Motivation:

An $\mathcal{O}_X$-module is a sheaf of groups with $\mathcal{O}_X$. multiplication. Before we do it formally, I give examples

Example 5.1.1: On $\mathbb{CP}^n$ the variety: $\mathbb{C}^{n+1} \setminus \{\underline{0}\} / \mathbb{C}^*$

Consider $\mathcal{O}_{\mathbb{P}^n}(d)(U) = \{ P(\underline{X}) / Q(\underline{X})$ Rational homogeneous function st degree is d and regular on all points of $U\}$

Notice $\mathcal{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n) =$ Degree d homogeneous polynomials

Recall $\mathcal{O}_{\mathbb{P}^n}(U)$ are rational, ie. ratios of poly's same degree so we have a multiplication map!

Note: If $d<0$, no global sections but still pretty interesting!

Example 5.1.2: Given a module M over A, define the sheaf $\mathcal{F}_M(U_f) = M_f$ by localization. Gluing is identical to what we know. Notation: sometimes $\tilde{M}$.

§ 5.2 DEFINITIONS OF $\mathcal{O}_X$-MODULES

Fix $(X, \mathcal{O}_X)$ a ringed space.

DEFINITION 5.2.1: A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ of groups st for $U \subseteq X$ open there is a multiplication $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ compatible w/ restriction.

A sheaf of $\mathcal{O}_X$-algebras is defined similarly

Standard Notions: Kernel, image, cokernel, direct sum, direct product, submodule, ideal sheaf

Also: Tensor product & Hom — tensor requires sheafification!

Moving between spaces

$X \xrightarrow{f} Y$ a ringed space morphism. Given $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules the pushforward $f_*\mathcal{F}$ is a $f_*\mathcal{O}_X$-module.

But we have $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ giving an $\mathcal{O}_Y$-module. structure

Conversely for $\mathcal{G}$ a sheaf of $\mathcal{O}_Y$-modules, define $f^*\mathcal{G} = f^{-1}\mathcal{G} \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$ via

the adjoint $f^\#$ $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

Basic Fact 5.2.2 f^* and f_* are adjoint functors for modules over ringed spaces.

(cf Ex Sh I Q14)

§5.3 $\mathcal{O}_X$-MODULES ON SCHEMES X a SCHEME

DEFINITION 5.3.1 A quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is one such that there exists open cover $\{U_i\}$ by affines with $\mathcal{F}|_{U_i}$ the sheaf associated to a module M_i over $\Gamma(U_i, \mathcal{O}_X|_{U_i})$. It is coherent if M_i are fg modules. * More than simply a condition on $\mathcal{F}(U_i)$.

Basic Examples: $\mathcal{O}_X$ on any scheme, similarly $\mathcal{O}_X^{\oplus n}$. For $Y \overset{i}{\hookrightarrow} X$, $i_* \mathcal{O}_Y$ is coherent. Affine case: this is the sheaf associated to A/I.

REFERENCE: Hartshorne II §5.

PROPOSITION 5.3.2 An $\mathcal{O}_X$-module $\mathcal{F}$ is q-coh if and only if on any $U = \operatorname{Spec} A$ affine, $\mathcal{F}|_U$ is the sheaf assoc. to an A-module. If X noetherian, then $\mathcal{F}$ is coherent if and only if M's are finitely generated.

KEY COROLLARY 5.3.3: q-coherent $\mathcal{O}_X$-modules on X affine are equiv. to modules over $\mathcal{O}_X(X)$.

Proof strategy: Condition on random opens $\rightsquigarrow$ Condition on distinguished random opens $\rightsquigarrow$ Condition on any affie of your choosing

KEY LEMMA 5.3.3: $X = \operatorname{Spec} A$, $f \in A$, and $\mathcal{F}$ q-coh. Let $s \in \Gamma(X, \mathcal{F})$ Then

(i) If s restricts to 0 on U_f then $f^n s = 0$ for some $n > 0$

(ii) If $t \in \mathcal{F}(U_f)$ then $f^m t$ is the restriction of a global section for some $m > 0$

Both clear when $\mathcal{F} = M^{sh}$ for M an A-module.

Proof: There is some cover by things of the form $\operatorname{Spec} B = V$, st $\mathcal{F}|_V = M^{sh}$ for M a B-module. Cover each V with things of form U_g, so $\mathcal{F}|_U = (M \otimes_B A_g)^{sh}$. Finitely many g_i & M_i on U_{g_i} cover. Now use properties of localization. $\square$

PROOF OF 5.3.2: Given $U = \mathrm{Spec}\, A \subseteq X$ and $\mathcal{F}$ on X q-coh, observe that $\mathcal{F}|_U$ is still quasi-coh.
[why?] There is a basis of opens where $\mathcal{F}|_U$ is sheaf assoc. to module. Reduce to $X = \mathrm{Spec}\, A$.
Now, take $M = \mathcal{F}(X)$ and let M^{sh} be the sheaf on $X = \mathrm{Spec}\, A$ associated to it.
we have a map $M^{sh} \longrightarrow \mathcal{F}$. By the lemma, on a distinguished open U_g the map is an isomorphism, therefore an isomorphism globally.

LESSON: Quasi-coherence is local.

STRAIGHTFORWARD (proofs omitted)

- Images, kernels, cokernels stay q-coherent/coh.
- Pullbacks stay q-coh/coh
- Pushforward of q-coh is q-coh.
- If $f: X \to S$ is proper then f_* preserves coherence

Now want to give you a feeling for coherent sheaves
There is a "Proj" version. Take $A_\bullet$ a $\mathbb{N}$

graded ring and $M_\bullet$ a graded $A_\bullet$-module. Then for $U = \mathbb{V}(f)^c \subseteq \operatorname{Proj} A_\bullet$ with $f \in A_1$, we know $U \cong \operatorname{Spec}(A_\bullet[1/f])_0$. Now

$$U \longmapsto (M_\bullet[1/f])_0 \quad \text{glues to}$$

a sheaf $M_\bullet^{sh}$ on $\operatorname{Proj} A_\bullet$.

WORDS: A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is a vector bundle if it there exists a Zariski open cover st $\mathcal{F}$ restricts to a FREE module on each.

Line bundle (aka invertible sheaf) is locally free of rank 1.

An ideal sheaf is locally an ideal in $\mathcal{O}_X$.

ASIDE: On a scheme/top space X, always have the constant sheaf $\mathbb{Z}$. Not a quasi-coherent sheaf in a natural way. The structure sheaf $\mathcal{O}_X$ is coherent. These are very different but equally important!

§5.4 COHERENT SHEAVES via PROJ

On $\mathbb{CP}^n$ the old skool variety we defined $d \in \mathbb{Z}$

$$\mathcal{O}_{\mathbb{P}^n}(d)(U) = \left\{ f/g \;\middle|\; \begin{array}{l}\text{homog rational function}\\ \text{degree } d, \text{ no poles on } U\end{array} \right\}$$

SCHEME THEORETICALLY: Instead of degree 0 elements in localizations, take degree d elements.

DEFINITION 5.4.1: Fix $A_\bullet$ graded, $M_\bullet$ a graded $M_\bullet$-module. Define $M_\bullet(d)$ the module whose degree k piece is M_{k+d}. The sheaf $(A_\bullet(d))^{sh}$ on $\operatorname{Proj} A_\bullet$ is $\mathcal{O}_X(d)$.

Note: $\mathcal{O}_X(d) = \mathcal{O}_X(1)^{\otimes d}$

PROPOSITION 5.4.2: $\mathcal{O}_X(d)$ is a line bundle.

Proof: Easy exercise.

Remark 5.4.3 If $A_\bullet$ is generated over A_0 in degree 1, then $\operatorname{Proj} A_\bullet \hookrightarrow \mathbb{P}^n$ and those are restrictions of homogeneous rational functions.

Construction 5.4.3 : (Source of line bundles)

Given* $X \xrightarrow[f]{} \mathbb{P}^n$ over any base we get a line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ on X. Moreover, we get write the homogeneous coordinates of $\mathbb{P}^n$ by $x_0, \dots, x_n$ we get sections

*: very concrete!

$$s_1, \dots, s_n \in \Gamma(X, f^*\mathcal{O}(1))$$

DEFINITION 5.4.4 Let L be a line bundle on X. Then L is basepoint free if there exists $X \to \mathbb{P}^r$ st $f^*\mathcal{O}(1) = L$. It is very ample if in additn $X \hookrightarrow \mathbb{P}^r$ is a locally closed embedding. L is ample if $L^{\otimes n}$ is very ample for $n \gg 0$.

Example 5.4.4 Take $A_{\bullet,\bullet} = \mathbb{C}[X, Y, Z, W]$ and $\mathcal{O}(1,1)$ the shifted $A_{\bullet,\bullet}(1,1)$-sheaf. Then $\mathrm{BiProj}\, A_{\bullet,\bullet} \xhookrightarrow[i]{} \mathbb{P}^3$ with $\mathcal{O}(1,1) = i^*\mathcal{O}_{\mathbb{P}^3}(1)$. SEGRE EMBEDDING.

In commutative algebra, your intuition for modules comes from quotients of free modules.

DEFINITION 5.4.5 An $\mathcal{O}_X$-module is called globally generated if it can be written as a quotient of $\mathcal{O}_X^{\oplus n}$ for some n.

Equivalently, there exist $\{s_i\} \subseteq \Gamma(X, \mathcal{F})$ whose images in $\mathcal{F}_p$ generate for all $p \in X$.

Let $i : X \hookrightarrow \mathbb{P}^n_R$ closed and let $\mathcal{O}_X(1)$ be restriction of $\mathcal{O}_{\mathbb{P}^n}(1)$. Note: $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$

THEOREM 5.4.6: Let $\mathcal{F}$ be a coherent sheaf on X. There exists d_0 st for all $d \geq d_0$, $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$ is globally generated.

Proof: Formal properties: equivalent to show $i_* \mathcal{F}(n)$ is globally generated on $\mathbb{P}^n$. Reduce to $X = \mathbb{P}^n_R$

Write: $\mathbb{P}^n_R = \mathrm{Proj}\ R[x_0, \dots, x_n]$. Cover $\mathbb{P}^n$ by standard opens U_i where

$$U_i = \mathrm{Spec}\ R[x_j/x_i]$$ call this ring B_i

$\mathcal{F}|_{U_i} = M_i^{sh}$ for some f.g B_i-module.

Take $s_{ij} \in M_i$ that generate. I claim that $x_i^d \cdot s_{ij}$ which is a section of $\mathcal{F}(d)(U_i)$, for $d \gg 0$ is the restriction of a global section $t_{ij} \in \Gamma(\mathbb{P}^n, \mathcal{F}(d))$. [Exercise: I will explain the idea]

Why generate? On U_i the s_{ij} generate M_i^{sh}. But on U_i multiplication by x_i^d

$$\cdot x_i^d : \mathcal{F} \to \mathcal{F}(d)$$ restricts to an

isomorphism between: $\mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}(d)|_{U_i}$ and

so $x_i^d \cdot s_{ij} = x_i^d\, t_{ij}$ generate. $\square$

COROLLARY: Every coherent sheaf is a quotient of $\mathcal{O}_X(d)^{\oplus N}$ (on X closed in projective, d likely very negative)

Aside (How to do exercise)

s_{ij} is a section in $M_i = \mathcal{F}_i(U)$.

$x_i \in \mathcal{O}_X(1)$

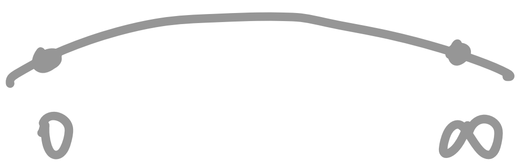
so $\quad x_i^d \cdot s_{ij} \in (\mathcal{F} \otimes \mathcal{O}(d))(U)$.

This is restriction of $t_{ij} \in \mathcal{F}(d)(X)$.

(why?. Try $\mathbb{P}^1$ first;

$U_1 = \mathbb{P}^1 \setminus 0$

$U_2 = \mathbb{P}^1 \setminus \infty$



$s_{ij} \in \mathcal{F}(U_1)$; restrict to $U_1 \cap U_2$

But we proved in LEMMA 5.3.3

that for large d, $x_1^d \cdot s_{1j}$ extends

§6 DIVISORS ON SCHEMES

Why? In rings, principal ideals are key. They are examples of height 1 primes. We will globalize.

Recall: Height of $\mathfrak{p} \subseteq R$ is longest chain $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_n = \mathfrak{p}$ of primes in R.

We now discuss WEIL DIVISORS and in such discussion assume X is Noetherian, integral separated.

If X is integral then in any affine open Spec A, the ideal $(0) \subseteq A$ is prime. Gives generic point in X. One point independent of choice of Spec A.

If $Y \subseteq X$ integral codim 1, we assume $\mathcal{O}_{X,\eta_Y}$ a DVR.

§6.1 TOPOLOGICAL PRELIMINARIES:

(i) Dimension of X is length n of longest chain of nonempty closed irred. subsets

$$Z_0 \subsetneq \cdots \subsetneq Z_n \text{ in } X$$

Follows from normality.

Dimension of $\mathbb{A}^n_k$ is n.

(ii) Codimension of $Z \hookrightarrow X$ closed irred. defined similarly: $Z = Z_0 \subsetneq \cdots \subsetneq Z_n$ in X

If A is a fg k-algebra & integral then

Krull Dim A = height $\mathfrak{p}$ + krull dim $A/\mathfrak{p}$

Most intuition from here fails in general.

(iii) If X is a noetherian topological space, then every closed $Z \subseteq X$ has a finite irred comp. decomp.

§6.2 WEIL DIVISORS

DEFINITION 6.2.1 A prime divisor is a closed integral subscheme of codimension 1. A weil divisor is an element of the free abelian group on prime divisors $\underline{\mathrm{Div}\, X}$.

Divisor D is effective if all coeffs are ≥ 0.

CONSTRUCTION 6.2.2: Let $f \in k(X)^\times$. [what is this practically?]. Then take

$$\mathrm{div}(f) = \sum_{\substack{Y \subseteq X \\ \text{prime}}} n_Y(f)\,[Y] \quad \text{where}$$

$n_Y(f)$ is the valuation of f in $\mathcal{O}_{X,\eta_Y}$.

PROPOSITION 6.2.3 : The element $\mathrm{div}(f)$ is a divisor, i.e. the sum is FINITE.

FACT: If $Y \subseteq X$ integral codim 1, $\mathcal{O}_{X,\eta_Y}$ DVR w/ fraction field $K(X)$.

PROOF: Take $U \subseteq X$ affine ; $U = \mathrm{Spec}\, A$ st f is regular i.e. $f \in A \subseteq K(X)$. Then $X \setminus U = Z$ is closed of codim ≥ 1. Thus only finitely many Y_i's are in U^c. On the rest, any Y_i for which $\nu_{Y_i}(f) > 0$ is contained in $\mathbb{V}(f)$. Those are contained in $\mathbb{V}(f)$ so we're done. $\square$

⚠ We used something here. Given a closed subset $Z \hookrightarrow X$ there is a unique reduced scheme structure on it. $\boxed{\text{Hartshorne Ex. 3.2.6}}$

DEFINITION 6.2.4 : A divisor of the form $\mathrm{div}(f)$ is principal. They form a group. The quotient

$$\mathrm{Div}\, X / \mathrm{Prin}\, X := \mathrm{Cl}(X).$$

The class group is (i) interesting (ii) Hard to calculate - simplest of the Chow groups.

Basic Calculations 6.2.5

(i) $X = \operatorname{Spec} A$ integral. Then A UFD $\iff$ A is integrally closed and $Cl(X) = 0$.

(ii) In particular, $Cl(\mathbb{A}^n_k) = 0$.

(iii) If $Z \hookrightarrow X$ closed with $U = Z^c$ open, then $Cl(X) \twoheadrightarrow Cl(U)$ given by intersection with U. If codim $(Z) \geq 2$ this is an isomorphism. If Z is codim 1 & irred. then $\mathbb{Z} \to Cl(X) \to Cl(U) \to 0$ is exact.

Excision Sequence.

Class Group of $\mathbb{P}^n$: work over k

1. If $D \subseteq \mathbb{P}^n$ integral & codimension 1, then

$$D = \mathbb{V}(f) \text{ ; } f \text{ homogeneous degree } d.$$

Define $\deg(D) = \deg(f)$.

2. Extend linearly to get

$$\deg : \operatorname{Div} \mathbb{P}^n \longrightarrow \mathbb{Z}$$

Claim: deg is an isomorphism

$$\deg : Cl(\mathbb{P}^n) \longrightarrow \mathbb{Z}$$

well-defined because if $f = g/h$ is rational
div(f) degree 0

3. Surjective: take $H = \mathbb{V}(x_0)$.

Injective: If $D = \sum n_{Y_i}[Y_i]$. If $\sum n_{Y_i} \cdot (\deg(Y_i)) = 0$, with $Y_i = \mathbb{V}(g_i)$,

take $f = \prod g_i^{n_{Y_i}}$. Then $\underline{\operatorname{div}(f) = D}$

Proof of Excision: $Z \hookrightarrow X \hookleftarrow U$; Z irred

i. The map $Cl(X) \longrightarrow Cl(U)$ induced by

$$\sum n_i Y_i \longmapsto \sum n_i Y_i \cap U$$ is

well-defined b/c if $f \in K(X)^\times$ we can view $f \in K(U)^\times$ so principal maps to principal.

Surjective b/c every integral codim 1 in U is restriction of its closure.

ii. Obvious.

iii. Kernel is divisors with support in Z.

$\square$

§ 6.3 CARTIER DIVISORS

Commutative Algebra: A is a UFD iff all height $\bar{1}$ primes are principal. On a scheme X st $\mathcal{O}_{X,x}$ all UFD's Weil divisors are nice.

Intuitively, a Cartier divisor is "locally" a principal ideal

We need a few preliminary notions.

For X a scheme, take the presheaf assigning

affine $U\ (= \operatorname{Spec} A) \longmapsto S^{-1}A$, $S =$ all nonzero divisors

and sheafify to get $\mathcal{K}_X$. Similarly take

$$U = \operatorname{Spec} A \longmapsto A^*$$ to be

to get $\mathcal{O}_X^*$. Now $\mathcal{K}_X^* \subseteq \mathcal{K}_X$ subsheaf of invertibles.

DEFINITION 6.3.1 A Cartier divisor is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$

But care is required here!

Remark 6.3.2 : what does this mean practically ?
Given a cover $\{U_i\}$ with rational functions f_i on each such that on overlaps $f_i f_j^{-1}$ lie in $\mathcal{O}_X^*(U_i \cap U_j)$.

Image of $\Gamma(X, K_X^*) \longrightarrow \Gamma(X, K_X^*/\mathcal{O}_X^*)$ are principal div's.

Construction 6.3.3: If X is regular in codim 1 and [integral, noetherian, separated] then given D a Cartier divisor we get a Weil divisor by the rule, for $Y \subseteq X$ codim 1 & integral, and D represented by $\{(U_i, f_i)\}$ with $\eta_Y \in U_i$, take $n_Y = \nu_Y(f_i)$.

[well defined: $f_i / f_j \in \mathcal{O}_X(U_{ij})^*$ so has valuation equal to 0.
by the principal divisor construction!]

PROPOSITION 6.3.4 If X is noetherian integral sep. with all local rings UFD's ($\Rightarrow$ regular in codim 1) then the association

$$\begin{Bmatrix} \text{Cartier} \\ \text{Divisors} \end{Bmatrix} \xrightarrow[6.3.3]{\text{Construction}} \begin{Bmatrix} \text{Weil} \\ \text{divisors} \end{Bmatrix}$$

respects principal divisors and is a bijection.

PROOF: Follow nose & look at Hartshorne. Key: if A is UFD then height 1 primes are principal. If $x \in X$, then $\mathcal{O}_{X,x}$ is a UFD & so for $D \in \mathrm{Div}\, X$, $D \cap \mathrm{Spec}\, \mathcal{O}_{X,x}$ is $\mathrm{div}(f_x)$. Extends to an open U_x where D & $\mathrm{div}(f_x)$ agree.

Exercise: verify it is bijective.

PROPOSITION 6.3.5: If X is normal, integral, sep, noetherian then Cartier divisors are Weil divisors that are locally principal.

Construction 6.3.6 Given $\mathcal{D}$ Cartier w/ representative $\{(U_i, f_i)\}$ let $\mathcal{L}(\mathcal{D}) \subseteq \mathcal{K}_X$ be the sub-$\mathcal{O}_X$ module generated by f_i^{-1} on U_i.

[well-defined b/c f_i/f_j invertible on $U_i \cap U_j$]

PROPOSITION 6.3.7: The sheaf $\mathcal{L}(\mathcal{D})$ is a line bundle.

Pf: On U_i, we have $\mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}(\mathcal{D})$ by $1 \mapsto 1/f_i$ $\square$

In fact, Cartier divisors up to equiv. are almost same as line bundles

Important Exercise: $X = \mathbb{P}^n$, $\mathcal{D}$ = hyperplane, show $\mathcal{L}(\mathcal{D}) \cong \mathcal{O}(1)$

A locally free sheaf of rank 1 (line bundle) L has an "inverse" $\mathrm{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X) =: L^{-1}$ and $L \otimes L^{-1} = \mathcal{O}$

The Picard group is

$$\mathrm{Pic}(X) = \left\{ \begin{array}{c} \text{Line bundles up to} \\ \text{isomorphism} \end{array} \right\}$$ Group under $\otimes$

Under very mild assumptions [eg projective over k; integral]

The map $\boxed{\mathrm{Cartier}(X) \longrightarrow \mathrm{Pic}(X)}$ is surjective with kernel exactly the principal divisors.

Calculating these groups is hard, but they are critical to understanding schemes.

More structure lurking: If X proper dimension n then there is a natural map

$$\mathrm{Pic}(X)^{\otimes n} \longrightarrow \mathbb{Z}$$

In good cases, we intersect n Cartier divisors and count the number of points.

§7 SHEAF COHOMOLOGY: Hartshorne Ch. III.

Given a sheaf of abelian groups on a top space X, the group $\Gamma(X, \mathcal{F})$ is natural, but loses lots of information.

EG: • $X = \mathbb{P}^n_{\mathbb{C}}$ & $\mathcal{F}_1 = \mathcal{O}_X$ & $\mathcal{F}_2 = \underline{\mathbb{C}}$ (constant) have same global sections.

• Take $X = \mathbb{A}^2$ & $Y = \mathbb{A}^2 \setminus (0,0)$, we saw $\Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$. Where has the extra information gone?

§7.1: Overview

Given $(X, \mathcal{F})$, sheaf cohomology will give collection of groups $H^i(X, \mathcal{F})$ $i \in \mathbb{N}$ with the following features:

1. The group $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
2. It's functorial. If $f: X \to Y$ and $\mathcal{F}$ is a sheaf on Y then

$$H^i(X, f^{-1}\mathcal{F}) \leftarrow H^i(Y, \mathcal{F}) : f^*$$

3. If $\underline{\mathbb{Z}}$ is the constant sheaf on a CW complex/nice top space, $H^i(X, \underline{\mathbb{Z}})$ is the "usual" topological cohomology theory.

4. It will take a SES

$$\{0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$ and output

an exact sequence:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to$$
$$H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to \cdots$$

We will find interesting new invariants of schemes that depend on the algebraic structure:

eg: $H^i(X, \mathcal{O}_X)$ or if $X \to \operatorname{Spec} k$ is a scheme with a coherent sheaf $\mathcal{F}$

$$\chi(X, \mathcal{F}) = \sum (-1)^i \dim_k h^i(X, \mathcal{F})$$

Euler characteristic of a sheaf.

Two facts: If X affine & $\mathcal{F}$ q-coh. then $H^i(X, \mathcal{F}) = 0$ for $i > 0$. If $X = \mathbb{A}^2 \setminus pt$ $H^1(X, \mathcal{O}_X) \neq 0$.

§7.2 FORMAL ASPECTS

DEFINITION 7.1.1: An abelian group I is injective if given

$$\begin{tikzcd} & I \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[ul, dashed, "\exists \text{ lifting.}"'] \end{tikzcd} \qquad (*)$$

Remark 7.1.2: For abelian groups, injective means DIVISIBLE; G divisible means $\forall g \in G$ & $\forall n \in \mathbb{N}$ there is $h \in G$ st $nh = g$.

Ex: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{C}^*$, arbitrary direct products.

Nontrivial finitely generated is never divisible.

Injective sheaf of abelian groups is defined via analogous lifting property

⚠ Constant sheaf $\underline{\mathbb{Q}}$ is not always injective!

DEFINITION 7.1.3 An injective resolution of A is an exact sequence $\underbrace{A \to I_0 \to I_1 \to \cdots}_{\text{possibly infinite}}$ with I_j injective.

PROPOSITION 7.1.4: Injective resolutions of abelian groups exist.

Proof: Every ab group injects into a divisible group.
[Use $G = \bigoplus_I \mathbb{Z}/K$; take $\bigoplus_I \mathbb{Z} \hookrightarrow \prod_I \mathbb{Q}$ & quot by K.

Now iterate:

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & A \arrow[r] & I_0 \arrow[rr] \arrow[dr] & & I_1 \arrow[r] & \cdots \\ & & & C_0 \arrow[ur] \arrow[dr] & & \\ & & 0 \arrow[ur] & & 0 & \end{tikzcd}$$

$\square$

COROLLARY 7.1.5: A sheaf of ab. groups can be emb. into an injective sheaf.

PROOF: $0 \to \mathcal{F}_x \to I_x$ for each stalk. Now take $\iota_x : \{x\} \hookrightarrow X$ and consider $(\iota_x)_* I_x$ and

$$\boxed{\mathcal{F} \hookrightarrow \prod_{x \in X} (\iota_x)_* I_x} \rightsquigarrow \mathcal{I}$$

. Now use fact that $\mathrm{Hom}_{sh}(\mathcal{G}, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}_{gp}(\mathcal{G}_x, I_x)$. $\square$

Exercise: If $\mathcal{F}$ is injective then given $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ global sections stays exact!

Given $\mathcal{F}$, replace by a complex of injectives $0 \to \mathcal{F} \to I_\bullet$ and define

$$H^i(X, \mathcal{F}) := \frac{\text{kernel}}{\text{image}} \text{ of } \Gamma(I_\bullet) \text{ at } i\text{th step.}$$

$$= \frac{\ker(\Gamma(I_i) \to \Gamma(I_{i+1}))}{\operatorname{im}(\Gamma(I_{i-1}) \to \Gamma(I_i))}.$$

Attack Plan: Replace a sheaf $\mathcal{F}$ on which you want to apply $\Gamma(X,-)$ with an $\underbrace{\text{injective}}_{\text{why?}}$ resolution. Now apply $\Gamma(X,-)$.

PROPERTIES 7.2.1

(i) $H^i(X,-)$ is independent of resolution

(ii) Given $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we get connecting homomorphisms $H^i(\mathcal{F}'') \to H^{i+1}(\mathcal{F}')$ giving the promised LES.

Theorem 7.2.2 (Grothendieck vanishing) If X is noetherian of dimension n and $\mathcal{F}$ a sheaf of ab. groups on X, $H^i(X,\mathcal{F}) = 0$ if $i > n$.

§7.3 ČECH COHOMOLOGY

X: topological space & $\mathcal{F}$ a sheaf on X.

$\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X. Well-order I

write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$.

The group of Čech p-cochains is

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$$

There is a differential

$$C^p \xrightarrow{d} C^{p+1}; \quad \text{given } \alpha \in C^p$$

then $$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i_k} \cdots i_{p+1}} \Big|_{U_{i_0 \cdots i_{p+1}}}.$$

Exercise: $d^2 = 0$

DEFINITION 7.3.1: The Čech cohomology groups are $\check{H}^p(\mathcal{U}, \mathcal{F})$ the cohomology groups of the above cochain complex.

∇ If $\mathcal{U}$ sucks then $\check{H}^*$ will also suck. For example if $\mathcal{U} = \{X\}$ then you only detect H^0.

Examples 7.3.2: $X = S^1$ with $\mathcal{F} = \underline{\mathbb{Z}}$ the constant sheaf. Take $\mathcal{U} = \{U, V\}$ to be

Then $C^0 = \mathbb{Z}^2$ and $C^1 = \mathbb{Z}^2$ with

Čech cochain complex
$$\begin{cases} d : C^0 \longrightarrow C^1 \\ (a,b) \longmapsto (b-a, b-a) \end{cases}$$

$$\check{H}^0 = \check{H}^1 = \mathbb{Z}$$ [kernel & cokernel of d]

This is super explicit!

Assume all $U_{i_0 \ldots i_p}$ affine open & $\mathcal{F}$ q-coh.
Then Čech computes cohomology.
In particular q-coh cohomology vanishes of affines.

Work on $\mathbb{P}^n_k$:

THEOREM 7.3.3: Let $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)$. Then as graded vector spaces:

- $H^0(\mathbb{P}^n, \mathcal{F}) \cong k[x_0, \ldots, x_n]$

- $H^n(\mathbb{P}^n, \mathcal{F}) \cong \frac{1}{x_0 \cdots x_n} k[x_0^{-1}, \dots, x_n^{-1}]$

- $H^p(\mathbb{P}^n, \mathcal{F}) = 0$ for all other p.

So $h^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d}$ for $d \geq 0$.

$h^n(\mathbb{P}^n, \mathcal{O}(d)) = \binom{-d-1}{n}$ $d \leq -n-1$.

PROOF: First part is trivial / follows from def'n.

- Second part: Standard cover $U_i = \mathbb{V}(x_i)^c$.

Observe: $\mathcal{F}(U_{i_0 \cdots i_p}) = k[x_0, \dots, x_n]_{x_{i_0} \cdots x_{i_p}}$.

k-spanned by monomials $x_0^{k_0} \cdots x_n^{k_n}$ with $k_{i_0}, \dots, k_{i_p} \in \mathbb{Z}$ & rest in $\mathbb{N}_0$.

Vector spaces are:
$$\begin{cases} \check{C}^{n-1} = \bigoplus_{i=0}^{n} k[x_0, \dots, x_n]_{x_0 \cdots \hat{x}_i \cdots x_n} \\ \check{C}^n = k[x_0, \dots, x_n]_{x_0 \cdots x_n} \end{cases}$$

Since $\check{C}^{n+1} = 0$ we get:

$$H^n(\mathbb{P}^n, \mathcal{F}) = C^n / \mathrm{im}(C^{n-1} \to C^n)$$

$$= \frac{k\text{-span}\{x_0^{k_0} \cdots x_n^{k_n} : k_i \in \mathbb{Z}\}}{k\text{-span}\{x_0^{k_0} \cdots x_n^{k_n} : \text{at least one } k_i \geq 0\}}$$

$$= k\text{-span}\{\text{monomials as} : \text{all } k_i < 0\}$$

This is the claimed answer.

(c) Induction on dimension:

View $i: \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ as $\mathbb{V}(x_0)$. Exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\cdot x_0} \mathcal{O}_{\mathbb{P}^n} \to i_* \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$

Build this as an exact sequence of graded modules over $k[x_0, \ldots, x_n]$. Tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$.

By design, we have a LES; assuming result for dimensions up to $n-1$, we get 3 seq.

$$0 \to H^0(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^0(\mathbb{P}^n, \mathcal{F}) \to H^0(\mathbb{P}^{n-1}, \mathcal{F}) \to H^1(\mathbb{P}^n, \mathcal{F})$$

sheaf on $\mathbb{P}^{n-1}$

$$\xrightarrow{\cdot x_0} H^1(\mathbb{P}^n, \mathcal{F}) \to 0 \quad \text{A}$$

$$0 \to H^p(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^p(\mathbb{P}^n, \mathcal{F}) \to 0 \quad 1 < p < n-1$$

B

$$0 \to H^{n-1}(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^{n-1}(\mathbb{P}^n, \mathcal{F}) \to H^{n-1}(\mathbb{P}^{n-1}, \mathcal{F})$$

$$\to H^n(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^n(\mathbb{P}^n, \mathcal{F}) \to 0 \quad \text{C}$$

The second sequence is also an isomorphism for $p = 1 \,\&\, n-1$ by explicitly writing the first sequence.

Now, $\cdot x_0$ makes $H^p(\mathbb{P}^n, \mathcal{F})$ a $k[x_0]$-module. Calculate localization of this at x_0 by localizing the complex: $H^p(\mathbb{P}^n, \mathcal{F})_{x_0} = H^p(U_0, \mathcal{F}|_{U_0}) = 0$

Thus, for any $\alpha \in H^p(\mathbb{P}^n, \mathcal{F})$, then $x_0^k \cdot \alpha = 0$ for some k. But $\cdot x_0$ is an isomorphism.

$\square$

SIMILAR CALCULATION:

- If $X = \mathbb{A}^2 \setminus \{(0,0)\}$ then $H^1(X, \mathcal{O}_X)$ is infinite dimensional.

FACT 7.3.4: Let $X \to \operatorname{Spec} k$ be proper & $\mathcal{F}$ coherent on X. Then $H^p(X, \mathcal{F})$ is finite dimensional over k

$k[x_0, x_1, x_2]$

- If $X = \mathbb{V}(f_d) \subseteq \mathbb{P}^2_k$ f homog. of degree d

Assume $(1:0:0) \notin X$. Then say

$$U = X \cap \mathbb{V}(x_1)^c \ \& \ V = X \cap \mathbb{V}(x_2)^c$$

Can similarly write out Čech complex.

Get
$$\dim_k H^0(X, \mathcal{O}) = 1$$
$$\dim_k H^1(X, \mathcal{O}) = \binom{d-1}{2}.$$

Calculating cohomology is hard. A simpler but useful invariant is the Euler characteristic.

Easier to compute: Euler characteristics

Given X/k proper & $\mathcal{F}$ coherent on X.

Set $$\chi(X,\mathcal{F}) = \sum_{p=0}^{\infty} (-1)^p \dim_k H^p(X,\mathcal{F})$$

Since $0 \to \mathcal{F}' \hookrightarrow \mathcal{F} \to \mathcal{F}'' \to 0$ gives LES:

$$\boxed{\chi(X,\mathcal{F}) = \chi(X,\mathcal{F}') + \chi(X,\mathcal{F}'').}$$

Let X be a 1 dimensional scheme. The (arithmetic) genus of X is $p_a(X) = 1 - \chi(C,\mathcal{O}_C)$

PROPERTY: Suppose $Z = X \times_{\operatorname{Spec} k} Y$. Then If $\mathcal{F}$ & $\mathcal{G}$ are sheaves on X & Y and $\mathcal{P} = p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}$. Then

$$\chi(Z,\mathcal{P}) = \chi(X,\mathcal{F}) \cdot \chi(Y,\mathcal{G}).$$

NICE COROLLARY: No product of curves of genus ≥ 1 is a hypersurface in $\mathbb{P}^3$.

§7.4 CHOICE OF COVER

THEOREM 7.4.1: Let X be affine, noetherian, $\mathcal{F}$ quasi-coherent. For any cover $\mathcal{U} = \{U_i\}$, the groups $\check{H}^i(X, \mathcal{F}) = 0$ for $i > 0$.

Proof: Define the sheafified Čech complex:

$$\mathcal{C}^p(\mathcal{F}) = \prod_{i_0 < \dots < i_p} i_* \mathcal{F}|_{U_{i_0 \dots i_p}}$$

where $i: U_{i_0 \dots i_p} \hookrightarrow X$ is the inclusion.

The sheaves $\mathcal{C}^p(\mathcal{F})$ are quasi-coherent and $\Gamma(X, \mathcal{C}^p(\mathcal{F})) = C^p(\mathcal{F})$ the usual group of p-chains.

Differentials $\mathcal{C}^p(\mathcal{F}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{F})$ as usual

Now on affines, taking global sections preserves exactness (Ex Sh IV Q10)

Now it suffices to prove exactness of

$$\mathcal{C}^0(\mathcal{F}) \to \mathcal{C}^1(\mathcal{F}) \to \mathcal{C}^2(\mathcal{F}) \to \cdots$$

Now can check exactness at stalk-level:

Let $q \in X$ and let $q \in U_j$. Now define:

$$k : \mathcal{C}^p_q(\mathcal{F}) \longrightarrow \mathcal{C}^{p-1}_q(\mathcal{F})$$

$$\alpha \longmapsto k(\alpha).$$

The $(i_0 \dots i_{p-1})$-factor of $k(\alpha)$ is equal to

$\alpha_{j i_0 \dots i_{p-1}}$ where if the indices are wrong order and $\sigma \in S_{p+1}$ puts it in the right order this means:

$$\operatorname{sgn}(\sigma) \cdot \alpha_{\sigma(j), \sigma(i_0) \dots \sigma(i_{p-1})}.$$

By direct calculation (Exercise): $dk + kd = \mathrm{id}$.

We know $\operatorname{im}(\mathcal{C}^{p-1} \to \mathcal{C}^p) \subseteq \ker(\mathcal{C}^p \to \mathcal{C}^{p+1})$

Conversely if $\alpha \in \ker(\mathcal{C}^p \to \mathcal{C}^{p+1})$ then

$$\alpha = (kd + dk)(\alpha) = d(k\alpha) \in \operatorname{im}(\mathcal{C}^{p-1} \to \mathcal{C}^p)$$

$\square$

LEMMA 7.4.2: Let X affine and $\mathcal{F}$ q-coherent. Fix $\mathcal{U}=\{U_1,\dots,U_k\}$ and $\hat{\mathcal{U}}=\{U_0,\dots,U_k\}$. Then cohomology groups $\check{H}(\mathcal{U},\mathcal{F})=\check{H}(\hat{\mathcal{U}},\mathcal{F})$.

Proof Sketch: Let $C^p(\mathcal{F})$ and $\tilde{C}^p(\mathcal{F})$ be the Čech groups for these covers. There are maps:

$$\tilde{C}^p \longrightarrow C^p \quad \text{and}$$

$$H^p(\hat{\mathcal{U}},\mathcal{F}) \to H^p(\mathcal{U},\mathcal{F})$$

$\tilde{\alpha} \in \tilde{C}^p(\mathcal{F})$ is (α, α_0), where $\alpha \in C^p$ & $\alpha_0 \in C^{p-1}(\mathcal{F}|_{U_0})$

by "dropping U_0 data". Commutes w/ differential so gives a cohomology map.

Exercise: Use Thm 7.4.1 to prove injectivity and surjectivity, by reducing to affine schemes.

□

COROLLARY 7.4.3: For a scheme X and $\mathcal{F}$ q-coh, the groups $\check{H}^i(\mathcal{U},\mathcal{F})$ are independent of cover $\mathcal{U}$ of X.

§7.5 FURTHER TOPICS in COHOMOLOGY

• One can extract concrete consequences from sheaf cohomology. For example, let

$$X_d \subseteq \mathbb{P}^3_k \quad \text{be} \quad V(f_d), \text{ homog. deg } d.$$

Then if $d \geq 3$ then X_d is not isom. to a product (over Spec k) of schemes of dim 1.

• Similarly, schemes X_d for different d are never isomorphic (Calculate $\chi(X, \mathcal{O}_X)$)

<u>DUALITY THEORY</u>: Given $Z \overset{\iota}{\hookrightarrow} X$ a closed subscheme then $\mathcal{I} := \ker(\iota^\#: \mathcal{O}_X \to \mathcal{O}_Z)$ this is also coherent!

DEFINITION 7.4.3 : The conormal sheaf is given by $\iota^*(\mathcal{I}/\mathcal{I}^2)$, where $\mathcal{I}^2$ is the sheafification of the presheaf $U \mapsto \mathcal{I}(U)^2 \subseteq \mathcal{O}_X(U)$. <u>Notation</u> : $N^\vee_{Z/X}$.

If X & Z have all regular local rings then $N^\vee_{Z/X}$ is a locally free sheaf of rank $\mathrm{codim}(Z,X)$.

The normal bundle is $N_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(N^\vee_{Z/X}, \mathcal{O}_Z)$.

DEFINITION 7.4.4: If $\begin{array}{c} X \\ \downarrow \\ Y \end{array}$ is separated, define the then we define

$$\Omega_{X/Y} := N^\vee_{\Delta_{X/Y}}.$$

Motivation from topology: Normal bundle of X in $X \times X$ is naturally T_X.

If X is non-singular then this is a bundle [locally free]

The "determinant" bundle is $\underbrace{\Lambda^{\dim X} \Omega_X}_{\text{SHEAF ASSOC. TO THE PRESHEAF}} = \omega_X$

THEOREM 7.4.5 (Serre Duality) If X is nonsingular projective over k of dimension n. If $\mathcal{F}$ is locally free of finite rank [$\mathcal{O}_X$-module] then:

$$H^i(X, \mathcal{F}) \xrightarrow{\cong} H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee.$$

Where might you go from here?