Algebraic Geometry

PART III

Autumn 2022

LECTURE NOTES.

§0: Preliminary Remarks

0.1: Goals & Non-Goals

- The course is a STARTER KIT
- Mastery of scheme theory is NOT a goal
- scheme theory represents a SPECTACULAR revolution in pure mathematics; I will try to guide you towards an understanding of why.
- Example sheets are crucial!

0.2 The plan:

- Spectrum of a ring & basics of sheaves
- Definitions of schemes & morphisms
- Properties of schemes & morphisms
- Rapid Introduction to sheaf cohomology.

0.3 : Prerequisites :

- Basic undergraduate maths [algebra, topology, etc]
- Commutative algebra [co-requisite or willingness to read].

0.4 Resources:

- Dhruv's course page [notes]
- Texts: Hartshorne; Vakil ; Intuition: Eisenbud-Harris
- Commutative Algebra: Atiyah-MacDonald & PART III
- Web: MathOverflow & MathstackExchange

(YouTube) AGITTOC "pseudo lectures" by Ravi

- Example sheets/classes ☆: SAGES reading group

0.5 Why suffer through this?

Let $f \in \mathbb{Z}[\underline{X}]$ be a homogeneous polynomial

Two Worlds (Weil 1949)

1. $X = \mathbb{V}(f) \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$ a projective hypersurface.

assume X is smooth.

no point on X where $\frac{\partial f}{\partial x_i}(p) = 0$ for all i

X is a compact topological space in Euclidean top.

Numbers: $b_0(X), b_1(X), \ldots, b_{2n}$ Betti Numbers

In the $\mathbb{C}$-Euclidean topology X can be triangulated

2. Fix prime number p with X smooth over $\overline{\mathbb{F}}_p$

Define $N_m := \# X(\mathbb{F}_{p^m})$

Now package this:

$$\zeta(X,t) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} t^m\right)$$

the Weil Zeta Function

UNBELIEVABLE THEOREM ! [Grothendieck]

1. $\zeta(X,t)$ is a ratio of polynomials:

$$= \boxed{\frac{P_0(t)\, P_2(t) \cdots P_{2n}(t)}{P_1(t)\, P_3(t) \cdots P_{2n-1}(t)}}$$

2. The degree of $P_i(t)$ is equal to the Betti number b_i.

The topology of X over $\mathbb{C}$ is connected to the number of points on X over $\mathbb{F}_q$.

This is completely absurd and to understand it we need to develop technology.

§1 : Beyond algebraic varieties

DAY 2

1.1 Summary of varieties (affine case)

k = algebraically closed field

$\left\{ \text{subsets of } \mathbb{A}^n_k \text{ of the form } \mathbb{V}(\text{polynomials}) \right\}$

$\updownarrow$ upto isomorphism

$\left\{ \text{fin. generated } k\text{-algebras without nilpotent elements} \right\}$

$\left\{ \text{Subsets of } \mathbb{A}^n \text{ of the form } \mathbb{V}(I) \right\}$

$\updownarrow$

$\left\{ \text{Radical ideals in } k[x_1, \ldots, x_n] \right\}$

1.1.1 Basic structures.

$$\boxed{\begin{array}{l} \mathfrak{J} \subseteq k[\underline{x}] \ ; \quad \mathbb{V}(\mathfrak{J}) := V \subseteq \mathbb{A}^n_k \\ \text{radical ideal} \qquad \mathcal{O}_V = k[V] = k[\underline{x}]/\mathfrak{J} \\ \qquad\qquad\qquad\qquad \text{coordinate ring} \end{array}}$$

1.1.2 Topology: $V = \mathbb{V}(S) \subseteq \mathbb{A}^k_k$

Zariski Topology:

closed sets $= \mathbb{V}(S)$ for $S \subseteq k[V]$

$= \mathbb{V}(\text{ideal gen. by } S)$.

Closed sets are where functions vanish.

(Exercise: check this!)

1.1.3 Nullstellensatz: Fix $V = \mathbb{V}(S)$

Given $p \in V$ we have

$\boxed{ev_p : k[V] \longrightarrow k}$; $m_p = \text{kernel}(ev_p)$

evaluate functions at p.

and conversely, by Hilbert's Nullstellensatz

$\{\text{points of } V\} = \{\text{maximal ideals of } k[V]\}$

Points of variety corr. to maximal ideals

1.1.4 Function theory: Fix $V = \mathbb{V}(\mathfrak{I})$. Then $k[V]$ is the ring of alg. functions on V

$\boxed{\text{Each } f \in k[V] \text{ gives } \begin{array}{l} f: V \longrightarrow \mathbb{A}^1_k = k \\ \quad p \longmapsto f(p) = \bar{f} \text{ in } k[V]/\mathfrak{m}_p \end{array}}$

1.1.4 Morphisms: Given V and $W \subseteq \mathbb{A}^m_k$

$$\varphi : V \longrightarrow W \subseteq \mathbb{A}^m_k$$
$$\| $$
$$(f_1, .., f_m) \ ; \quad f_i \in k[V] = \mathcal{O}_V$$

whose image lies in W

Equivalently: a pullback map

$$\varphi^* : k[W] \longrightarrow k[V] \quad \text{that}$$

preserves k

- Morphisms $V \to W$ are built from functions on V subject to a set-theoretic condition.
- $V \cong W$ as varieties $\iff$ $k[V] \cong k[W]$ as k-algebras

1.2 LIMITATIONS:

Question 1.2.1: What is an abstract variety?

Should be something that is "LOCALLY" an affine variety.

Example 1.2.2 (non-algebraically closed fields)

$$I = (x^2+y^2+1) \subseteq \mathbb{R}[x,y]$$

then $\mathbb{V}(I) = \emptyset \subseteq \mathbb{R}^2$

But I is prime, therefore radical.

Nullstellensatz fails since I is NOT unit.

Question 1.2.3: On what topological space

X is $\mathbb{R}[x,y]/(x^2+y^2+1)$ NATURALLY

the space of functions?

Question 1.2.4 (Similar) On what topological space

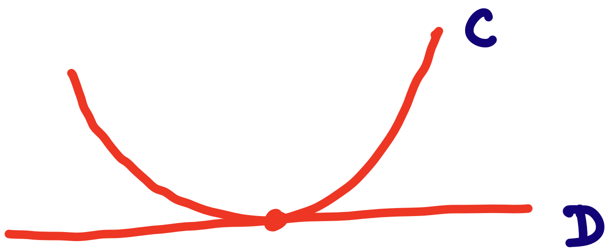
is $\mathbb{R}[x]$ the ring of functions? And the

rings $\mathbb{Z}$, $\mathbb{Z}[x]$?

Example 1.2.5 (why restrict to radical ideals?)

Take $C = \mathbb{V}(y-x^2) \subseteq \mathbb{A}^2_k$

$D = \mathbb{V}(y)$



$C \cap D = \mathbb{V}(y,\ y-x^2) = \boxed{\mathbb{V}(x^2, y) = \mathbb{V}(x,y)}$

1 POINT

Now if $D_\delta = \mathbb{V}(y+\delta)$ $\delta \in C$

$C \cap D_\delta = \{\pm\sqrt{\delta}\}$ 2 POINTS

> Intersections of varieties don't want to be varieties.

Remark 1.2.6 (Moduli) If $X \xrightarrow{\pi} B$ is a morphism of varieties how is geometry of $\pi^{-1}(a)$ & $\pi^{-1}(b)$ for $a, b \in B$ related? How do you parameterize varieties

§ 1.3 SPECTRUM OF A RING

Let A be a commutative ring with identity. We will define X_A a topological space on which A is the ring of functions.

Definition 1.3.1 The Zariski spectrum of A is $\operatorname{Spec} A = \{ \mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal} \}$

Better than maxSpec: $A \to B \Rightarrow \operatorname{Spec} B \to \operatorname{Spec} A$ by taking preimage. Construction is "FUNCTORIAL"

Why? This is the smallest thing X_A COULD be subject one requirement:

Given p in X_A, we should have:

$$ev_p : A \longrightarrow K_p \text{, with } K_{\mathfrak{p}} \text{ a field}$$

Now note, $\ker(ev_p)$ is a PRIME IDEAL $\mathfrak{p}$

and ev_p is just

$$A \longrightarrow A/\mathfrak{p} \subseteq FF(A/\mathfrak{p}).$$

Example 1.3.2 $A = \mathbb{Z}$ then $\operatorname{Spec} \mathbb{Z}$ is the set of prime mubers plus 0.

Pick a $\boxed{\text{FUNCTION}}$ e.g $132 \in \mathbb{Z}$

$\boxed{\text{EVALUATE}}$ $132(p) := 132 \bmod p$

The codomain of the functio changes from point-to-point

Example 1.3.3 $A = \mathbb{R}[x]$ then

$$\operatorname{Spec} \mathbb{R}[x] = \mathbb{C} / \underbrace{\text{complex conjugation}}_{\text{Galois group!}} \ \& \ (0)$$

$$= \text{Upper half plane in } \mathbb{R}^2 \ \& \ (0)$$

Exercise 1.3.4 Draw Spec A for $A = \mathbb{Z}[x]$ and $A = k[x]$ for k arbitrary field.

Why not maximal ideals? Functoriality & Experience

§1.4 TOPOLOGY ON SPEC(A)

Zariski topology = Zero sets of functions

Fix $f \in A$ & $\mathfrak{p} \in \operatorname{Spec}(A)$; then

$$\mathbb{V}(f) = \{ \mathfrak{p} \in \operatorname{Spec}(A) : \bar{f} = 0 \bmod \mathfrak{p} \text{ i.e. } f \in \mathfrak{p} \}$$

Points where f vanishes.

Similarly for $I \subseteq A$ an ideal

$$\mathbb{V}(I) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p} \text{ for all } f \in I \}$$

i.e. $I \subseteq \mathfrak{p}$

PROPOSITION 1.4.1: The sets $\mathbb{V}(I) \subseteq \operatorname{Spec} A$ for all ideals $I \subseteq A$ form the closed sets of a topology — the Zariski topology.

Proof: Easy facts: $\emptyset$ and $\operatorname{Spec} A$ are closed.

Since $\mathbb{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathbb{V}(I_\alpha)$ arbitrary intersections are closed. [I_1+I_2 is the smallest ideal containing $I_1 \cup I_2$]

REQUIRES SOME THOUGHT:

$$\mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$$

$\supseteq$: clear

Claim: $\mathbb{V}(I_1 \cap I_2) \subseteq \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$

$I_1 I_2 \subseteq I_1 \cap I_2 \subseteq \mathfrak{p}$ with $\mathfrak{p}$ prime then

I_1 or I_2 is contained in $\mathfrak{p}$.

we used primality ! $\square$

Example 1.4.2 (How does this compare with old skool)

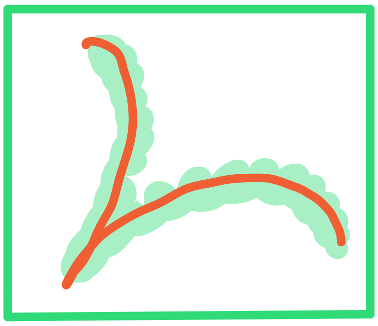
Let $k = \bar{k}$ and consider Spec $k[X,Y]$ is

k^2 plus one point for each irreducible curve

and an extra point corresponding to (0).

what are closures of the weird points ?

ROUGH PICTURE:



$y^2 = x^3$

(0) is dense; (y^2-x^3) closure includes all of $\mathbb{V}(y^2-x^3)$.

Similar for $y=x^2$, $x-y$, etc.

POINTS aren't CLOSED !

§1.5 FUNCTIONS ON OPENS

Let $f \in A$. Then define
$U_f = (\mathrm{Spec}\ A) \setminus \mathbb{V}(f)$

$\boxed{\text{Distinguished open.}}$

LEMMA 1.5.1 The distinguished opens form a basis for the topology on Spec A.

Proof: Exercise (Example Sheet I) $\square$

The localization of A at f is $A_f = A[1/f]$

LEMMA 1.5.2 The subspace U_f is (naturally) homeomorphic to Spec A_f

Proof: Primes in A_f are primes in A that miss f; (why?) Let $f: A \to A_f$

be the canonical map. Every ideal in A_f is gen. by $f(I)$ for $I \subseteq A$ an ideal $\square$

Sets up a bijection [see comm. algebra]

<u>Example 1.5.3</u> Let $A = \mathbb{C}[X,Y]$ and $f = xy$. Then $U_f = (\mathbb{C}^2 \setminus \text{axes})$ + weird points.

Fix A a ring; $X = \operatorname{Spec} A$. We have an association

$$\left\{ \begin{array}{lcl} \text{Distinguished Open Sets} & \longrightarrow & \text{Rings} \\ U_f & \longmapsto & A_f := \mathcal{O}_X(U_f) \end{array} \right.$$

<u>Meaning</u>: Open Set $\longmapsto$ Functions on that open.

"<u>Functoriality</u>":

If $U_{f_1} \subseteq U_{f_2}$ we get a homomorphism.

$$A_{f_2} \longrightarrow A_{f_1}$$ restriction of functions

GUIDEPOST for the next steps:

Given an $\boxed{\text{arbitrary open}}$ set $U \subseteq \operatorname{Spec} A$

define:

$$\mathcal{O}_X(U) = \left\{ \begin{array}{l} \text{families } (f_V)_{V \subseteq U;\, V \text{ distinguished}} \\ \text{of functions st if } V \subseteq W \subseteq U \\ \text{with } W \text{ distinguished then } f_W \text{ restricts} \\ \text{to } f_V \end{array} \right\}.$$

Unnecessary but fancy: $\mathcal{O}_X(U) = \mathcal{O}_X(\bigcup_\lambda B_\lambda) = \varprojlim_\lambda \mathcal{O}_X(B_\lambda)$

OBSERVE : This is naturally a ring. Every open set in Spec A has a ring of functions; if $U \subseteq U'$ then there is a restriction

$$\mathcal{O}_X(U') \longrightarrow \mathcal{O}_X(U).$$

PUNCHLINE: A scheme is SOMETHING OBTAINED BY GLUEING THE DATA STRUCTURES ABOVE.

VECTOR SPACES $\rightsquigarrow$ MANIFOLDS.

§2 SHEAVES:

Sheaves formalize objects that you know and the behaviour we just saw.

2.1 PRESHEAVES

The basic instance:

If X is a topological space:

$$\begin{aligned} \text{Open Sets} &\longrightarrow \text{Ab. Groups.} \\ U &\longmapsto \{f : U \to \mathbb{R} \mid f \text{ continuous}\} \end{aligned}$$

DEFINITION 2.1.1 A $\boxed{\text{presheaf}}$ of abelian groups on a topological space X is an association

$$\begin{aligned} \mathcal{F} : \text{Opens in } X &\longrightarrow \text{Ab. Groups} \\ U &\longmapsto \mathcal{F}(U) \end{aligned}$$

such that if $U \subseteq V$ there is a homomorphism $res^V_U : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ with $res^U_U = id$ and $res^V_U \circ res^W_V = res^W_U$. for $U \subseteq V \subseteq W$ opens.

Similarly presheaf of sets, rings, etc...

Remark 2.1.2 (Language) A presheaf is therefore a FUNCTOR from the CATEGORY $\underbrace{\text{Open}(X)}$ to abelian groups.

Objects : Opens
Morphisms : Inclusions

Morphisms between presheaves

what should it be? Definition by "DUH"

DEFINITION 2.1.3 A morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of presheaves on X is, for each U, a homomorphism

$$\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$ commuting with restrictions:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle res^U_V}\downarrow & & \downarrow{\scriptstyle res^U_V} \\ \mathcal{F}(V) & \xrightarrow[\varphi_V]{} & \mathcal{G}(V) \end{array}$$

$V \subseteq U$ opens

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves is injective/surjective if $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective/surjective for all U

§2.2 SHEAVES: DEFINITIONS & EXAMPLES

What additional properties does the sheaf of continuous functions satisfy?

DEFINITION 2.2.1: A sheaf $\mathcal{F}$ is a presheaf such that:

S1: If $U \subseteq X$ is open and $\{U_i\}$ is an open cover of U then for $s \in \mathcal{F}(U)$ with $s|_{U_i} := res^{U}_{U_i}(s) = 0$ for all i, then $s = 0$.

S2: If U & $\{U_i\}$ as above, given $s_i \in \mathcal{F}(U_i)$ with $\boxed{s_i = s_j \text{ on } U_i \cap U_j}$ s_i glue

AMUSING DEDUCTION: If $\mathcal{F}$ is a sheaf on X then $\mathcal{F}(\emptyset) = \{e\}$.

A morphism of sheaves is a morphism of the underlying presheaves.

Sheaves on X form a category.

Example 2.2.2 If X is a topological space the sheaf of continuous functions:

$$\mathcal{F}(U) = \{ f : U \to \mathbb{R} \, ; \text{ continuous} \}$$

is a sheaf.

Non-Example 2.2.3 Let $X = \mathbb{C}$ with euclidean topology. set

$$\mathcal{F}(U) = \{ f : U \to \mathbb{C} : f \text{ bounded \& analytic} \}$$

This is not a sheaf; bounded doesn't GLUE.

Non-Example 2.2.4 Fix a group G and set

$$\mathcal{F}(U) = G.$$

If U_1 & U_2 are disjoint, then by sheaf axioms $\mathcal{F}(U_1 \cup U_2)$ is forced to be $G \times G$.

But it should be G.

Example 2.2.5 (the constant sheaf) Fix G and set $\mathcal{F}(U) = \{ f : U \to G \mid f \text{ locally constant} \}$

This is the sheaf that 2.2.4 wants to be.

Example 2.2.6: If V is an affine/projective/quasi projective irreducible variety, set

$$\mathcal{O}_V(U) = \{ f \in k(V) \mid f \text{ is regular at } p \text{ for all } p \in U \}$$

$k(V) = \operatorname{Frac} k[V]$; regular means near p, can write $f = r/s$ with $s(p) \neq 0$.

This is called the STRUCTURE SHEAF $\mathcal{O}_X$

Check sheaf axioms [obvious!]

In "VARIETY THEORY" $k(V)$ gets used a LOT. The sheaf is the same data but with better/more flexible user interface!

§2.3 BASIC CONTRUCTIONS $\mathcal{F}$ a sheaf on X

Terminology: A section of $\mathcal{F}$ over U is some element
$s \in \mathcal{F}(U)$.

Construction 2.3.1 (stalks) Fix p in X. Then

$$\mathcal{F}_p = \text{stalk at } p$$
$$= \{(s,U) \mid s \in \mathcal{F}(U)\} / \sim$$

with $(s,U) \sim (s',U')$ if there exists nonempty $W \subseteq U' \cap U'$ such that

$$s'|_W = s|_W$$

We call elements of $\mathcal{F}_p$ a germ at p.

Example 2.3.2: Calculate $\mathcal{O}_{\mathbb{A}^1,0}$ – the stalk of the structure sheaf of $\mathbb{A}^1$ at 0.

Using "variety theory" Ex. 2.2.6. Answer: it is rational functions $f(t)/g(t)$ with $g(0) \neq 0$

The following shows the power of the sheaf axioms

PROPOSITION 2.3.3: If $f: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X such that for all p

$$f_p : \mathcal{F}_p \to \mathcal{G}_p \text{ is an isomorphism.}$$

Then f is an $\underbrace{\text{isomorphism}}_{\text{Meaning what?}}$.

PROOF: We will show that

$$f_U : \mathcal{F}(U) \to \mathcal{G}(U) \text{ is an isomorphism}$$

for all U; define f^{-1} via f_U^{-1}.

Exercise: Show that this defines an inverse map of sheaves i.e. compatibility with restriction.

Injectivity: Suppose $s \in \mathcal{F}(U)$ with $f_U(s) = 0$. Then the germ of s is 0 in every stalk $\mathcal{F}_p$ for $p \in U$, by injectivity of f_p.

Unwind definition: there exist opens U_p around every p with $s|_{U_p} = 0$. Cover U by U_p. Use sheaf axioms.

Surjective: Let $t \in \mathcal{G}(U)$; we will build $s \in \mathcal{F}(U)$.

At the level of stalks, we have an iso, so this determines stalks in $\mathcal{F}_p$ for all $p \in X$. Now choose representatives (s_p, U_p) with $s_p \in \mathcal{F}(U_p)$.

By shrinking U_p if necessary, we can assume that $\boxed{f_{U_p}(t|_{U_p}) = s_p}$ by def'n of equiv. relation.

Now, injectivity shows that these glue. So writing $U_{pq} = U_p \cap U_q$: $f_{U_{pq}}(s_p|_{U_{pq}} - s_q|_{U_{pq}})$

$$= t|_{U_{pq}} - t|_{U_{pq}} = 0$$

By sheaf axioms these glue. By the sheaf axioms, the resulting section maps to t.

$\square$

⚡ Take note of the logic: injectivity was needed for proving surjectivity

REMARK 2.3.4 Even easier (Exercises)

(i) $\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p$ is injective by S1.

(ii) Given $\mathcal{F} \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathcal{G}$ with $\varphi_p = \psi_p$ for all p

then $\varphi = \psi$.

DEFINITION 2.3.5 (sheafification) If $\mathcal{F}$ is a presheaf on X then a morphism

$$sh : \mathcal{F} \longrightarrow \mathcal{F}^{sh}$$ is a sheafification

if $\mathcal{F}^{sh}$ is a sheaf and for any "map"

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$$ with $\mathcal{G}$ a sheaf

there is a unique diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "\varphi"'] & \mathcal{F}^{sh} \arrow[d] \\ & \mathcal{G} \end{tikzcd}$$ that commutes

Remark 2.3.6: (i) Unique if it exists – if $\mathcal{F}^+$ were another, we get a diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{sh} \arrow[d, shift right] \\ & \mathcal{F}^+ \arrow[u, shift right] \end{tikzcd}$$

Now use uniqueness to show isom.

(ii) Presheaf morphisms induce morphisms of sheafification

Trivial exercise

[Stalks of presheaves make sense; $f \in \mathcal{F}_p$ is germ at p]

Proposition/Construction 2.3.7

Sheafification exists. Given a presheaf $\mathcal{F}$ on X

define:

$\mathcal{F}(U) = \{ (f_p)_{p \in U} : f_p \in \mathcal{F}_p$ & for every p there exists an open $V_p \subseteq U$ containing p and a section $s \in \mathcal{F}(V_p)$ st $s_q = f_q$ for all $q \in V_p \}$.

PROOF THIS WORKS:

- Restriction maps are clear; clearly a sheaf!
- The map $\mathcal{F} \to \mathcal{F}^{sh}$ is obvious.
- Exercise: Verify the universal property. $\square$

Note: $(\mathcal{F}^{sh})^{sh} = \mathcal{F}^{sh}$

Corollary 2.3.8: The stalks of $\mathcal{F}$ & $\mathcal{F}^{sh}$ coincide.

Exercise 2.3.9: Find a nonzero presheaf whose sheafification is zero. [This is actually rather stupid]

§2.4 KERNELS, COKERNELS, ETC

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves.

The presheaf KERNEL/IMAGE/COKERNEL assigns

$$U \longmapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U)) ; \text{ etc.}$$

If φ is a map of sheaves then

Exercise 2.4.1: The presheaf kernel of a map of sheaves is a sheaf.

Beware of the cokernel:

Example 2.4.2: $X = \mathbb{C}$, $\mathcal{O}_X = \left(\begin{array}{l}\text{holomorphic} \\ \text{functions,}\end{array} + \right)$

and $\mathcal{O}_X^* = \left(\begin{array}{l}\text{nowhere } 0 \\ \text{holomorphic} \\ \text{functions,}\end{array} \times \right)$. Now define

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^* ; \quad \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)^*$$

$\ker(\exp) =$ constant sheaf $2\pi i \mathbb{Z}$.

Cokernel is not a sheaf! Take

$U_1 = \mathbb{C} \setminus [0,\infty)$; $U_2 = \mathbb{C} \setminus [0,-\infty)$

$$U = U_1 \cup U_2 = \mathbb{C} \setminus \{0\}.$$

Take $f = z$ in $\mathcal{O}_X(U)$. This lies in the presheaf cokernel of exp. But on U_i the cokernel is 0 because logarithm exists.

$\square$.

DEFINITION 2.4.3 For a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves, the sheaf cokernel/image is the sheafification of the presheaf cokernel/image.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is injective/surjective if $\ker \varphi = 0$ / $\operatorname{im} \varphi = \mathcal{G}$.

Remark 2.4.4 (crucial!) the sequence

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \text{ is}$$

exact as sheaves; in fact for X a $\mathbb{C}$-manifold

Remark 2.4.5: Do the kernel & cokernel deserve their name? What properties should they satisfy? The kernel of $\varphi : A \longrightarrow B$ is the data

$$\ker\varphi \longrightarrow A \quad \text{universal for}$$

things becoming 0: for any diagram

$$\begin{tikzcd} & K \arrow[dl, dashed, "\exists!"'] \arrow[d] \arrow[dr, "0"] & \\ \ker\varphi \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

sending K to 0, there is a unique filled in diagram.

Proximate Notions 2.4.6

(i) Subsheaf: $\mathcal{F} \subseteq \mathcal{G}$ if there are inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ compatible with restrictions

(ii) Quotient sheaf: the sheafification of

$$U \longmapsto \mathcal{G}(U)/\mathcal{F}(U).$$

Warning 2.4.7 If $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is surjective the maps on any particular open need not be.

FACTS 2.4.7 These follow from the same kind of arguments we've used. I will not provide proofs.

(i) Stalks of kernel and image are kernel and image of the stalk maps.

(ii) Injectivity and surjectivity are stalk local properties BUT this does not mean fn's are always surjective Exponential Sequence!

§2.5 MOVING BETWEEN SPACES

Given $f : X \to Y$ with $\left.\begin{array}{l}\mathcal{F} \text{ on } X \\ \mathcal{G} \text{ on } Y\end{array}\right\}$ sheaves

DEFINITION 2.5.1 (pushforward) Define the presheaf $f_* \mathcal{F}$ on Y by $U \mapsto \mathcal{F}(\underbrace{f^{-1}(U)}_{\text{OPEN}})$

PROPOSITION 2.5.2 The pushforward is a sheaf.

Proof: Trivial. $\square$

DEFINITION 2.5.1 (inverse image) The inverse image presheaf is defined by

$$f^{-1}\mathcal{G}^{pre}(V) = \{(s_U, U) : U \text{ open containing } f(V) \text{ \& } s_U \in \mathcal{G}(U)\}/\sim$$

where $\sim$ identifies pairs that agree on a smaller open. The inverse image is

$$f^{-1}\mathcal{G} = (f^{-1}\mathcal{G}^{pre})^{sh}.$$

Remark 2.5.2 (Why is the sheafification necessary?)

Take $f : X \to Y$ with $X := Y \amalg Y$.

Take $\mathcal{G}$ on Y and $\mathcal{F} := f^{-1}\mathcal{G}^{pre}$.

Then for $V \subseteq Y$ open and $U = f^{-1}(V)$

$$\mathcal{F}(U) = \mathcal{G}(V)$$

But $U = V \amalg V \subseteq Y \amalg Y$ so

$$\mathcal{F}^{sh}(U) = \mathcal{G}(V) \times \mathcal{G}(V) \text{ by sheaf axioms.}$$

Notice similarity with constant sheaf & constant pre-sheaf.

§3 SCHEMES Spec A has a sheaf; we globalize

§3.1 AFFINE SCHEMES

Let A be a ring and $S \subseteq A$ multiplicatively closed. Then

$$S^{-1}A = \{(a,s) : s \in S,\ a \in A\}/\sim$$

with $(a,s) \sim (a',s') \iff s''(as' - a's) = 0$ in A. for some $s'' \in S$.

Example 3.1.1: (i) Take $S = \{1, f, f^2, \dots\}$

(ii) Take $S = S \setminus \mathfrak{p}$ with $\mathfrak{p}$ a prime.

$A_\mathfrak{p}$ will be the stalk of the structure sheaf at $\mathfrak{p}$. I now take the route of Vakil — not Hartshorne

SHEAF on a BASE:

Sheaf $\mathcal{F}$ on X gives $\mathcal{F}: \{\text{Base opens}\} \to \text{Groups}$ + natural restrictions

Reverse this Given a base $\{B_i\}$ with $\mathcal{F}(B_i)$ assignments & $\operatorname{res}^{B_i}_{B_j}$ satisfying

SB1: if $B = \cup B_i$ with B in the base and $res^B_{B_i}(f) = res^B_{B_j}(g)$ for all f & g then $f \equiv g$.

SB2: If $B = \cup B_i$ with $f_i \in F(B_i)$ and agreeing on overlaps then there exists $f \in F(B)$ with $f|_{B_i} = f_i$. Go look at the discussion at the end of §1.

Call this a SHEAF on a BASE $\mathcal{B} = \{B_\alpha\}$

PROPOSITION 3.1.2 A sheaf on a base F with base $\mathcal{B}$ determines a sheaf $\mathcal{F}$ by

$$\mathcal{F}(B_i) = F(B_i)$$

agreeing with restriction maps, where $B_i \in \mathcal{B}$. It is unique up to unique isomorphism.

PROOF: (i) Determine the stalks $\mathcal{F}_p$ via the basis.

(ii) Use "sheafification trick" and define

$F(U) = \{ (f_p \in \mathcal{F}_p)_{p \in U} \mid$ for all $p \in U$ there exists B a basis open around p and $s \in F(B)$ with $s_q = f_q$ for all $q \in B\}$. sheaf axioms are clear.

(iii) Natural maps $F(B) \to \mathcal{F}(B)$ are isomorphism.

PROPOSITION 3.1.3 Let A be a ring. The assignment

$$U_f = \{p \in \operatorname{Spec} A \mid f \notin p\} \mapsto A_f$$

is a sheaf on the base of distinguished opens in Spec A, with the restriction maps given by localization

Note how this depends on U_f rather than f.

PROOF: We check SB1 & SB2 on the base & set $B = \operatorname{Spec} A$ in the verification for simplicity; general case is similar.

SB1: Basic commutative algebra - Spec A is "quasi-compact" [terminology!] - proof is simple!

Write $\operatorname{Spec} A = \bigcup_{i=1}^{n} U_{f_i}$; $U_{f_i} = \operatorname{Spec} A \setminus \mathbb{V}(f_i)$.

Given $s \in A$ with $s|_{U_{f_i}} = 0$ for all i

then $f_i^m s = 0$ for appropriate m. [by defin. of localization]

But $(f_1^m, \dots, f_n^m) = (1) = A$ [b/c U_{f_i} COVER Spec A]

$1 = \left(\sum r_i f_i^{m_i} \right)$. Now clear that

$s \cdot 1 = s = 0$.

SB2: Say $\operatorname{Spec} A = \bigcup_{i \in I} U_{f_i}$. and elements in

A_{f_i} agreeing in $A_{f_i f_j}$ — do they glue?

First suppose I is finite.

On $U_{f_i} \rightsquigarrow$ have $\dfrac{a_i}{f_i^{l_i}}$. Write $g_i = f_i^{l_i}$

noting $U_{f_i} = U_{g_i}$. Overlaps: $U_{g_i g_j}$ (why?)

Overlap condition: $(g_i g_j)^{m_{ij}} \cdot (a_i g_j - a_j g_i) = 0$

Rewrite using algebra & fact that $U_f = U_{f^k}$, for $k \geq 1$

Assume $m = m_{ij}$ by taking the largest.

Write $b_i = a_i g_i^m$; $h_i = g_i^{m+1}$.

On each U_{h_i} have b_i / h_i.

Overlap condition:

$$h_j b_i = h_i b_j$$

But U_{h_i} cover Spec A so

$$1 = \sum r_i h_i \qquad r_i \in A.$$

Now write $r = \sum r_i b_i$ with r_i, b_i as above

Now verify this restricts correctly — elementary algebra

When I is infinite, pick a finite subcover with $(f_1, \dots, f_n) = A$ and U_{f_i} a cover.

Construct r as above. Need that this satisfies all restrictions

Now given $(f_1, \dots, f_n, f_\alpha)$ we get a "new" r'. By SB1 $r' = r$

$\square$.

DEFINITION 3.1.4 The structure sheaf on Spec $A = X$ is the sheaf associated to the sheaf on the base

$$U_f \longmapsto A_f. \quad \text{Denoted } \mathcal{O}_{\operatorname{Spec} A} = \mathcal{O}_X.$$

Note that the stalks are A_p.

We are now basically there – a scheme is a pair $(X, \mathcal{O}_X)$ with $\mathcal{O}_X$ a sheaf of rings locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ provided we can understand "isomorphisms".

Terminology 3.1.5: · A ringed space $(X, \mathcal{O}_X)$ is a topological space with a sheaf of rings.

· An isomorphism $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism & an isomorphism

$$\mathcal{O}_Y \xrightarrow{\simeq} \pi_* \mathcal{O}_X$$

· An affine scheme $(X, \mathcal{O}_X)$ is a ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

DEFINITION 3.1.6 A scheme is a ringed space $(X, \mathcal{O}_X)$ locally isomorphic to an affine scheme.

We will now work to understand the analogues of

· Hausdorff · Compact · Smooth · Dimension · etc

But first:

§ 3.2 EXAMPLES OF SCHEMES

EXAMPLES 3.2.1 (INTERESTING RINGS)

- $k[x_1,\dots,x_n]$
- Quotients by ideals — THE GOLD STANDARD
- Monoid rings: A toric monoid P is the positive integer span of finitely many elements $\{v_1,\dots,v_k\} \subseteq \mathbb{Z}^n$. The MONOID RING over $\mathbb{Z}$ is

$$\mathbb{Z}[P] = \{ \textstyle\sum a_u x^u \mid a_u \in \mathbb{Z};\ u \in P\}$$

Dummy

$$P = \mathbb{N}^2 \subseteq \mathbb{Z}^2 \quad \text{then} \quad \mathbb{Z}[P] \cong \mathbb{Z}[X,Y]$$

$$P = \mathbb{Z}^2 \quad \text{then} \quad \mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[X^{\pm}, Y^{\pm}]$$

- Hypersurface rings: Rings of the form $\mathbb{Z}[\underline{X}]/(f)$
- Invariant rings: R^G or $k[x_1,\dots,x_n]^G$ — Quotients of varieties.
- Artinian rings: for instance $k[t]/t^2$ or $k[t_1/t_2]/(t_1^a, t_2^b)$, etc. "zero dimensional" schemes

Examples 3.2.2 (Open subschemes) Let $U \subseteq X$ be an open and take $\mathcal{O}_U = \mathcal{O}_X|_U$ as its structure sheaf.

Inverse image

PROPOSITION 3.2.3 The space $(U, \mathcal{O}_X|_U)$ is a scheme.

[Every point has a distinguished open around it]

Example 3.2.4: Take $U = \mathbb{A}^{n^2}_k \setminus \{\text{determinant}=0\}$

GL_n is a scheme & a group.

But these can be non-affine. Take

$$\left\{ \begin{array}{l} \mathbb{A}^2_k := \operatorname{Spec} k[X,Y] \quad \& \\ \\ U = \mathbb{A}^2_k \setminus \{(0,0)\}. \end{array} \right.$$

Note $U = \operatorname{Spec} k[X, X^{-1}, Y] \cup \operatorname{Spec} k[X, Y, Y^{-1}]$

Ring inclusions:

$$\begin{tikzcd}
 & k[X,Y,Y^{-1}] \arrow[dr, hook] & \\
k[X,Y] \arrow[ur] \arrow[dr, hook] & & k[X^{\pm}, Y^{\pm}] \\
 & k[X,X^{-1},Y] \arrow[ur, hook] &
\end{tikzcd}$$

Now evaluate $\mathcal{O}_U(U) = k[X,Y]$ as the intersection of the two rings. If U were affine then $U \cong \mathbb{A}^2_k$. But $V(x,y) = \emptyset$ in U but nonempty in $\mathbb{A}^2_k$.

§3.3 INTERLUDE ON GLUING SHEAVES

X a topological space with a cover $\{U_\alpha\}$ and sheaves $\mathcal{F}_\alpha$ st:

Pullback sheaf.

$$\phi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$$

satisfying $\boxed{\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}}$ on $U_{\alpha\beta\gamma}$.

COCYCLES

Then this glues to a sheaf $\mathcal{F}$ on X.

Construction 3.3.1 Given $V \subseteq X$ define $\mathcal{F}(V)$ as tuples (s_α) with $s_\alpha \in \mathcal{F}_\alpha(V \cap U_\alpha)$. with

$$\phi_{\alpha\beta}(s_\alpha|_{V \cap U_{\alpha\beta}}) = s_\beta|_{V \cap U_{\alpha\beta}}.$$

— COMPATIBILITY

PROPOSITION 3.3.2 : $\mathcal{F}$ is a sheaf and $\mathcal{F}|_{U_\alpha}$ is $\mathcal{F}_\alpha$.

PROOF : The fact that this is a sheaf is clear. The tricky part is $\mathcal{F}_\alpha = \mathcal{F}|_{U_\alpha}$ on U_α and here we use cocycle condition.

What is the isomorphism?

Given $V \subseteq U_\gamma$ and $s \in \mathcal{F}_\gamma(V)$ take its image in $\mathcal{F}(V)$ to be

$$\left(\phi_{\gamma\alpha}\left(s|_{V \cap U_\alpha}\right)\right)_\alpha$$

But to check compatibility :

$$\phi_{\alpha\beta} \circ \phi_{\gamma\alpha}\left(s|_{V \cap U_\alpha \cap U_\beta}\right) = \phi_{\gamma\beta}\left(s|_{V \cap U_\alpha \cap U_\beta}\right)$$

$\square$

§ 3.4 More schemes

Take schemes $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ with opens $U \subseteq X$ & $V \subseteq Y$ with an isomorphism

$$(U, \mathcal{O}_X|_U) \xrightarrow{\cong} (V, \mathcal{O}_Y|_V)$$ meaning what?

Then we can glue!

$$X \amalg Y / U \sim V$$ with the glued structure sheaf.

This generalizes cleanly – Ex Sh I ; Q 13

Example 3.4.1 : (Bug-eyed line) Let

$X = \operatorname{Spec} k[t]$ & $Y = \operatorname{Spec} k[u]$ both $\mathbb{A}^1$.

$U = \operatorname{Spec} k[t, t^{-1}]$ & $V = \operatorname{Spec} k[u, u^{-1}]$. both "$\mathbb{A}^1 \setminus pt$"

Glue via $t \longleftrightarrow u$

Compare from topology: $\mathbb{R}_x \amalg \mathbb{R}_y / x \sim y$ for $x = y \neq 0$.

This is the canonical example of Hausdorff failing. But schemes are already not Hausdorff. But still...

This scheme is not affine. Calculate that $\mathcal{O}_X(X) = k[t]$ but there is an extra point!

Example 3.4.2: (Projective line)

$X = \operatorname{Spec} k[t]$ & $Y = \operatorname{Spec} k[u]$ both $\mathbb{A}^1$.

$U = \operatorname{Spec} k[t, t^{-1}]$ & $V = \operatorname{Spec} k[u, u^{-1}]$. both "$\mathbb{A}^1 \setminus pt$"

Glue via $t \longleftrightarrow u^{-1}$.

PROPOSITION 3.4.3 $\mathbb{P}^1_k$ has only constants as the global functions; in particular $\mathbb{P}^1$ is not affine.

Proof: The only polynomials in t that are polynomials in $1/t$ are constant.

[why is this a proof? Use sheaf axioms!].

Example 3.4.4 Take $\mathbb{A}^2_k$ with doubled origin.

Notice intersection of two affines is not.

A HEADS UP: An important condition for us will be separated. It will be the analogue of Hausdorff. It will imply that (affine $\cap$ affine) is affine.

§ Appendix THE PROJ CONSTRUCTION — motivation

- A few words of motivation — it is actually hard to produce schemes that are not "open in proj" — i.e. quasi-projective i.e. PART II AG

- "separated" will be the "AG Hausdorff" condition. "Proper" will be the "AG compact" condition. Proj constructions will always give us proper ($\Rightarrow$ separated) things.

DEFINITION 3.5.1: A $\mathbb{Z}$-grading on a ring A is a decomposition
$$A = \bigoplus_{n \in \mathbb{Z}} A_n$$
such that

$A_k \cdot A_j \subseteq A_{j+k}$ "multiplication respects grading"

But what is the geometry behind graded rings?

PROPOSITION 3.5.2 (for motivation) Let A be a (finitely generated nilpotent free) $k = \bar{k}$ – algebra. Let $V = \mathrm{mSpec}(A)$ i.e. the variety of A. Then a k^* – action on V given by a morphism

$$k^* \times V \longrightarrow V$$

is the same thing as a grading of A by $\mathbb{Z}$.

Variety Theory: Define $\mathbb{P}^n_k = \mathbb{A}^{n+1}_k \setminus \underline{0} / k^*$.

Only homogeneous polynomials make sense:

i.e. $\sum_\alpha a_\alpha \underline{x}^\alpha$ $\quad \alpha \in \mathbb{N}^{n+1}$ with degree $(\underline{x}^\alpha) = d$. In other words:

$$k[x_0, .., x_n] = \bigoplus_{d \geq 0} S_d \ ; \quad S_d \text{ is } k\text{-span}$$

of degree d monomials.

Observe how both "graded" and k^*-action appear naturally.

This works not just for $\mathbb{P}^n_k$ but for projective varieties:

"irrelevant point"

$$\begin{tikzcd} \pi^{-1}(V) = \tilde{V} \arrow[r] \arrow[d] & \mathbb{A}^n_k \setminus \underline{0} \arrow[d, "\pi"', "k^*\text{-quotient}"] \\ V \arrow[r, phantom, "\subseteq"] & \mathbb{P}^n_k \end{tikzcd}$$

$V = \mathbb{V}(\text{homogeneous poly's})$

$\bar{V}$ = closure of $\tilde{V}$ in $\mathbb{A}^n$

Notice that $\tilde{V}$ is k^*-invariant as is $\bar{V}$! why?

Therefore to get a projective variety:

(i) Take $\bar{V} \subseteq \mathbb{A}^n_{k+1}$ a k^*-invariant variety

(ii) Throw out junk i.e. $\underline{0}$ b/c it is dumb.

(iii) Take a quotient.

§3.5 THE PROJ CONSTRUCTION

We've lifted $\mathbb{A}^n_k$ into scheme theory

Want to do the same for $\mathbb{P}^n_k$

Spec $k[x_1, \dots, x_n]$ gives the "new" scheme theory $\mathbb{A}^n_k$. $\underbrace{\text{standard}}$ grading

Similarly Proj $k[x_0, \dots, x_n]$ will give the "new" $\mathbb{P}^n_k$

DEFINITION 3.5.1: A $\mathbb{Z}$-grading on a ring A is a decomposition $A = \bigoplus_{n \in \mathbb{Z}} A_n$ such that $A_k \cdot A_j \subseteq A_{j+k}$ "multiplication respects grading"

⚠ I will only treat Proj in a simplified setting. Assume A is $\mathbb{Z}_{\geq 0}$-graded. Moreover, A is generated over A_0 by degree 1 elements.

- $A_+ = \bigoplus_{i=1}^{\infty} A_i$ is an ideal — the irrelevant ideal of positive degree elements.

DEFINITION 3.5.2: The set $\mathrm{Proj}\, A_\bullet$ is the set of $\underbrace{\text{homogeneous primes}}$ of $A_\bullet$ not containing A_+.

generated by homogeneous elements

If $I \subseteq A_\bullet$ is homogeneous:

$$\mathbb{V}(I) := \{ \mathfrak{p} \in \mathrm{Proj}\, A_\bullet \mid \mathfrak{p} \text{ contains } I \}$$

The Zariski topology has closed sets given by $\mathbb{V}(I)$.

We now cover by affines using degree 1 elts.

Let $f \in A_1$ and $U_f = \mathrm{Proj}\, A_\bullet \setminus \mathbb{V}(f)$

PROPOSITION 3.5.2 There are natural bijections:

i. $U \longleftrightarrow$ Homog. primes in $A_\bullet[1/f]$

ii. Homog. primes in $A_\bullet[1/f]$

$\longleftrightarrow$ ALL primes in $A_\bullet[1/f]_0$.

Proof: i. is straightforward. For ii. define the bijection as follows:

$$\left\{\begin{matrix}\text{Homog. primes} \\ \text{in} \\ A_\bullet[1/f] \\ \text{(missing irrelev.)}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Primes in} \\ (A_\bullet[1/f])_0\end{matrix}\right\}$$

$$\mathfrak{q} \longmapsto \mathfrak{q} \cap (A_\bullet[1/f])_0$$

To see it is a bijection, use fact that given $I \subseteq A_\bullet$ homog., the ideal

$$(I \cdot A_\bullet[1/f]) \cap (A_\bullet[1/f])_0 \text{ is}$$

generated by taking generators for I, then dividing each generator by f raised to its degree. $\Rightarrow$ left-to-right is injective

For surjectivity, pick generators for $\mathfrak{p} \subseteq A_\bullet[1/f]_0$ and use f to clear denominators & get ideal in $A_\bullet[1/f]$. Call it $\mathfrak{q}$. By basic algebra, $\mathfrak{q}$ is prime & $\mathfrak{q} \cap A_\bullet[1/f]_0$ is $\mathfrak{p}$. $\square$

Believable! $k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right]$ is what you get from $k[x_0, \dots, x_n]$ with $f = x_0$.

Remark 3.5.3: A basis for the Z-topology on $\text{Proj}\, A_\bullet$ is given by opens $U_f = \mathbb{V}(f)^c$. $f \in A_+$

Notice, we have a natural identification

$$U_f = \text{Spec}\,(A_\bullet[1/f])_0 \text{ by above}$$

State of the union: $\text{Proj}\, A_\bullet$ is a set of homogeneous prime ideals. It is covered by U_f which are Spec of a ring and have structure sheaves, where by hypothesis, f can be taken to be degree 1.

If $(\text{Proj}\, A_\bullet)_f = \mathbb{V}(f)^c$, and $f, g \in A_1$, we have

$$(\text{Proj}\, A_\bullet)_f \cap (\text{Proj}\, A_\bullet)_g \quad \text{is}$$

$$\text{Spec}\,(A_\bullet[1/f])_0[f/g] = \text{Spec}\,(A_\bullet[f^{-1}, g^{-1}])_0$$

This gives gluing data — cocycle condition is immediate from properties of localization.

§4 MORPHISMS

We have now lots of examples of schemes coming from "variety theory". We want MAPS

§4.1 LOCALLY RINGED SPACES & MORPHISMS

Varieties (and manifolds) have tangent spaces.

$(X, \mathcal{O}_X)$ a variety and $p \in X$ the stalk

$\mathcal{O}_{X,p}$ is a local ring – everything non-unit is an ideal ; automatic that it is maximal.

- $m_p \subseteq \mathcal{O}_{X,p}$ are functions vanishing at p.
- m_p/m_p^2 are LINEAR PARTS – $\mathrm{Hom}_k(m_p/m_p^2, k)$ is $T_{X,p}$.
- $\varprojlim \mathcal{O}_{X,p}/m_p^2 := \hat{\mathcal{O}}_{X,p}$ is Taylor Expansion

DEFINITION 4.1.1 A morphism of ringed spaces

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

with $f^{top} : X \longrightarrow Y$ continuous

$f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ on Y as sheaves of rings.

NOTATION! By adjunction, $[\mathcal{O}_Y \to f_* \mathcal{O}_X]$ &
$[f^{-1}\mathcal{O}_Y \to \mathcal{O}_X]$ are equivalent. Both are denoted $f^{\#}$.

DEFINITION 4.1.2 $(X, \mathcal{O}_X)$ is locally ringed if stalks $\mathcal{O}_{X,p}$ are local [automatic for schemes]. A morphism of locally ringed spaces:

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

with $\underbrace{f^{\#}(m_p) \subseteq m_{f(p)}}_{\text{"Local homomorphism"}}$ in stalks.

DEFINITION 4.1.3: If $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ are schemes, a morphism of schemes is a morphism as locally ringed spaces

[What does it buy us?] If $\varphi: X \to Y$ morphism of schemes, if $s \in \mathcal{O}_{Y,\varphi(p)}$ is invertible then $\varphi^{\#}(s) \in \mathcal{O}_{X,p}$ is too. You can tell where functions vanish by composition.

THEOREM 4.1.9: There is a natural bijection

$$\left\{ \begin{array}{l} \text{Morphisms from} \\ \operatorname{Spec} B \to \operatorname{Spec} A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Ring homomorphisms} \\ A \to B \end{array} \right\}$$

PROOF: 1. $A \to B$ induces a scheme map

2. Every scheme map arises this way

$\Rightarrow$

Given $\varphi: A \to B$, $\varphi^{top}: \operatorname{Spec} B \to \operatorname{Spec} A$ sends $\mathfrak{p}$ to $\varphi^{-1}(\mathfrak{p})$. TOPOLOGICAL LEVEL. Continuity follows by manipulating symbols to show:

$$(\varphi^{top})^{-1}(V(I)) = V(\varphi(I)).$$

Now we build:

$$\varphi^{\#}: \mathcal{O}_{\operatorname{Spec} A} \to \varphi^{top}_{*}\mathcal{O}_{\operatorname{Spec} B}.$$

Think at stalk level:

$$\begin{array}{ccc} A_{\varphi^{-1}(p)} & \longrightarrow & B_{\mathfrak{p}} \\ \dfrac{a}{s} & \longmapsto & \dfrac{\varphi(a)}{\varphi(s)} \end{array}$$

If $s \notin \varphi^{-1}(p)$ then $\varphi(s) \notin \mathfrak{p}$.

It is automatically local! The maximals are $\mathfrak{p}B_{\mathfrak{p}}$ and $\varphi^{-1}(\mathfrak{p})A_{\varphi^{-1}(\mathfrak{p})}$.

Now think on opens: Given $U \subseteq \operatorname{Spec} A$ open define $\varphi^{\#}: \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} B}(\varphi_{top}^{-1}(U))$

A section is a reasonable assignment of stalk elts to points.

$$\underset{\text{in } U \ \vdots \ \text{in } A_{\mathfrak{p}}}{[p \mapsto s(p)]} \mapsto \underset{\text{in } \varphi^{-1}_{top}(U)}{[q \mapsto \varphi_q(s(\varphi_{top}(q)))]}$$

Does this glue?: if s is locally at p written $\dfrac{a}{h}$ then $\varphi_{\#}^{U}(s)$ is $\varphi(a)/\varphi(h)$. This tells us that $\varphi_{\#}$ glues!

Therefore $A \to B$ yields a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$.

Conversely: $(f^{top}, f^{\#}) : \operatorname{Spec} B \to \operatorname{Spec} A$. How to build $A \to B$?

Take $g : \underbrace{\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)}_{A} \to \underbrace{\mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)}_{B}$.

$$A \longrightarrow B$$

We show g gives $(f, f_{\#})$.

Map on stalks is compatible with global section maps:

Notation: $\mathcal{F}$ on X : $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$

$$\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\
\mathcal{O}_{\operatorname{Spec} A, f(p)} \arrow[r] & \mathcal{O}_{\operatorname{Spec} B, p}.
\end{tikzcd}$$

Equivalently:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d] \\
A_{f(p)} \arrow[r, "f_p^{\#}"'] & B_p.
\end{tikzcd}$$

By locality $(f_p^{\#})^{-1}\, \mathfrak{p} B_{\mathfrak{p}} = f(p) A_{f(p)}$.

By commutativity $f(p) = g^{-1}(p)$. Thus, topologically we get the right map. They agree on stalks so we are done! $\square$

§4.2 A FEW BASIC NOTIONS

Note about $f^\#$ notation & adjunction.

DEFINITIONS 4.2.1

$f: X \to Y$ is an open immersion if f induces an isomorphism onto an open subscheme of Y i.e. $(U, \mathcal{O}_Y|_U)$; $U \subseteq Y$ open

$g: X \to Y$ is a closed immersion if g^{top} is a homeomorphism ont a closed subset and $g^\#: \mathcal{O}_Y \to g_* \mathcal{O}_X$ is surjective.

Example 4.2.2: Take $k[t] \longrightarrow k[t]/t^2$ and take Spec. This is closed.

AWKWARD DEFINITION 4.2.3: A closed subscheme is an equivalence class of closed immersions, where $[X \to Y] \sim [X' \to Y]$ if there is a triangle.

$$\begin{tikzcd} X' \arrow[rr, "\text{iso}"] \arrow[dr] & & X \arrow[dl] \\ & Y & \end{tikzcd}$$

Scheme theoretic points: Let K be any field. A K-valued point of a scheme X is a morphism

$$\mathrm{Spec}\, K \to X.$$

We write the set of all such maps as $X(K)$.

Example 4.2.6 Take $X = \mathbb{P}^n_{\mathbb{C}}$. Then $X(\mathbb{C})$ is the $\mathbb{P}^n_{\mathbb{C}}$ you know and love.

Remark 4.2.5: For any ring R we could define R-valued points similarly. Infact, we can do the same for S any scheme! We will therefore obtain

$$F_X : \text{Rings} \longrightarrow \text{Sets}$$
$$R \longmapsto X(R).$$

This "functor of points" is eventually very useful, but I want to stay close to geometry.

Very concrete! Given $p \in X$, there is an affine open U around p. Setting $k(p) = FF(\mathcal{O}_{X,p})$ we get

$$\mathrm{Spec}\, k(p) \to U \hookrightarrow X$$

Every point is a scheme theoretic point

§ 4.3 FIBRES & FIBRE PRODUCTS

Motivation: Fibre products are a common generalization of several operations: (0) The right notion of product.

(i) $X_1 \hookrightarrow Y$ & $X_2 \hookrightarrow Y$ closed subschemes
Intersection "$X_1 \cap X_2$" is a fibre product

(ii) Given $X \xrightarrow{f} Y$ a morphism and $y \in Y$, the fibre $f^{-1}(y)$ is a scheme

(iii) The intuitive statement that $\mathbb{P}^n_{\mathbb{C}}$ is obtained from $\mathbb{P}^n_{\mathbb{Z}}$ and $\mathbb{Z} \hookrightarrow \mathbb{C}$.

DEFINITION 4.3.1 Let
$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$
be morphisms of schemes. The fibre product is a scheme $X \times_S Y$ with maps
$$\begin{tikzcd} X \times Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$
commuting,

such that for any

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

commuting, there is a unique map

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

with all diagrams commuting. If exists, unique upto unique iso

Makes sense in any category. If X, Y, B were just sets then $X \times_B Y \subseteq X \times Y$ as pairs that project to the same point of B.

THEOREM 4.3.2 Fibre products of schemes exist.

PROOF: [Hartshorne Theorem 3.3 — do look it up!]

1. Affine Case: If X, Y, S are affine with rings A, B, R then $\mathrm{Spec}(A \otimes_R B)$ satisfies the

universal property. To verify, notice that by same ideas in previous lecture, a map

$$Z \longrightarrow \operatorname{Spec} A \otimes_R B \text{ is a ring map}$$

$$A \otimes B \longrightarrow \Gamma(Z, \mathcal{O}_Z).$$

Globalization: Slowly turn the 3 pieces into affines.

2. If $X \times_S Y$ exists and $U \subseteq X$ is open then $U \times_S Y$ exists: take $p_X^{-1}(U)$ with open subscheme structure.

3. If X is covered by $\{X_i\}$ then if $X_i \times_S Y$ exists they can be glued to $X \times_S Y$.

why? The schemes already glue to X, but the maps to Y can also be glued – this is easier than you think – no cocycle conditions !

4. For any X but Y & S affine, $X \times_S Y$ exists. Since X & Y are interchangeable, $X \times_S Y$

exists for affine S.

5. Cover S by affines $\{S_i\}$. Let X_i & Y_i be the p_X & p_Y preimages. $X_i \times_{S_i} Y_i$ exist.

But in fact, $X_i \times_{S_i} Y_i = X_i \times_S Y$ [Think about intersections!

You have flexibility!]

6. Now glue again ∇ !

$\square$

Notation:

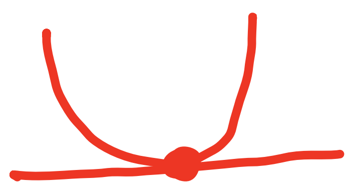
$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

Little box says Z is the fibre product.

Examples 4.3.3: with some honest geometry

(i) $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$. [why?]

(ii) Take $C = \operatorname{Spec} \mathbb{C}[x,y]/_{y-x^2}$



$L = \operatorname{Spec} \mathbb{C}[x,y]/_{(y)}$

Then $C \times_{\mathbb{A}^2} L = \operatorname{Spec} \mathbb{C}[x]/_{x^2}$

"FAT POINT"

(iii)

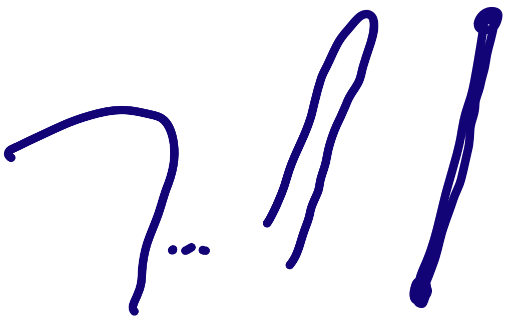
Total space of family

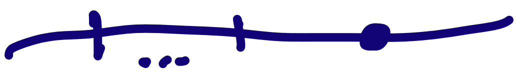
$$\begin{tikzcd} \operatorname{Spec} \dfrac{\mathbb{C}[x,y,t]}{(y^2+tx)} \arrow[r] & \operatorname{Spec} \mathbb{C}[t] \\ \operatorname{Spec} \dfrac{\mathbb{C}[x,y]}{y^2} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \operatorname{Spec} \mathbb{C}[t]/(t) \arrow[u] \end{tikzcd}$$

Family of mostly conics.

closed point 0.

"DOUBLED" X-Axis



More generally: 

(iv) Recall that given $p \in S$ we defined $\kappa(p) = FF(A/p)$ with $\operatorname{Spec} A \hookrightarrow S$ an open neighborhood. Given $X \to S$ the scheme theoretic fiber of $X \to S$ at p is

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \kappa(p) \arrow[r] & S \end{tikzcd}$$

SCHEME OVER $\kappa(p)$

⚠ If S is arithmetic eg $\operatorname{Spec} \mathbb{Z}$, fibres live in different FIELDS !!

(v) In Ex (iii) take $\operatorname{Spec} \mathbb{C}(t) \hookrightarrow \operatorname{Spec} \mathbb{C}[t]$

The generic fibre of π

$$\begin{tikzcd} \operatorname{Spec} \dfrac{\mathbb{C}(t)[x,y]}{(y^2+tx)} \arrow[r] \arrow[d] & \operatorname{Spec} \dfrac{\mathbb{C}[x,y,t]}{(y^2+tx)} \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{C}(t) \arrow[r] & \operatorname{Spec} \mathbb{C}[t] = S \end{tikzcd}$$

Consolidates information that is constant on an open set in the base S

Language 4.3.4 In scheme theory, we often fix a base scheme S and study the collection of schemes $X \to S$. If no such choice is implicitly made, we take $\underbrace{S = \operatorname{Spec} \mathbb{Z}}$

Terminal object.

In variety theory, $S = \operatorname{Spec} k$ $(k = \bar{k})$.

The product of varieties X & Y is

$$X \underset{\operatorname{Spec} k}{\times} Y.$$

In $\underbrace{\mathrm{Sch}/S}_{\text{schemes over } S}$, given X/S & Y/S the product in this category is $X \times_S Y$.

The "USUAL" PRODUCT never comes up [until you start using $\mathbb{C}$ + Euclidean topology].

§4.3½ : Example sheet II contains many basic notions — reduced, irreducible, integral, noetherian, finite type. — You should read the sheet at a minimum. (i) We will not need it for now (ii) I will supply a number of examples later

§4.4 SEPARATED MORPHISMS

Given X a scheme X_{top} is essentially never Hausdorff. But bug-eyed-line is worse than $\mathbb{A}^1$ or $\mathbb{P}^1$ — why?

Hausdorff is about separating pairs of points and so can be phrased in $\boxed{\text{topology}}$ as

$$X \text{ is Hausdorff} \iff \Delta_X \subseteq \underbrace{X \times X}_{\text{product topology}} \text{ is closed.}$$

DEFINITION 4.4.1: Given $X \to S$ a scheme map the diagonal $\Delta_{X/S}$ is the morphism below:

$$\begin{tikzcd}
X \arrow[dr, "\Delta_{X/S}"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

(*) [induced by universal property]

Write Δ insted of $\Delta_{X/S}$ when clear.

PROPOSITION 4.4.2: Let $X \to S$ be a scheme map. The diagonal is a locally closed immersion,

i.e.
$$\begin{tikzcd}
X \arrow[r, "\text{CLOSED}"] \arrow[rr, bend right, "\Delta"'] & U \arrow[r, "\text{OPEN}"] & X \times_S X
\end{tikzcd}$$

PROOF: We find open in $X \times_S X$ in which X is closed.

Say S is covered by $\{V_i\}$ affine opens and X is covered by affines $\{U_{ij}\}$; fix i, U_{ij} covers $p_X^{-1}(V_i)$

$U_{ij} \to V_i$ the maps induced by

$$\begin{tikzcd} U_{ij} \arrow[r] & X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V_i \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

Now $U_{ij} \times_{V_i} U_{ij}$ is affine & covers the diagonal

Also, $\Delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$ [Use Diagram ⊛ !]

For U_{ij} affine

$$U_{ij} \hookrightarrow U_{ij} \times_{V_i} U_{ij}$$ is $\boxed{\text{clearly}}$ if you remember the definitions.

a closed immersion. $\square$

PROPOSITION 4.4.3 If $X \to S$ is a map of affine schemes then $\Delta_{X/S}$ is a closed immersion

[Same claim — $A \otimes_B A \to A$ always surjective]

DEFINITION 4.4.4 spooky! A morphism $X \to S$ is separated if the diagonal is a closed immersion.

Easy fact: If $X \to Y$ is a locally closed immersion whose image is a closed topological subset, then it is closed [definition chasing]

Examples 4.4.5: (i) For any ring R the morphism $\mathbb{A}^n_R \to \operatorname{Spec} R$ is separated.

(ii) The bug-eyed line $\mathbb{A}^1_k \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1_k$ is NOT separated

(iii) For a ring R $\mathbb{P}^n_R \to \operatorname{Spec} R$ is separated.

(iv) Open & closed embeddings are separated. Composition of separated maps are too. Proofs are ok but

PROPOSITION 4.4.6: Let R be any ring. Then

$$\mathbb{P}^n_R \longrightarrow \operatorname{Spec} R \text{ is separated,}$$

[recalling $\mathbb{P}^n_R := \operatorname{Proj} R[x_0, .., x_n]$]

Proof: We want to show that

$$\begin{tikzcd} \mathbb{P}^n_R \arrow[r, "\Delta"'] & \mathbb{P}^n \times_R \mathbb{P}^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}^n \arrow[d] \\ & \mathbb{P}^n \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

is closed. Suffices to show after restricting to an open cover of $\mathbb{P}^n \times_R \mathbb{P}^n$.

Set $A_\bullet = R[\underline{x}]$ and $U_i = \operatorname{Spec}(A_\bullet[\frac{1}{x_i}])_0$

[recall from construction of $\mathbb{P}^n_R$]

The schemes $U_i \times_R U_j$ cover $\mathbb{P}^n \times_R \mathbb{P}^n$

Observe: restriction of diagonal is exactly

$$U_i \cap U_j \longrightarrow U_i \times_R U_j$$

$$R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]\left[\frac{x_i}{x_j}\right] \longleftarrow R\left[\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_1}{y_j}, \dots, \frac{y_n}{y_j}\right]$$

replace y's with x's. Map is clearly surjective, so Δ is closed.

$\square$

PROPOSITION 4.4.7: Let U, V be affine opens of a separated scheme X/S. Then $U \cap V$ is affine.

Proof: The following diagram is Cartesian

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta_{X/S}"'] & X \times_S X \end{tikzcd}$$

Since $\Delta_{X/S}$ is separated $U \cap V \to U \times_S V$ is closed.

Closed in affine is always affine [ExSh III]

$\square$

§4.6 PROPERNESS

Recall from ExSh II $f: X \to Y$ is finite type if for any $V \subseteq Y$ affine, $V = \operatorname{Spec} A$, $f^{-1}(V)$ has a finite cover by U_i affine such that $U_i = \operatorname{Spec} B_i$ and B_i is finitely generated over A.

DEFINITION 4.6.1: A scheme map $X \xrightarrow{f} Y$ is closed if f is a closed map. It is universally closed if f' is also closed for any diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

$\boxed{\text{WHY?}}$ Think about closed vs. compact.

A scheme map $f: X \to Y$ is proper if it is separated, finite type, and universally closed.

PROPOSITION 4.6.2: Suppose A is any commut. ring. Then $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper.

Proof: Soon!

ASSUMPTION from NOW FORWARD: All schemes are Noetherian unless stated otherwise.

We now discuss the valuative criteria.

A discrete valuation ring is a local PID.

Examples 4.4.6: $\mathbb{C}[[t]]$, $\mathcal{O}_{\mathbb{A}^1,0}$, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ p-adic integers.

If A is a DVR $\operatorname{Spec} A$ is a connected doubleton. GERMS OF CURVES

max'l ideal has a gen. OPEN CLOSED

called the uniformizer

There is a valuation: $A \setminus \{0\} \to \mathbb{Z}_{\geq 0}$

THEOREM 4.6.2 (VALUATIVE CRITERIA): Let $X \xrightarrow{f} Y$ be a scheme map. Then f is separated with $FF(A)=K$ is separated iff for any DVR A

given

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

there is at most 1 lift

making everything commute. It is universally closed if there is at least 1 lift. It is proper if there is exactly 1 lift.

COROLLARY 4.6.3

(i) $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper

(ii) $\mathbb{A}^n_A \to \operatorname{Spec} A$ for $n \geq 1$ is NOT proper

(iii) Closed subschemes of $\mathbb{P}^n_A$ are proper over Spec A.

(iv) Closed immersions are proper

(v) Composition of proper morphisms are proper.

(vi) Consider
$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ Y \arrow[r] & S \end{tikzcd}.$$
If f is proper then f' is proper.

(vii) Properness of $f: X \to Y$ is local on Y.

THEOREM 4.6.4: $\mathbb{P}^n_A \to \operatorname{Spec} A$ satisfies existence & uniqueness parts of valuative crit.

Proof: By base change, can take $A = \mathbb{Z}$.

We check the valuative criterion: take a DVR R. $T = \operatorname{Spec} R$, & let $U = \operatorname{Spec} K$ where $K = FF(R)$.

Consider the lifting diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

By induction: assume $U \subseteq \mathbb{V}(X_i)^c$ for all i, where $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \dots, x_n]$.

Now, x_i/x_j lie in the stalk at the image of U i.e. they are well-defined and let f_{ij} be the pullback in K and

note: $f_{ij} \cdot f_{jk} = f_{ik}$

Now let $d_0, \dots, d_n$ be $\nu(f_{i0})$. If d_k is the smallest, then $\nu(f_{ik}) \geq 0$.

Now define: $\mathbb{Z}\left[\frac{x_0}{x_k}, \dots, \frac{x_n}{x_k}\right] \longrightarrow R$

$$x_i/x_k \longmapsto f_{ik}.$$

$\square$

§4.6 – a brief interlude on other types of morphisms

$X \xrightarrow{f} Y$ a scheme map.

(i) FINITE: Cover Y by affines $\operatorname{Spec} B_i = U_i$ st $V_i = f^{-1}(U_i)$ is open affine $\operatorname{Spec} A_i$ and A_i is a fg B_i MODULE.

Examples: Non-constant maps of smooth curves

closed immersions

(ii) FLAT: At every $p \in X$ the map

$$f^{\#} : \mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X, p} \text{ makes } \mathcal{O}_{X,p}$$

a flat $\mathcal{O}_{Y, f(p)}$ module. [injectivity of $\mathcal{O}_{Y,f(p)}$ modules is preserved] "Everything is flat over a field"

UTILITY: Given

$$\begin{array}{ccccc} Z_\eta & \underset{\text{closed}}{\hookrightarrow} & \mathbb{P}^n_{\mathbb{C}((t))} & \hookrightarrow & \mathbb{P}^n_{\mathbb{C}[[t]]} \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec} \mathbb{C}((t)) & \to & \operatorname{Spec} \mathbb{C}[[t]] \end{array}$$

There exists a unique $Z \hookrightarrow \mathbb{P}^n_{\mathbb{C}[[t]]}$ that is FLAT OVER $\mathbb{C}[[t]]$

(iii) More sophisticated ring theory gives notions of étale map [covering spaces], unramified maps [immersions in topology], smooth map [submersions]. I have equipped you with enough background to understand such notions make sure the work to understand such notions is in the affine / ring case.

§5 MODULES OVER $\mathcal{O}_X$

§5.1 Motivation:

An $\mathcal{O}_X$-module is a sheaf of groups with $\mathcal{O}_X$-multiplication. Before we do it formally, I give examples

Example 5.1.1: On $\mathbb{P}^n$ the variety take the sheaf $\mathcal{O}_{\mathbb{P}^n}(d)(U) = \{ P(\underline{X})/Q(\underline{X})$ Rational homogeneous function st degree is d and regular on all points of $U\}$

Notice $\mathcal{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n) =$ Degree d homogeneous polynomials

Recall $\mathcal{O}_{\mathbb{P}^n}(U)$ are rational, i.e. ratios of poly's same degree so we have a multiplication map!

Note: If $d<0$, no global sections but still pretty interesting!

Example 5.1.2: Given a module M over A, define the sheaf $\mathcal{F}_M(U_f) = M_f$ by localization. Gluing is identical to what we know. Notation: sometimes $\tilde{M}$.

§ 5.2 DEFINITIONS OF $\mathcal{O}_X$-MODULES

Fix $(X, \mathcal{O}_X)$ a ringed space.

DEFINITION 5.2.1: A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ of groups st for $U \subseteq X$ open there is a multiplication $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ compatible w/ restriction.

A sheaf of $\mathcal{O}_X$-algebras is defined similarly.

Standard Notions: Kernel, image, cokernel, direct sum, direct product, submodule, ideal sheaf

Also: Tensor product & Hom — Require sheafification!

Moving between spaces

$X \xrightarrow{f} Y$ a ringed space morphism. Given $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules the pushforward $f_*\mathcal{F}$ is a $f_*\mathcal{O}_X$-module.

But we have $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ giving an $\mathcal{O}_Y$-module. structure

Conversely for $\mathcal{G}$ a sheaf of $\mathcal{O}_Y$-modules, define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ via the adjoint $f^\#$.

Basic Fact 5.2.2 f^* and f_* are adjoint functors for modules over ringed spaces.

§5.3 $\mathcal{O}_X$-MODULES ON SCHEMES $\quad X$ a SCHEME

DEFINITION 5.3.1 A quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is one such that there is an open cover $\{U_i\}$ by affines with $\mathcal{F}|_{U_i}$ the sheaf associated to a module M_i over $\Gamma(U_i, \mathcal{O}_X|_{U_i})$. It is coherent if M_i are fg modules.

Basic Examples: $\mathcal{O}_X$ on any scheme, similarly $\mathcal{O}_X^{\oplus n}$. For $Y \overset{i}{\hookrightarrow} X$, $i_* \mathcal{O}_Y$ is coherent.
Affine case: this is the sheaf associated to A/I.

REFERENCE: Hartshorne II §5.

PROPOSITION 5.3.2 An $\mathcal{O}_X$-module $\mathcal{F}$ is q-coh if and only if on any $U = \operatorname{Spec} A$ affine, $\mathcal{F}|_U$ is the sheaf assoc. to an A-module.
If X noetherian, then $\mathcal{F}$ is coherent if and only if M's are finitely generated.

KEY COROLLARY 5.3.3: q-coherent $\mathcal{O}_X$-modules on X affine are equiv. to modules over $\mathcal{O}_X(X)$.

Proof strategy: Condition on random opens $\rightsquigarrow$ condition on distinguished random opens $\rightsquigarrow$ condition on any affie of your choosing

KEY LEMMA 5.3.3: $X = \operatorname{Spec} A$, $f \in A$, and $\mathcal{F}$ q-coh. Let $s \in \Gamma(X, \mathcal{F})$ Then

(i) If s restricts to 0 on U_f then $f^n s = 0$ for some n

(ii) If $t \in \mathcal{F}(U_f)$ then $f^m t$ is the restriction of a global section for some m.

Proof: There exists an open cover where $\mathcal{F}$ restricts to the sheaf assoc. to a module on each one. Since $V(g)^c$ form a basis, we can find a cover by such $\{U_{g_i}\}$ st $\mathcal{F}|_{U_{g_i}}$ is a module over A_{g_i}. Finite number suffices. Now use properties of the localization.

$\square$

PROOF OF 5.3.2: Given $U = \operatorname{Spec} A \subseteq X$ and $\mathcal{F}$ on X, observe that $\mathcal{F}|_U$ is still quasi-coh. [Why?] Reduce to $X = \operatorname{Spec} A$.

Now, take $M = \mathcal{F}(X)$ and let M^{sh} be the sheaf on $X = \operatorname{Spec} A$ associated to it.

We have a map $M^{sh} \longrightarrow \mathcal{F}$. By the lemma, on a distinguished open $\mathbb{V}(g)^c$, the map is locally an isomorphism, therefore an isomorphism.

LESSON: Quasi-coherence is local.

There is a "Proj" version. Take $A_\bullet$ a $\mathbb{N}$ graded ring and $M_\bullet$ a graded $A_\bullet$-module. Then for $U = \mathbb{V}(f)^c \subseteq \operatorname{Proj} A_\bullet$ with $f \in A_1$, we know $U \cong \operatorname{Spec}(A_\bullet[1/f])_0$. Now

$$U \longmapsto (M_\bullet[1/f])_0$$

glues to a sheaf $M_\bullet^{sh}$ on $\operatorname{Proj} A_\bullet$.

WORDS: A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is a vector bundle if it there exists a Zariski open cover st $\mathcal{F}$ restricts to a FREE module on each.

Line bundle is locally free of rank 1. An ideal sheaf is locally an ideal in $\mathcal{O}_X$.

ASIDE: On a scheme/top space X, always have the constant sheaf $\underline{\mathbb{Z}}$. Not a quasi-coherent sheaf in a natural way. The structure sheaf $\mathcal{O}_X$ is coherent. These are very different but equally important!

§5.4 PROJECTIVE SCHEME THEORY.

On $\mathbb{P}^n_{\mathbb{C}}$ the old skool variety we defined $d \in \mathbb{Z}$

$$\mathcal{O}_{\mathbb{P}^n}(d)(U) = \{ f/g \mid \text{homog rational function degree } d, \text{ no poles on } U\}$$

SCHEME THEORETICALLY: Instead of degree 0 elements in localizations, take degree d elements.

DEFINITION 5.4.1: Fix $A_\bullet$ graded, $M_\bullet$ a graded $M_\bullet$-module. Define $M_\bullet(d)$ the module whose degree k piece is M_{k+d}. The sheaf $(A_\bullet(d))^{sh}$ on $\operatorname{Proj} A_\bullet$ is $\mathcal{O}_X(d)$.

PROPOSITION 5.4.2: $\mathcal{O}_X(d)$ is a line bundle.

Proof really IS trivial !

Remark 5.4.3 If $A_\bullet$ is generated over A_0 in degree 1, then $\operatorname{Proj} A_\bullet \hookrightarrow \mathbb{P}^n$ and those are restrictions of homogeneous rational functions.

Construction 5.4.3: (source of line bundles)

Given* $X \xrightarrow{f} \mathbb{P}^n$ over any base we get a line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ on X. Moreover, we get write the homogeneous coordinates of $\mathbb{P}^n$ by $x_0, \dots, x_n$ we get sections

$$s_1, \dots, s_n \in \Gamma(X, f^*\mathcal{O}(1))$$

*: very concrete!

Definitions by cheating: Let L be a line bundle on X. Then L is basepoint free if there exists $X \to \mathbb{P}^r$ st $f^*\mathcal{O}(1) = L$. It is very ample if in addition $X \hookrightarrow \mathbb{P}^r$ is a locally closed embedding.

L is ample if $L^{\otimes n}$ is very ample for $n \gg 0$.

Example 5.4.4 Take $A_{\bullet,\bullet} = \mathbb{C}[X, Y, Z, W]$ and $\mathcal{O}(1,1)$ the shifted $A_{\bullet,\bullet}(1,1)$-sheaf. Then $\mathrm{BiProj}\, A_{\bullet,\bullet} \xhookrightarrow{i} \mathbb{P}^3$ with $\mathcal{O}(1,1) = i^*\mathcal{O}_{\mathbb{P}^3}(1)$.

SEGRE EMBEDDING.

Example 5.4.5: Take $A_\bullet = \mathbb{C}[x_0, .., x_n]$ and take $L = \mathcal{O}_{\mathbb{P}^n}(d)$ on Proj $A_\bullet$. Then if $m = \binom{n+d}{d} - 1$, we get

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^m$$ given by degree d monomials.

VERONESE EMBEDDING.

On Ex Sh III we do global versions of Spec and Proj. Given $\mathcal{A}$ a sheaf of $\mathcal{O}_X$-algebras that is quasi-coherent we can take

$$\underline{\mathrm{Spec}}\, \mathcal{A} \longrightarrow X$$ & if $\mathcal{A}$ is graded

$$\underline{\mathrm{Proj}}\, \mathcal{A} \longrightarrow X.$$

Gives three beautiful geometric constructions.

(i) Given $\mathcal{E}$ a locally free sheaf of rank r consider $\mathrm{Sym}^\bullet \mathcal{E}^\vee$ a sheaf a algebras over $\mathcal{O}_X$. Recall: if E is a vector space of dimension n $\mathrm{Sym}^\bullet E$ is a polynomial ring!

Then $\underline{\mathrm{Spec}}\ \mathrm{Sym}^\bullet \mathcal{E}^\vee = \mathrm{Tot}(\mathcal{E})$ is

$$\begin{array}{c} \mathrm{Tot}(\mathcal{E}) \\ \pi \downarrow \\ X \end{array}$$

a scheme st $\pi^{-1}(p) \cong \mathbb{A}^n_{k(p)}$ residue field.

Given $s \in \mathcal{F}$, there is a tautological morphism

$$X \underset{s}{\overset{\pi}{\rightleftarrows}} \mathrm{Tot}(\mathcal{E})$$ i.e. a SECTION.

(ii) Similarly, take $\underline{\mathrm{Proj}}\ \mathrm{Sym}^\bullet \mathcal{E}^\vee$ to get a $\mathbb{P}^{n-1}$- bundle

(iii) Let $\mathcal{J} \subseteq \mathcal{O}_X$ be an ideal sheaf. Then $\bigoplus_{d \geq 0} \mathcal{J}^d = \mathcal{J}_\bullet$ is a graded $\mathcal{O}_X$-algebra.

Define $Bl_{\mathcal{J}} X := \underline{\mathrm{Proj}}\, \mathcal{J}_\bullet \longrightarrow X$ to be the blowup.

Example: Take $\mathcal{J} = (x,y)$ & $X = \mathbb{A}^2_{\mathbb{C}}$. Then $Bl_{\mathcal{J}} \mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^2$ is the blowup you know. Wonderful calculation ! and $\pi^{-1}\mathcal{J} \cdot \mathcal{O}_X$ is a line bundle !

If $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal sheaf there is a difference in general between $f^*\mathcal{J}$ and the ideal generated by $f^{-1}\mathcal{J}$ inside $\mathcal{O}_Y$.

* Degree 1 in $A_\bullet$ and Global Spec.

§6 DIVISORS ON SCHEMES "sheaves & subschemes"

why? In rings, height 1 primes — in good cases — are principal and are the easiest to probe.

We now discuss WEIL DIVISORS and in such discussion assume X is Noetherian, integral, separated & regular in codimension 1, i.e. all $\mathcal{O}_{X,x}$ of dim 1 are DVR's.

§6.1 TOPOLOGICAL PRELIMINARIES:

(i) Dimension of X is length n of longest chain of nonempty closed irred. subsets

$$Z_0 \subsetneq \cdots \subsetneq Z_n \text{ in } X$$

Follows from normality.

Dimension of $\mathbb{A}^n_k$ is n.

(ii) Codimension of $Z \hookrightarrow X$ closed irred. defined similarly: $Z = Z_0 \subsetneq \cdots \subsetneq Z_n$ in X

If A is a fg k-algebra & integral then

$$\text{Krull Dim } A = \text{height } \mathfrak{p} + \text{Krull dim } A/\mathfrak{p}$$

Most intuition from here fails in general.

(iii) If X is a noetherian topological space, then every closed $Z \subseteq X$ has a finite irred comp. decomp.

§6.2 WEIL DIVISORS

DEFINITION 6.2.1 A prime divisor is a closed integral subscheme of codimension 1. A Weil divisor is an element of the free abelian group on prime divisors $\underline{\mathrm{Div}\, X}$. Effective means positive coeffs.

If X is integral, then there is a point η, the ideal (0) in any affine open. Define $k(X) = \mathcal{O}_{X,\eta}$

CONSTRUCTION 6.2.2: Let $f \in k(X)^{\times}$. [what is this practically?]. Then take

$$\mathrm{div}(f) = \sum_{\substack{Y \subseteq X \\ \text{prime}}} n_Y(f)\, [Y] \quad \text{where}$$

$n_Y(f)$ is the valuation of f in $\mathcal{O}_{X,\eta_Y}$.

Generic Point of Y.

PROPOSITION 6.2.3 : The element $\mathrm{div}(f)$ is a divisor, i.e. the sum is FINITE.

FACT: If $Y \subseteq X$ integral codim 1, $\mathcal{O}_{X,\eta_Y}$ DVR w/ fraction field $K(X)$.

PROOF : Take $U \subseteq X$ affine ; $U = \operatorname{Spec} A$ st f is regular i.e. $f \in A \hookrightarrow K(X)$. Then $X \setminus U = Z$ is closed of codim ≥ 1. Thus only finitely many Y_i's are in U^c. On the rest, any Y_i for which $\nu_{Y_i}(f) > 0$ is contained in $V(f)$. Those are contained in $V(f)$ so we're done. $\square$

⚠ We used something here. Given a closed subset $Z \hookrightarrow X$ there is a unique reduced scheme structure on it. $\boxed{\text{Hartshorne Ex. 3.2.6}}$

DEFINITION 6.2.4 : A divisor of the form $\mathrm{div}(f)$ is principal. They form a group. The quotient

$$\mathrm{Div}\, X / \mathrm{Prin}\, X := \mathrm{Cl}(X).$$

The class group is (i) interesting (ii) Hard to calculate - simplest of the Chow groups.

Basic Calculations 6.2.5

(i) If $X = \operatorname{Spec} A$ w/ A a UFD then $Cl(X) = 0$

[if X is normal then $Cl(X) = 0 \Leftrightarrow$ UFD]

(ii)* $Cl(\mathbb{P}^n) \cong \mathbb{Z}$ generated by H. [why?]

(iii) If $Z \hookrightarrow X$ closed with $U = Z^c$ open,

then $Cl(X) \twoheadrightarrow Cl(U)$ given by intersection with U. If $\operatorname{codim}(Z) \geq 2$ this is an isomorphism. If Z is codim 1 & irred. then $\mathbb{Z} \to Cl(X) \to Cl(U) \to 0$ is exact.

Excision Sequence.

COROLLARY 6.2.6 If $X = \mathbb{P}^n \setminus H_d$ [a degree d hypersurface] then $Cl(X) \cong \mathbb{Z}/d\mathbb{Z}$.

Class Group of $\mathbb{P}^n$: work over k

1. If $D \subseteq \mathbb{P}^n$ integral & codimension 1, then

$$D = \mathbb{V}(f)\ ;\ f \text{ homogeneous degree } d.$$

Define $\deg(D) = \deg(f)$.

2. Extend linearly to get

$$\deg : \operatorname{Div} \mathbb{P}^n \longrightarrow \mathbb{Z}$$

Claim: deg is an isomorphism

$$\deg : \operatorname{Cl}(\mathbb{P}^n) \longrightarrow \mathbb{Z}$$

well-defined because $\deg(\operatorname{div}(f)) = 0$

3. Surjective: take $H = \mathbb{V}(x_0)$.

Injective: If $D = \sum n_{Y_i}[Y_i]$. If $\sum n_{Y_i} \cdot (\deg(Y_i)) = 0$, with $Y_i = \mathbb{V}(g_i)$, take $f = \prod g_i^{n_{Y_i}}$. Then $\operatorname{div}(f) = D$

Excision Sequence:

Proposition 6.5. *Let X satisfy* (∗), *let Z be a proper closed subset of X, and let $U = X - Z$. Then:*

(a) *there is a surjective homomorphism* $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$ *defined by* $D = \sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$, *where we ignore those $Y_i \cap U$ which are empty;*

(b) *if* $\mathrm{codim}(Z,X) \geqslant 2$, *then* $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$ *is an isomorphism;*

(c) *if Z is an irreducible subset of codimension* 1, *then there is an exact sequence*

$$\mathbf{Z} \to \mathrm{Cl}\, X \to \mathrm{Cl}\, U \to 0,$$

where the first map is defined by $1 \mapsto 1 \cdot Z$.

PROOF.

(a) If Y is a prime divisor on X, then $Y \cap U$ is either empty or a prime divisor on U. If $f \in K^*$, and $(f) = \sum n_i Y_i$, then considering f as a rational function on U, we have $(f)_U = \sum n_i (Y_i \cap U)$, so indeed we have a homomorphism $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$. It is surjective because every prime divisor of U is the restriction of its closure in X.

(b) The groups $\mathrm{Div}\, X$ and $\mathrm{Cl}\, X$ depend only on subsets of codimension 1, so removing a closed subset Z of codimension $\geqslant 2$ doesn't change anything.

(c) The kernel of $\mathrm{Cl}\, X \to \mathrm{Cl}\, U$ consists of divisors whose support is contained in Z. If Z is irreducible, the kernel is just the subgroup of $\mathrm{Cl}\, X$ generated by $1 \cdot Z$.

§6.3 CARTIER DIVISORS

Commutative Algebra: A is a UFD iff all height 1 primes are principal. On a scheme X st $\mathcal{O}_{X,x}$ all UFD's Weil divisors are nice.

In general, "locally principal" & "codim 1" diverge

DEFINITION 6.3.1 A Cartier divisor is a section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$

But care is required here!

For X a scheme, take the presheaf

$$U = \operatorname{Spec} A \longmapsto S^{-1}A, \quad S = \text{all nonzero divisors}$$

and sheafify to get $\mathcal{K}_X$ & $\mathcal{K}_X^*$. Similarly take

$$U = \operatorname{Spec} A \longmapsto A^* \text{ to be}$$

and sheafify to get $\mathcal{O}_X^*$.

$\mathcal{K}_X^*$ are invertible elements

Remark 6.3.2: what does this mean practically?
Given a cover $\{U_i\}$ with rational functions f_i on each such that on overlaps f_i/f_j the ratios are in $\mathcal{O}_X^*$.

Construction 6.3.3: If X is regular in codim 1 and [integral, noetherian, separated] then given D a Cartier divisor we get a Weil divisor by the rule, for $Y \subseteq X$ codim 1 & integral, and D represented by $\{(U_i, f_i)\}$ with $\eta_Y \in U_i$, take $n_Y = \nu_Y(f_i)$.

well defined: $f_i/f_j \in \mathcal{O}_X(U_{ij})^*$ so has valuation equal to 0.
[by the principal divisor construction!]

PROPOSITION 6.3.4 If X is noetherian integral sep. with all local rings UFD's ($\Rightarrow$ regular in codim 1) then the association

$$\begin{Bmatrix} \text{Cartier} \\ \text{Divisors} \end{Bmatrix} \xrightarrow[\text{6.3.3}]{\text{Construction}} \begin{Bmatrix} \text{Weil} \\ \text{divisors} \end{Bmatrix}$$

respects principal divisors and is a bijection.

PROOF: Follow nose & look at Hartshorne. Key: if A is UFD $\Leftrightarrow$ Height 1 primes are principal. If $x \in X$, then $\mathcal{O}_{X,x}$ is a UFD & so for $D \in \operatorname{Div} X$, $D \cap \operatorname{Spec} \mathcal{O}_{X,x}$ is $\operatorname{div}(f_x)$. Extends to an open U_x where D & $\operatorname{div}(f_x)$ agree $\square$

This gives the Cartier divisor.

PROPOSITION 6.3.5: If X is normal, integral, sep, noetherian then Cartier divisors are Weil divisors that are locally principal.

Construction 6.3.6 Given $\mathcal{D}$ Cartier we can consider $L(\mathcal{D}) \subseteq \mathcal{K}_X$ the subsheaf by taking representatives $\{(U_i, f_i)\}$ [Remark 6.3.2] and defining $L(\mathcal{D})$ to be the $\mathcal{O}_X$-module generated by $\frac{1}{f_i}$'s on the U_i's.

Example: Take $\mathcal{D} = H$ on $\mathbb{P}^n_{\mathbb{C}}$. Then $L(H)$ is homogeneous linear polynomials.

A locally free sheaf of rank 1 — a line bundle

L has an "inverse" $\mathrm{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X) =: L^{-1}$.

The Picard group is

$$\mathrm{Pic}(X) = \left\{ \begin{array}{c} \text{Line bundles up to} \\ \text{isomorphism} \end{array} \right\}$$

Group by $\otimes$

Under very mild assumptions [eg projective over k; integral]

The map $\boxed{\mathrm{Cartier}(X) \longrightarrow \mathrm{Pic}(X)}$ is surjective

with kernel exactly the principal divisors.

Calculating these groups is hard, but they are critical to understanding schemes. For example, if X is a surface then $\mathrm{Pic}(X)$ is a group with a pairing

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}.$$

Key to classifying algebraic surfaces.

§7 SHEAF COHOMOLOGY: Hartshorne Ch. III.

Given a sheaf of abelian groups on a top space X, the group $\Gamma(X, \mathcal{F})$ is natural, but loses lots of information.

EG: • $X = \mathbb{P}^n_{\mathbb{C}}$ & $\mathcal{F}_1 = \mathcal{O}_X$ & $\mathcal{F}_2 = \underline{\mathbb{C}}$ (constant) have same global sections.

• Take $X = \mathbb{A}^2$ & $Y = \mathbb{A}^2 \setminus (0,0)$, we saw $\Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$. Where has the extra information gone?

Given $(X, \mathcal{F})$, sheaf cohomology will give new invariants: $H^i(X, \mathcal{F})$ $i \in \mathbb{N}$ with the following features:

1. The group $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
2. It's functorial. If $f: X \to Y$ and $\mathcal{F}$ is a sheaf on Y then

$$H^i(X, f^{-1}\mathcal{F}) \leftarrow H^i(Y, \mathcal{F}) : f^*$$

3. If $\underline{\mathbb{Z}}$ is the constant sheaf on a CW complex / nice top space, $H^i(X, \underline{\mathbb{Z}})$ is the "usual" topological cohomology theory.

4. It will take a SES

$$\left\{ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \right. \quad \text{and output}$$

an exact sequence:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to$$
$$\boxed{H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to} \cdots$$

We will find interesting new invariants of schemes that depend on the algebraic structure:

eg: $H^i(X, \mathcal{O}_X)$ or if $X \to \operatorname{Spec} k$ is a scheme with a coherent sheaf $\mathcal{F}$

$$\chi(X, \mathcal{F}) = \sum (-1)^i \dim_k h^i(X, \mathcal{F})$$

Euler characteristic of a sheaf.

Cohomology provides a LES [How? For a large class of $\mathcal{F}$, the maps are surj. Now extend

$$\begin{cases} 0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \\ \to \cdots \end{cases}$$

Contravariant in X.

§7.1 INJECTIVE RESOLUTIONS

DEFINITION 7.1.1: An abelian group I is injective if given

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[ul, dashed, "\exists \text{ lifting.}"'] \end{tikzcd}$$

(*)

Remark 7.1.2: For abelian groups, injective means DIVISIBLE; G divisible means $\forall g \in G$ & $\forall n \in \mathbb{N}$ there is $h \in G$ st $nh = g$.

Ex: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{C}^*$, arbitrary direct products.

Nontrivial finitely generated is never divisible.

Injective sheaf of abelian groups is defined via same property.

⚠ Constant sheaf $\underline{\mathbb{Q}}$ is not always injective!

DEFINITION 7.1.3 An *injective* resolution of A is an exact sequence $\underbrace{A \to I_0 \to I_1 \to \cdots}_{\text{possibly infinite}}$ with I_j injective.

PROPOSITION 7.1.4: Injective resolutions of abelian groups exist.

Proof: Every ab group injects into a divisible group.
[Use $G = \bigoplus_I \mathbb{Z}/K$; take $\bigoplus_I \mathbb{Z} \hookrightarrow \prod_I \mathbb{Q}$ & quot by K.

Now *iterate*:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & A \arrow[r] & I_0 \arrow[rr] \arrow[dr] & & I_1 \arrow[r] & \cdots \\
& & & C_0 \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

$\square$

COROLLARY 7.1.5: A sheaf of ab. groups can be emb. into an injective sheaf.

PROOF: $0 \to \mathcal{F}_x \to I_x$ for each stalk. Now take $\iota_x : \{x\} \hookrightarrow X$ and consider $(\iota_x)_* I_x$ and

$$\boxed{\mathcal{F} \hookrightarrow \prod_{x \in X} (\iota_x)_* I_x} \rightsquigarrow \mathcal{I}$$

. Now use fact that $\mathrm{Hom}_{sh}(\mathcal{G}, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}_{gp}(\mathcal{G}_x, I_x)$. $\square$

Exercise: If $\mathcal{F}$ is injective then given $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ global sections stays exact!

§7.2 COHOMOLOGY: Assert: Our theory is trivial on injective sheaves.

Given $\mathcal{F}$, replace by a complex of injectives $0 \to \mathcal{F} \to I_\bullet$ and define

$$H^i(X,\mathcal{F}) := \frac{\text{kernel}}{\text{image}} \text{ of } \Gamma(I_\bullet) \text{ at } i\text{th step.}$$

$$= \frac{\ker(\Gamma(I_i) \to \Gamma(I_{i+1}))}{\operatorname{im}(\Gamma(I_{i-1}) \to \Gamma(I_i))}.$$

Attack Plan: Replace a sheaf $\mathcal{F}$ on which you want to apply $\Gamma(X,-)$ with an $\underbrace{\text{injective}}_{\text{why?}}$ resolution. Now apply $\Gamma(X,-)$.

PROPERTIES 7.2.1

(i) $H^i(X,-)$ is independent of resolution

(ii) Given $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we get connecting homomorphisms $H^i(\mathcal{F}'') \to H^{i+1}(\mathcal{F}')$ giving the promised LES.

(iii) Given a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ commutative square.

$$\begin{tikzcd}
H^i(\mathcal{F}'') \arrow[r, "\delta"] \arrow[d] & H^{i+1}(\mathcal{F}') \arrow[d] \\
\cdots & \cdots
\end{tikzcd}$$

(iv) If $\mathcal{F}$ is "acyclic for Γ" i.e. all restrictions are surjective, we can resolve by acyclic rather than injective. || Injective $\Longrightarrow$ Flasque := restriction maps are surjective.

(v) H^0 is global sections.

<u>Remark 7.2.2</u>: Given $(X, \mathcal{O}_X)$ ringed space we can define things similarly (but actually no need for this.)

<u>Theorem</u> 7.2.3 (Grothendieck vanishing) If X is noetherian of dimension n and $\mathcal{F}$ a sheaf of ab. groups on X, $H^i(X, \mathcal{F}) = 0$ if $i > n$.

<u>Typical Use</u>: Given $X \xrightarrow{\pi} Y$ and $\mathcal{F}$ on X, if $\mathcal{F}$ is locally free, is $\pi_* \mathcal{F}$ also locally free?

- Calculating dimensions of natural spaces. For example, the scheme $\mathrm{Mor}_k(X, Y)$ or $\mathrm{Hilb}(X)$ – or components thereof.
- Proving that a scheme is smooth etc.
- If X is proj. smooth curve then genus $g(X) = h^1(X, \mathcal{O}_X)$

§7.3 Čech Cohomology

X: topological space & $\mathcal{F}$ a sheaf on X.

$\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X. Well-order I

write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$.

The group of Čech p-cochains is

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$$

There is a differential

$$C^p \xrightarrow{d} C^{p+1} \ ; \quad \text{given } \alpha \in C^p$$

then $$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i_k} \cdots i_{p+1}} \Big|_{U_{i_0 \cdots i_{p+1}}}.$$

Exercise: $d^2 = 0$

DEFINITION 7.3.1: The Čech cohomology groups are $\check{H}^p(\mathcal{U}, \mathcal{F})$ the cohomology groups of the above cochain complex.

⚠ If $\mathcal{U}$ sucks then $\check{H}^*$ will also suck. For example if $\mathcal{U} = \{X\}$ then you only detect H^0.

Examples 7.3.2: $X = S^1$ with $\mathcal{F} = \underline{\mathbb{Z}}$ the constant sheaf. Take $\mathcal{U} = \{U, V\}$ to be

Then $C^0 = \mathbb{Z}^2$ and $C^1 = \mathbb{Z}^2$ with

Čech cochain complex
$$\begin{cases} d : C^0 \longrightarrow C^1 \\ (a,b) \longmapsto (b-a, b-a) \end{cases}$$

$$\check{H}^0 = \check{H}^1 = \mathbb{Z}$$ [kernel & cokernel of d]

This is super explicit!

Assume all $U_{i_0 \dots i_p}$ affine open & $\mathcal{F}$ q-coh. Then Čech computes cohomology.

Work on $\mathbb{P}^n_k$:

THEOREM 7.3.3: Let $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)$. Then as graded vector spaces:

- $H^0(\mathbb{P}^n, \mathcal{F}) \cong k[x_0, \dots, x_n]$

• $H^n(\mathbb{P}^n, \mathcal{F}) \cong \frac{1}{x_0 \cdots x_n} k[x_0^{-1}, \ldots, x_n^{-1}]$

• $H^p(\mathbb{P}^n, \mathcal{F}) = 0$ for all other p.

So $h^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d}$ for $d \geq 0$.

$h^n(\mathbb{P}^n, \mathcal{O}(d)) = \binom{-d-1}{n}$ $d \leq -n-1$.

<u>PROOF</u>: First part is trivial / follows from def'n.

• Second part: Standard cover $U_i = \mathbb{V}(x_i)^c$.

Observe: $\mathcal{F}(U_{i_0 \cdots i_p}) = k[x_0, \ldots, x_n]_{x_{i_0} \cdots x_{i_p}}$.

k-spanned by monomials $x_0^{k_0} \cdots x_n^{k_n}$ with

$k_{i_0}, \ldots, k_{i_p} \in \mathbb{Z}$ & rest in $\mathbb{N}_0$.

Vector spaces are: $\begin{cases} \check{C}^{n-1} = \bigoplus_{i=0}^{n} k[x_0, \ldots, x_n]_{x_0 \cdots \hat{x}_i \cdots x_n} \\ \check{C}^n = k[x_0, \ldots, x_n]_{x_0 \cdots x_n} \end{cases}$

Since $\check{C}^{n+1} = 0$ we get:

$$H^n(\mathbb{P}^n, \mathcal{F}) = C^n / \mathrm{im}(C^{n-1} \to C^n)$$

$$= \frac{k\text{-span}\{x_0^{k_0} \cdots x_n^{k_n} : k_i \in \mathbb{Z}\}}{k\text{-span}\{x_0^{k_0} \cdots x_n^{k_n} : \text{at least one } k_i \geq 0\}}$$

$$= k\text{-span}\{\text{monomials as} : \text{all } k_i < 0\}$$

This is the claimed answer.

(c) <u>Induction on dimension</u>:

View $i : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ as $\mathbb{V}(x_0)$. Exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\cdot x_0} \mathcal{O}_{\mathbb{P}^n} \to i_* \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$

Build this as an exact sequence of graded modules over $k[x_0, \ldots, x_n]$. Tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$.

By design, we have a LES; assuming result for dimensions up to $n-1$, we get 3 seq.

$$0 \to H^0(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^0(\mathbb{P}^n, \mathcal{F}) \to H^0(\mathbb{P}^{n-1}, \mathcal{F}) \to H^1(\mathbb{P}^n, \mathcal{F})$$

sheaf on $\mathbb{P}^{n-1}$

$$\xrightarrow{\cdot x_0} H^1(\mathbb{P}^n, \mathcal{F}) \to 0 \quad \text{A}$$

$$0 \to H^p(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^p(\mathbb{P}^n, \mathcal{F}) \to 0 \quad 1 < p < n-1$$

B

$$0 \to H^{n-1}(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^{n-1}(\mathbb{P}^n, \mathcal{F}) \to H^{n-1}(\mathbb{P}^{n-1}, \mathcal{F})$$

$$\to H^n(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} H^n(\mathbb{P}^n, \mathcal{F}) \to 0 \quad \text{C}$$

The second sequence is also an isomorphism for $p=1$ by explicitly writing the first sequence.

Now, $\cdot x_0$ makes $H^p(\mathbb{P}^n, \mathcal{F})$ a $k[x_0]$-module. Calculate localization of this at x_0 by localizing the complex: $H^p(\mathbb{P}^n, \mathcal{F})_{x_0} = H^p(U_0, \mathcal{F}|_{U_0}) = 0$

Thus, for any $\alpha \in H^p(\mathbb{P}^n, \mathcal{F})$, then $x_0^k \cdot \alpha = 0$ for some k. But $\cdot x_0$ is an isomorphism.

$\square$

SIMILAR CALCULATION:

- If $X = \mathbb{A}^2 \setminus \{(0,0)\}$ then $H^1(X, \mathcal{O}_X)$ is infinite dimensional.

FACT 7.3.4: Let $X \to \operatorname{Spec} k$ be proper & $\mathcal{F}$ coherent on X. Then $H^p(X, \mathcal{F})$ is finite dimensional over k

- If $X = \mathbb{V}(f_d) \subseteq \mathbb{P}^2_k$ f homog. of degree d

$k[x_0, x_1, x_2]$

Assume $(1:0:0) \notin X$. Then say

$$U = X \cap \mathbb{V}(x_1)^c \text{ \& } V = X \cap \mathbb{V}(x_2)^c$$

Can similarly write out Čech complex.

Get
$$\dim_k H^0(X, \mathcal{O}) = 1$$
$$\dim_k H^1(X, \mathcal{O}) = \binom{d-1}{2}.$$

Calculating cohomology is hard. A simpler but useful invariant is the Euler characteristic.

Easier to compute: Euler characteristics

Given X/k proper & $\mathcal{F}$ coherent on X.

Set $$\chi(X,\mathcal{F}) = \sum_{p=0}^{\infty} (-1)^p \dim_k H^p(X,\mathcal{F})$$

Since $0 \to \mathcal{F}' \hookrightarrow \mathcal{F} \to \mathcal{F}'' \to 0$ gives LES:

$$\boxed{\chi(X,\mathcal{F}) = \chi(X,\mathcal{F}') + \chi(X,\mathcal{F}'').}$$

Let X be a 1 dimensional scheme. The (arithmetic) genus of X is $p_a(X) = 1 - \chi(C,\mathcal{O}_C)$

PROPERTY: Suppose $Z = X \times_{\operatorname{Spec} k} Y$. Then

If $\mathcal{F}$ & $\mathcal{G}$ are sheaves on X & Y and

$\mathcal{P} = p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}$. Then

$$\chi(Z,\mathcal{P}) = \chi(X,\mathcal{F}) \cdot \chi(Y,\mathcal{G}).$$

NICE COROLLARY: No product of curves of genus ≥ 1 is a hypersurface in $\mathbb{P}^3$.

Let's define $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$

Easy Calculation: $\left(\begin{matrix}\text{Ideal sheaf} \\ \text{sequence}\end{matrix}\right)$ $X = \mathbb{V}(f_d) \subseteq \mathbb{P}^n$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

implicit pushforward

In particular, $h^1(X, \mathcal{O}_X) = 0$. But also

$$\chi(X, \mathcal{O}_X) = 1 - \binom{3-d}{3}.$$

so once $d \geq 3$, different surfaces are never isomorphic.

DUALITY THEORY: Given $Z \overset{\iota}{\hookrightarrow} X$ a closed subscheme then $\mathcal{I} := \ker(\iota^* : \mathcal{O}_X \to \mathcal{O}_Z)$ this is also coherent!

DEFINITION 7.4.3: The conormal sheaf is given by $\iota^*(\mathcal{I}/\mathcal{I}^2)$, where $\mathcal{I}^2$ is the sheafification of the presheaf

$U \mapsto \mathcal{I}(U)^2 \subseteq \mathcal{O}_X(U)$. Notation: $N^\vee_{Z/X}$.

If X & Z have all regular local rings then $N^\vee_{Z/X}$ is a locally free sheaf of rank codim(Z, X).

The normal bundle is $N_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(N^\vee_{Z/X}, \mathcal{O}_Z)$.

DEFINITION 7.4.4: If $\begin{array}{c} X \\ \downarrow \\ Y \end{array}$ is separated, define the then we define

$$\Omega_{X/Y} := N^\vee_{\Delta_{X/Y}}.$$

Motivation from topology: Normal bundle of X in $X \times X$ is naturally T_X.

If X is non-singular then this is a bundle [locally free]

The "determinant" bundle is $\underbrace{\Lambda^{\dim X} \Omega_X}_{\text{SHEAF ASSOC. TO THE PRESHEAF}} = \omega_X$

THEOREM 7.4.5 (Serre Duality) If X is nonsingular projective over k of dimension n. If $\mathcal{F}$ is locally free of finite rank [$\mathcal{O}_X$-module] then:

$$H^i(X, \mathcal{F}) \xrightarrow{\simeq} H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee.$$