Tropical Curves and Superabundancy

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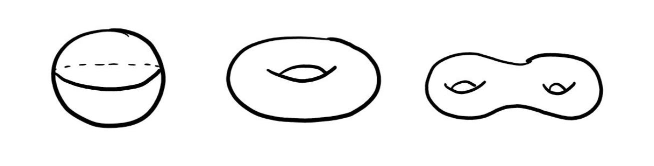
Algebraic geometry studies spaces that can be described as the zeros of some polynomials.

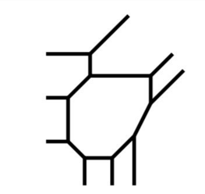
Here we will be interested in *algebraic curves*, which are such spaces with complex dimension 1. As topological spaces, they look like surfaces.



From Algebraic Curves to Tropical Curves

We are interested in algebraic curves, but algebraic curves are complicated.

Tropicalization turns an algebraic curve into a combinatorical object, called a *tropical curve.* They are comparatively very simple, but they also retain a lot of structure. e.g. number of holes, degree.



There are lots of powerful theorems relating properties of tropical curves to properties of algebraic curves.

e.g. plane curve

Suppose we have an algebraic curve C in $\mathbb{P}^2$.

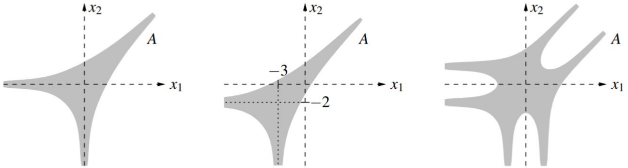
1. Restrict to $(\mathbb{C}^*)^2$ and apply the map

$$\begin{aligned} \text{Log}: \quad & (\mathbb{C}^*)^2 \to \mathbb{R}^2 \\ & (z_1, z_2) \mapsto (\log|z_1|, \log|z_2|) \end{aligned}$$

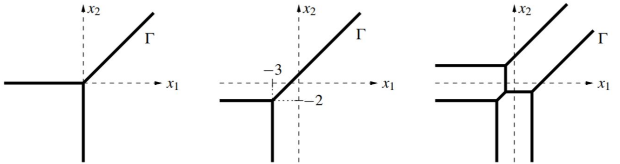
   to obtain an *amoeba*.

2. 'Squish' the tentacles.

See *Tropical Algebraic Geometry* by Andreas Gathmann for details.



(a) $C = \{z;\ z_1 + z_2 = 1\}$ (b) $C = \{z;\ e^3 z_1 + e^2 z_2 = 1\}$ (c) $C =$ a generic conic



(a) $C = \{z;\ z_1 + z_2 = 1\}$ (b) $C_t = \{z;\ t^{-3} z_1 + t^{-2} z_2 = 1\}$ (c) C_t : a family of conics

Tropical Curves

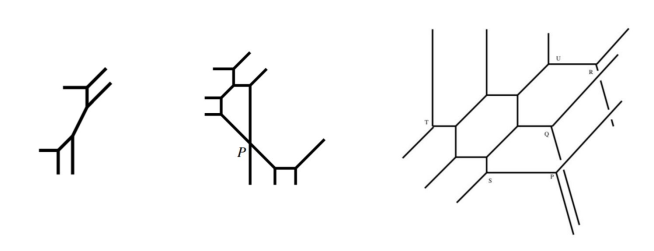
What kind of objects can arise from this process?

Definition
A *tropical curve* is a one-dimensional curve in $\mathbb{R}^n$ with the structure of a finite, connected metric graph, with edges being straight lines and vertices where they join, potentially with edges going off to infinity. These satisfy:

1. Rational slopes
2. Balancing condition

Examples

Some examples



Realizability

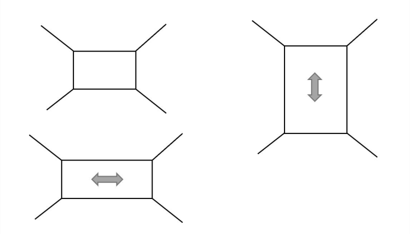
The inverse problem: **can a tropical curve be faithfully realised as the tropicalization of an algebraic curve?**

Cheung, Fantini, Park and Ulirsch (2016): **yes... with conditions**
(Faithful Realizability of Tropical Curves)

One of these conditions is that the tropical curve is *non-superabundant.*

Dimension of moduli space of deformations

To explain what superabundancy means, let's take a step back and ask a different question: **Given a tropical curve, in how many independent ways can you deform it?**



In this example, we have 2 independent 'stretches' and 2 independent translations, so the dimension of the space of deformations is 4.

Dimension of moduli space of deformations

Proposition
Suppose (Γ, h) *is a* ***trivalent****, tropical curve in* $\mathbb{R}^n$*, with* ***genus*** g *and* x ***ends****. Then the dimension of the space of deformations is at least* $x + (n-3)(1-g)$*.*

Definition
A tropical curve is *superabundant* if the dimension of the space of deformations is strictly greater than $x + (n-3)(1-g)$.

Results

In this project...

New families of superabundant curves

This + other new results $\implies$

Families of tropical curves which *cannot* be realised as the tropicalization of algebraic curves.

How to look for superabudant tropical curves

Consider the space of maps from a smooth, projective, connected algebraic curve C of genus g to $\mathbb{P}^n$.

Riemann-Roch predicts that this space has dimension at least $x + (n-3)(1-g)$ where $x = d(n+1)$, the number of ends we expect from the tropicalization.

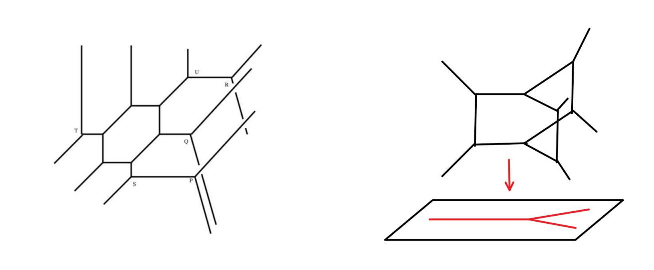
The space is 'too big' when the term $L(K_C - D)$ in Riemann-Roch is greater than 0.

Times when this happen:

- $D = 0, g > 0$
- $D = K_C$, which gives rise to the *canonical embedding*.
- Hyperelliptic curves - the existence of a '2:1' map from C to $\mathbb{P}^1$.

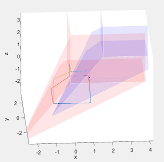
Examples

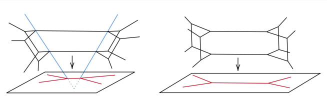
- Planar superabundance
- Three fins



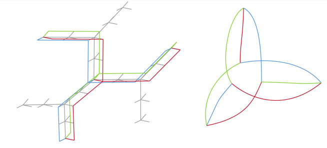
More Examples

- Genus three curve in a tropical plane
- More ‘2:1’ maps to trees



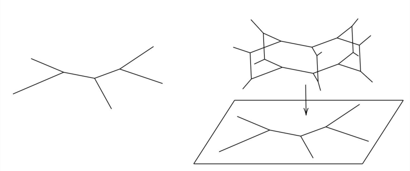


- Genus four curve in the tropicalisation of $\mathbb{P}^1 \times \mathbb{P}^1$ realized as a tropical conic x tropical line in $\mathbb{R}^4$.



Even more examples

Whole family of '2:1' maps to trees.



… and these are all examples of tropical curves which cannot be realised as the tropicalization of an algebraic curve.

Acknowledgements

Thank you to my supervisor, the department, and my fellow students for invaluable advice and support.

Thank you, for being here. I hope this was interesting, useful or both.

Questions?