Student Algebraic Geometry Enrichment Seminar

The following document contains suggestions for a student-run reading seminar for students in Part III Algebraic Geometry in Michaelmas 2022. The Part III course is devoted to a theoretical development of sheaves and schemes. While this is extremely powerful and versatile, it does not permit us to have the time to study a large range of examples in detail.

Do I need this for the exam? The material here is chosen intentionally to be completely orthogonal to the examinable material. The purpose is to give you an opportunity to see the beautiful geometric examples that make algebraic geometry an exciting subject. You will not gain anything on an exam by knowing the material suggested below.

Why should you do this? If you're fairly convinced you like algebraic geometry already you might enjoy this. If you've not seen undergraduate algebraic geometry before, you might enjoy seeing something more concrete. You might also choose to do this if you like learning mathematics by discussing it with other people.

How should you use this? It is really up to you! Ideally, there would be either a few small study groups who read things and meet to discuss, or a larger organized reading seminar.

Will the instructor be involved? I will not be present for any of these activities, as much as I would like to be. I am, however, happy to discuss aspects of any of the topics below occasionally. The PhD students in algebraic geometry may also be interested in discussing things from time to time.

Suggested topics:

In addition to what is suggested below, basically anything from Joe Harris's textbook "Algebraic Geometry: A First Course" is fantastic. Another great source is the book "3264 and all that" by Eisenbud-Harris.

Grassmannians and flag varieties. The Grassmannian is a generalization of projective space. While one studies linear subspaces of dimension 1 in order to obtain the projectivization, higher dimensional subspaces give rise to Grassmannians. A basic goal is to prove that the Grassmannian is naturally a projective variety, i.e. cut out by a homogeneous ideal in projective space. One can explicitly write down this ideal using minors of matrices. Flag varieties are natural generalizations. The reference below contains a beautiful discussion.

See: Harris "Algebraic Geometry: A First Course" Lecture 8

Toric varieties. Toric varieties form another generalization of projective space, orthogonal from the one above. The appeal of toric varieties is that they can be controlled entirely using combinatorics, via an object called the fan. This is something you can really draw and visualize, and lots of calculations that are basically impossible for arbitrary varieties become very concrete for toric varieties. Examples include projective spaces and affine spaces of all dimensions and products of these. They also include "weighted" projective spaces (see Wikipedia).

See: David Cox's "Lectures on Toric Varieties" here: https://dacox.people.amherst.edu/lectures/coxcimpa.pdf.
See also: Bill Fulton's "Introduction to Toric Varieties".

Quadrics and cubics. Hypersurfaces in projective space defined by equations of degree 2 and 3 are the most basic objects of algebraic geometry after projective space itself. A nice lecture would examine quadric hypersurfaces and prove that they are always birational to projective space. For cubics, the most interesting accessible result is the fact that there are 27 lines on any smooth cubic surface. The hardest part is proving that there is only one line. If you assume this, then it's an enjoyable calculation to show that there are 27 lines. The rationality of hypersurfaces of cubics of dimension 4 and higher is one of the most fundamental open problems in algebraic geometry!

See: Various. One treatment is given in Reid's "Undergraduate Algebraic Geometry".

Algebraic groups. One thing that the first two examples above have in common is that in both cases, there are large groups of symmetries that act on Grassmannians and toric varieties. The general study of groups "within algebraic geometry", i.e. algebraic groups, and the properties of their quotients, is a big subject. A nice lecture could be a tour natural group actions on algebraic varieties and a tour of quotients in algebraic geometry, including symmetric products and quotients by tori.

See: Harris "Algebraic Geometry: A First Course" Lecture 10

Algebraic curves. The typical undergraduate course in algebraic geometry studies curves. At the same time, some of the most compelling open problems concern algebraic curves. An excellent lecture here would define the genus of an algebraic curve and prove the following basic fact: that there exists a curve of any positive genus. Students who took Part II Algebraic Geometry will have a quick proof looking through the example sheets in the 2022 edition of the course. Another nice potential lecture could explain Riemann's parameter count for the moduli space of algebraic curves, which can be done via Riemann-Roch/Riemann-Hurwitz by taking on faith that the moduli space of curves exists. Write to me if you'd like to pursue this.

See: Bill Fulton's "Algebraic Curves", freely available online.
See also: Dhruv's Notes for Part II Algebraic Geometry here: https://www.dpmms.cam.ac.uk/~dr508/AGIINotes.pdf

Blowups. Perhaps the most important example of a morphism in all of algebraic geometry is the blowup. Roughly, the idea is to start with a variety X and produce a new variety Y that is birational to it. This means that they should be isomorphic almost everywhere. The blowup takes as input a subvariety of X and "blows it up" to produce Y. Blowups are related to things like rational maps and also singularities. Any singular variety in characteristic 0 can be blown up to produce a smooth variety (Hironaka). If there has already been a discussion of toric varieties, the resolution of singularities algorithm for toric surfaces has a nice relationship with continued fractions. Write to me if you'd like to pursue this. Further topics could include the weak factorization theorem.

See: Wikipedia for a start, and Harris "Algebraic Geometry: A First Course" Lecture 7

Moduli spaces. Loosely, a moduli space is a space where you care about the points: each point has a geometric meaning. For example, in projective space, every point "means" a line through the origin in the associated vector space. The Grassmannian is similar. Moduli spaces (of curves, of vector bundles, of…) are ways to study families of algebraic varieties. They are one of the things that schemes are really great for. A good lecture would be to explain the first example of a moduli space after the Grassmannian: the moduli space of n-points on P1 and its compactification. The moduli space of higher genus curves is more ambitious, but extremely interesting.

See: Notes of Renzo Cavalieri "Moduli Spaces of Pointed Rational Curves" available here: https://www.math.colostate.edu/~renzo/teaching/Moduli16/Fields.pdf
See also: "Algebraic Geometry: A First Course" Lecture 21