Part III Algebraic Geometry: October Engine Check

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The following document is a collection of basic tasks/exercises for you to work through. If you feel you understand the material on this sheet well, it means that you've understood the material in this first part of the course to a satisfactory extent. If any of this material seems unfamiliar or confusing, now is a good time to speak with friends, revise the notes, and/or consult with other materials to make sure the foundations are set.

1. If V is an irreducible affine variety over an algebraically closed field k, i.e. the vanishing locus of a prime ideal $\mathfrak{p}$ of polynomials in n variables in $\mathbb{A}^n_k$ it has a natural *structure sheaf*. If we say that $k(V)$ is the fraction field of $k[\underline{x}]/\mathfrak{p}$, and call it the *rational function field*, then the sheaf associates to to a Zariski open set U, the ring $\mathcal{O}_X(U)$ of rational functions that are *regular at all points of* U. Recall that a function regular at a point p means that it can be expressed as a fraction whose denominator is nonzero at the point.

Task 1: Make sure you understand the definition of this sheaf, convince yourself that you understand the restriction maps, the stalks at every point, and why it satisfies the sheaf axioms. This means that you suddenly have a very large class of examples of ringed spaces, which are very close related to schemes.

Task 2: Compare the space and structure sheaf above to the affine scheme $\operatorname{Spec} k[\underline{x}]/\mathfrak{p}$.

At the end of the course, we will completely replace this rudimentary notion by saying that they are scheme that satisfy certain properties (which we haven't defined yet). Nevertheless, they are a great source of examples.

2. Review the notion of sheaffification and the definition of the cokernel and image of a morphism of sheaves on a topological space.

Task: Suppose that $\mathcal{F} \to \mathcal{G}$ is a surjective morphism of sheaves on a topological space. Choose a section f of $\mathcal{G}$ over an open neighborhood around a point. Observe that there is not necessarily a lifting of this section to a section of $\mathcal{F}$ over this open set. However, convince yourself that if you're allowed to shrink the neighborhood, you *can* find a lift.

3. Try to understand the inverse image and pushforward sheaf in the following concrete example. Let X be a topological space and let $j : \{x\} \hookrightarrow X$ be the inclusion of some point. Let $p : X \to \mathsf{pt}$ be the unique map to a point.

Task 1: Take the constant sheaf $\underline{\mathbb{Z}}$ on $\{x\}$ and describe the sheaf $j_\star\underline{\mathbb{Z}}$. That is, describe its values on all open sets of X.

Task 2: Let $\mathcal{F}$ be a some sheaf on X. Calculate the sheaf inverse image $j^{-1}\mathcal{F}$, i.e. describe the value of this inverse image sheaf on $\{x\}$ in terms of the stalks of $\mathcal{F}$.

Task 3: Modify the task for the map p: start with a sheaf on a point and examine its inverse image under p. Similarly, start with a sheaf on X and pushforward to a point.

4. Let X be a scheme and $U \subset X$ be a Zariski open. We endowed U with a structure sheaf by taking the inverse image sheaf of the structure sheaf $\mathcal{O}_X$ under the inclusion of the open.

Task 1: Check that the resulting ringed space U is actually a scheme (this was claimed very rapidly in lecture).

Task 2: Check that if U is a distinguished open subset of an affine scheme X, then the resulting scheme structure on U makes it isomorphic to an affine scheme.

Task 3: What would happen in the above discussion if U was allowed to be something other than Zariski open? For example, take the inclusion of the x-axis into $\mathbb{A}^2$ and try to describe the inverse image sheaf associated to $\mathcal{O}_X$. Alternatively, take the inclusion of a closed point.