There and back again: a tale of expansions in Gromov–Witten theory

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In this talk, I described a theory of logarithmic Gromov–Witten invariants with expansions, and a proof of the degeneration formula for Gromov–Witten invariants in simple normal crossings (or toroidal) degenerations. The construction is based on "virtual semistable reduction" – for the universal curve, universal map, and (constant) target family – over moduli space of logarithmic stable maps. The technical ingredients that go into this are weak semistable reduction for toroidal morphisms, as developed by Abramovich and Karu in the late 1990's, and the modern perspective on the relationship between logarithmic modifications, subdivisions of tropical moduli spaces, and the effect they have on the virtual fundamental class [3].

0.1. **The main results.** Let X be a smooth projective algebraic variety and let $D \subset X$ be a simple normal crossings divisor with irreducible components $D_1, \ldots, D_k$. Relative Gromov–Witten theory is concerned with maps of pairs

$$(C, p_1, \ldots, p_n) \to (X, D),$$

where C is a smooth genus g curve in a homology class β, meeting the component D_i at p_j with contact order $c_{ij} \in \mathbb{N}$. For each i, the contact order c_{ij} is nonzero for at most one index i. These numerical data are packaged in the symbol Γ and the divisor D is understood as part of the notation X. There is a Deligne–Mumford stack $\mathsf{K}^{\circ}_{\Gamma}(X)$, with quasi-projective coarse moduli space, parameterizing such maps. The main geometric construction is the following.

Theorem A (Logarithmic maps with expansions). *There exists a proper moduli stack $\mathsf{K}_{\Gamma}(X)$ containing $\mathsf{K}^{\circ}_{\Gamma}(X)$, and, for a logarithmic point S, whose S-points are flat families*

$$\mathcal{C} \to \mathcal{X} \to X \times S,$$

of maps from nodal curves to logarithmic modifications $\mathcal{X} \to X \times S$. The map $\mathcal{C} \to \mathcal{X}$ is logarithmically transverse – the curve is disjoint from the codimension 2 strata and the generic points of components of $\mathcal{C}$ map to the generic points of $\mathcal{X}$.

This space is built as a logarithmically étale modification of the space of logarithmic stable maps constructed by Abramovich–Chen–Gross–Siebert. It comes equipped with a virtual fundamental class, and the canonical morphism

$$\mathsf{K}_{\Gamma}(X) \to \mathsf{ACGS}_{\Gamma}(X)$$

identifies virtual fundamental classes, so the invariants of the former determine invariants of the latter.

0.2. **A word about the proof.** The moduli space constructed by Abramovich–Chen and Gross–Siebert encode "transversality" in logarithmic and tropical data, and our goal is simply to make this visible geometrically. Over the moduli space $\mathsf{ACGS}_{\Gamma}(X)$, there exists a universal map

$$\mathcal{C} \to X \times \mathsf{ACGS}_{\Gamma}(X).$$

After passing to the associated family of Artin fans, one can use toroidal geometry to make this map *weakly semistable*. That is, we blowup the target in such a way that the curve is disjoint from the codimension 2 strata. The resulting target family $\mathcal{X}$ will no longer be flat over $\mathsf{ACGS}_\Gamma(X)$, but a modification of the latter produces $\mathsf{K}_\Gamma(X)$. These modifications are all done using tropical geometry, following the path laid out by Abramovich and Karu [2]. A further modification to $\mathcal{C}$ produces the desired moduli space. A key insight required here is that logarithmically, modifications are subfunctors/subcategories, lending them a tautological modular interpretation. The arguments concerning obstruction theories developed by Abramovich and Wise quickly give rise to the space we require [3].

0.3. **Using the spaces.** The main advantage of the new spaces is the logarithmic transversality. This allows one to generalize well-known arguments, used by Jun Li and others, to relate the virtual fundamental class of the space of maps to a variety, with spaces of maps to components of the central fiber of a degeneration.

Theorem B (The degeneration formula). *Let $\mathcal{Y} \to \mathbb{A}^1$ be a toroidal degeneration without self intersections, general fiber Y_η, and special fiber Y_0. There exist moduli spaces $\mathsf{K}_\Gamma(\cdot)$ of maps to expansions with the following properties.*

(1) **Virtual deformation invariance.** *There is an equality of virtual classes*
$$[\mathsf{K}_\Gamma(Y_0)] = [\mathsf{K}_\Gamma(Y_\eta)]$$
in the Chow group of the space of maps $\mathsf{K}_\Gamma(\mathcal{Y})$.

(2) **Decomposition.** *The virtual class of maps to Y_0 decomposes as a sum over combinatorial splittings of the discrete data (i.e. tropical stable maps)*
$$[\mathsf{K}_\Gamma(Y_0)] = \sum_\rho m_\rho [\mathsf{K}_\rho(Y_0)]$$
where $m_\rho \in \mathbb{Q}$ are explicit combinatorial multiplicities depending on the splitting ρ. Each space $\mathsf{K}_\rho(Y_0)$ is a space of maps to expansions, marked by splitting type.

(3) **Gluing.** *For each splitting ρ with graph type G, there are moduli spaces $\mathsf{K}_\rho(X_v)$ of maps to expansions of components X_v of Y_0. There is a virtual birational model of their product*
$$\widetilde{\prod_v} \mathsf{K}_\rho(X_v) \to \prod_v \mathsf{K}_\rho(Y_v),$$
and an explicit formula relating the virtual class $[\mathsf{K}_\rho(Y_0)]$ with the virtual class $[\widetilde{\prod}_v \mathsf{K}_\rho(X_v)]$ and the class of the relative diagonal of the universal divisor expansion.

0.4. **Some things are the same, some things are not.** For those who are fond with Jun Li's theory of stable maps to expansions of smooth pairs, much of this theory will be familiar [6, 7]. In particular, the transversality ensures that logarithmic structures play a minimal role, and when the target is a smooth pair, the expanded theory (essentially) returns Jun Li's theory.

The main change from the old theory comes from the fact that certain statements are only after a virtual birational modification. The precise nature of this modification is fairly explicit, but certainly adds a large degree of combinatorial complexity. The need for such modifications in part (3) above is clear – in the product geometry, the degeneration of one of the targets does not force the degeneration of the targets to which it is glued. As a consequence, there is no well-defined evaluation space for boundary markings to produce a morphism to. In Part (3) above, this can be fixed by "tying the fates" of the different components of a degeneration together. This amounts to a tropical subdivision, which produces a virtual birational model.

The result of the birational modifications however, is that the familiar numerical degeneration formula, appearing as a convolution, is not immediately visible in the logarithmic setting. Indeed it appears too much to expect it. The diagonal classes above are strict transforms of the usual diagonal under birational modifications, and consequently do not satisfy a Künneth splitting. A solution to this issue nevertheless seems possible. While the theory is still calculable in principle, it awaits developments in logarithmic intersection theory.

References

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The Tropological Vertex

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(joint work with Norm Do)

Complex manifolds with normal crossing divisors, and normal crossing degenerations, have a natural large scale with a tropical, or piecewise integral-affine structure. Moreover, relative Gromov-Witten invariants of these spaces correspond to counts of tropical curves in this tropical large scale. I will draw some pictures, and explain some beautiful aspects of this tropical correspondence in the Calabi-Yao 3-fold setting, related to the StromingerYaoZaslow approach to mirror symmetry: In this case, the large scale is a 3-dimensional integral affine manifold with singularities along a 1-dimensional graph. The vertices of this singular graph come in two types: positive and negative, and the relative GromovWitten invariants