Abstracts

Tropical curve counting and double ramification cycles

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In this talk, I described the relationship between tropical and logarithmic curve counting and recent developments concerning the double ramification cycle. The main result, joint with Ajith Urundolil Kumaran, is a complete solution to the logarithmic Gromov–Witten (GW) theory of toric varieties, relative to the full toric boundary. The solution is in terms of the intersection theory of logarithmic tautological classes, and is not practically implementable. However, in recent work of Kennedy-Hunt, Shafi, and Urundolil Kumaran, a simple tropical correspondence theorem is proved in special cases that gives a link to refined tropical curve counting.

0.1. **The problem.** We fix a toric variety X with toric boundary divisor D, itself a union of components $D_1, \dots, D_k$. In logarithmic GW theory, one is interested in curves in X that meet D with fixed tangency data. Precisely, we study maps of pairs:

$$(C, p_1, \dots, p_n) \xrightarrow{f} (X, D),$$

from smooth pointed curves to X, and fix:

- the genus g of C,
- the curve class β for $f_\star[C]$ in $H_2(X)$,
- matrix of tangency orders c_{ij} of p_i along D_j.

When (X, D) is a toric pair, there are a couple of simplifications. First, the curve class β is determined by the matrix of tangency orders; we will still keep the symbol β reserved for this curve class when we need it. Second, if we make an identification of the cocharacter lattice the dense torus with $\mathbb{Z}^r$, then at each marked point, we can record a point in this lattice. Precisely, if the point p_i has positive tangency order with some subset of divisors, it picks out a cone in the fan Σ, dual to the intersection of these. The multiplicities then give a lattice point in this cone.

Putting these vectors together, in this setting, we will denote this $r \times n$ matrix by the symbol Λ. Logarithmic GW theory gives rise to a proper Deligne–Mumford stack $\mathsf{M}_\Lambda(X|D)$ parameterizing such maps, and "logarithmic degenerations". We will not say too much here about the nature of these degenerate objects; the details can be found in the foundational papers of Abramovich–Chen and Gross–Siebert [1, 4, 5]. An important feature of the space is that every point in $\mathsf{M}_\Lambda(X|D)$ determines two things:

- A stable map from an n-pointed nodal curve $C \to X$, of class β and
- A *tropical map* $\Gamma \to \mathbb{R}^n$, i.e. a piecewise linear map from a metric graph to $\mathbb{R}^r$, enhancing the dual graph of C.

The data have to be compatible in various ways, which can be found in the original sources. For now, we encourage the reader to take $\mathsf{M}_\Lambda(X|D)$ to be a "good compactification" of the space of tangent curves described above.

Associated to the moduli space $\mathsf{M}_\Lambda(X|D)$ are certain tautological structures. First, at every marked point p_i, we can evaluate the stable map to obtain:

$$\mathsf{ev}_i : \mathsf{M}_\Lambda(X|D) \to X.$$

We can put these together to form

$$\mathsf{ev} : \mathsf{M}_\Lambda(X|D) \to \mathsf{Ev}_\Lambda(X).$$

The target space is, to first approximation, the product of n copies of X, though it is often useful to refine this. There is also a tautological map to the moduli space of curves:

$$\pi : \mathsf{M}_\Lambda(X|D) \to \overline{\mathsf{M}}_{g,n}.$$

Finally, the space $\mathsf{M}_\Lambda(X|D)$ has a "virtual class". It usually takes some technical machinery to say what this means, however this particular case we are lucky. For $(X|D)$ toric, it turns out that there is a canonical expression of $\mathsf{M}_\Lambda(X|D)$ as an intersection of two smooth schemes inside of a third scheme, each of predictable dimension see [11]. As a consequence, the space $\mathsf{M}_\Lambda(X|D)$ has a distinguished class in Chow homology. That is, if the expression is

$$\mathsf{M}_\Lambda(X|D) = \mathsf{M}_1 \cap \mathsf{M}_2 \text{ inside } \mathsf{B},$$

we can define $[\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}$ to be the refined intersection class, in the sense of Fulton–Macpherson. The homology class lives in the "expected" or "virtual dimension":

$$\mathsf{vdim}\ \mathsf{M}_\Lambda(X|D) = (r-3)(1-g) + n.$$

The goal of logarithmic GW theory is, in some sense, to calculate the classes

$$\pi_\star \left(\mathsf{ev}^\star \gamma \cap [\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}\right).$$

Special interest is paid to the intersection numbers of these classes with the ψ-classes of the moduli space $\overline{\mathsf{M}}_{g,n}$. The pushforwards are called *logarithmic Gromov–Witten classes*, while the numbers are called *logarithmic Gromov–Witten invariants.*

0.2. **The main result and some specializations.** The Chow ring of $\overline{\mathsf{M}}_{g,n}$ contains a subring known as the *tautological ring.* It includes two sets of classes in particular: substacks paramaterizing curves of fixed topological type (e.g. the singular curves) and Chern classes ψ_i of the cotangent line bundles.

The main theorem proved with Urundolil Kumaran is the following:

Main Theorem. *All logarithmic GW classes of (X, D) lie in the tautological ring of the moduli space of curves $\overline{\mathsf{M}}_{g,n}$.*

The proof of the theorem is effective: it actually produces an expression that calculates any such class in terms of the standard generators of the tautological ring. It also therefore gives the first complete method for calculating all logarithmic GW invariants of (X, D).

The main new input in the theorem is a method to reduce such calculations to a variant of the double ramification cycle; see [6] for an introduction. New methods in logarithmic intersection theory, developed with Molcho, play the key role [10].

0.3. **Specializations, and refined curve counting.** In the stated generality, it is not really practical to "do calculations" in this way. But let us conclude by explaining how the result comes alive when specializing to special geometries and special sectors.

Before doing this, we point out the terminology for two types of GW invariants. The *primary* invariants are those obtained by taking degrees of classes of the form $\pi_\star\left(\mathsf{ev}^\star\gamma \cap [\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}\right)$.The *descendant GW invariants* are obtained by first capping $\pi_\star\left(\mathsf{ev}^\star\gamma \cap [\mathsf{M}_\Lambda(X|D)]^{\mathsf{vir}}\right)$ with a polynomial in the classes ψ_i on $\overline{\mathsf{M}}_{g,n}$, and then taking degree.

(1) When $r = 2$, the primary log GW invariants are computed by Mikhalkin's tropical correspondence theorem [9].
(2) When $g = 0$, the primary invariants are calculated by tropical correspondence theorems, by Nishinou–Siebert, and the descendants were treated by Mandel–Ruddat [8, 7].
(3) When $r = 3$, but the matrix Λ has rank 2, without descendants, Bousseau has shown that these invariants can be again computed by tropical correspondence using Block–Göttsche's refined multiplicities [3].

In recent work, Kennedy-Hunt, Shafi, and Urundolil Kumaran show that the main theorem rapidly specializes to all these results using intersection theory on $\overline{\mathsf{M}}_{g,n}$, quite different from the original proofs.

They also use the result to give a geometric interpretation of Blechman–Shustin's refined descendant tropical curve counting, which had earlier been defined purely combinatorially [2].

References

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