Abstracts

The Hilbert scheme in logarithmic geometry, after Kennedy-Hunt

Dhruv Ranganathan

In this talk, I discussed shortly forthcoming work of Patrick Kennedy-Hunt (Cambridge) concerning the Hilbert scheme in logarithmic geometry [1].

0.1. **Degenerations.** The motivation for studying such an object are as follows. Suppose

$$\pi : \mathcal{X} \to B,$$

is a simple normal crossings degeneration over a smooth curve B. That is, a flat and proper morphism with smooth domain, with π a smooth fibration away from a single point $0 \in B$, and such that $X_0 = \pi^{-1}(0)$ is a reduced snc variety. Let $\mathcal{X}^\circ$ denote the complement of the singular fiber. We might then ask the following slightly vague question:

How does the relative Hilbert scheme $\mathsf{Hilb}(\mathcal{X}^\circ/B^\circ)$ *degenerate?*

The question being asked is really "what should we put in the special fiber of such family?" There is an obvious candidate, which is the relative Hilbert scheme $\mathsf{Hilb}(\mathcal{X}/B)$. However, this is rather poorly behaved. For example, if π has relative dimension 2, and if we consider Hilbert scheme of points, the family $\mathsf{Hilb}(\mathcal{X}^\circ/B^\circ)$ is smooth over B° – so in particular, very nice. One might expect that the proposed mystery family $\mathsf{L}(X/B) \to B$ that completes this should be correspondingly very nice. For example, one might ask that this family is again simple normal crossings. At the very least, one could ask for $\mathsf{L}(X/B) \to B$ to be flat over B. The relative Hilbert scheme achieves neither.

0.2. **Pairs.** An intimately tied mystery to the one above is encapsulated by the following, equally vague question. Let Y be a smooth projective variety and let $D \subset Y$ be a simple normal crossings divisor.

How does the presence of D affect the Hilbert scheme $\mathsf{Hilb}(Y)$*?*

Again, the simplest answer is that "it doesn't". But also, it clearly does e.g. we specify that subschemes must intersect D or its strata in particular dimensions, one ends up stratifying the Hilbert scheme of Y. This is not dissimilar to the construction of Schubert cells, or the matroid stratification of the Grassmannian. Our version of this is as follows. Define

$$\mathsf{Hilb}^\circ(Y|D) \subset \mathsf{Hilb}(Y)$$

to be the subfunctor parameterizing points $[Z \subset Y]$ of $\mathsf{Hilb}(Y)$ such that the pullback of D to Y is *regular crossings* – the pullbacks of equations for the different irreducible components of D form regular sequences on Z. In other words, from the point of view of algebra, the subschemes $Z \subset Y$ are transverse. We call such subschemes *algebraically transverse*. Since the transversality condition can be phrased in terms of the vanishing of higher Tor's, the subfunctor above is *open*.

This transversality condition is natural from the point of view of logarithmic geometry. It is precisely the condition that Z is *log flat over a a point* when equipped with the pullback log structure from Y to Z. This is the first hint that the questions above should really be asked, and answered, in logarithmic geometry.

0.3. **A pair of paragraphs on logarithmic geometry.** Logarithmic schemes are enhancements of schemes by combinatorial information. While a scheme comes with a notion of a *polynomial* function, logarithmic schemes come also with the notion of a *monomial* function. Precisely, it is a scheme Y equipped with a sheaf of monoids $\mathcal{M}_Y$ that record the "monomials". Part of the data is a map

$$\mathcal{M}_Y \to \mathcal{O}_Y$$

of sheaves of monoids, that tells us how to take an element of the monomial sheaf and think about it as a polynomial. An artefect of the theory is that this map is merely a morphism of sheaves of monoids – it does not have to be injective. In particular, *there can be more monomials than polynomials.* Logarithmic schemes can be assembled into a category with good geometric properties.

Numerous schemes come with natural notions of monomial, and give examples of logarithmic schemes. If Y is a toric variety, the monomials on Y for a sheaf of monoids. A pair (Y, D) as above of a variety and an snc divisor also has a sheaf of monoids – the functions locally defined by a monomial in defining equations for the components of D. The "more monomials than polynomials" phenomena come via taking pullback in this category.

0.4. **The solution of Kennedy-Hunt.** In his thesis, Kennedy-Hunt proposes a moduli functor on the category of logarithmic schemes $\mathsf{Hilb}^{\log}(Y|D)$ that contains $\mathsf{Hilb}^{\circ}(Y|D)$ as a subfunctor. He establishes several of its basic properties. The moduli space is reverse engineered from an "imagined proof" of properness via the valuative criterion, which comes from a 2007 theorem of Tevelev [4].

Tevelev's Theorem. *Let $Z \hookrightarrow Y$ be a subvariety of a smooth toric variety. There exists a toric blowup $Y' \to Y$ such that the strict transform $Z' \hookrightarrow Y'$ is algebraically transverse.*

A particular example of the theorem applies in the following context. Consider the toric variety $Y \times \mathbb{A}^1$, and an algebraically transverse subscheme

$$Z \hookrightarrow Y \times \mathbb{G}_m,$$

flat over $\mathbb{G}_m$. The theorem explains how to complete such families to ones that remain algebraically transverse. The special fibers in these families are not simply blowups of Y, but reducible varieties obtained by gluing together torus bundles over strata in Y together. We call such an an object an *expansion* of Y along D.

As a simple example, the deformation to the normal cone of a union of strata of D, inside Y, is an example of such a special fiber. There is some ambiguity in the limit, because there is not always a *most efficient blowup.*

A *logarithmic subscheme of* (Y, D) is a subscheme of an expansion of Y along D that is algebraically transverse.

By using this definition in families, and imposing an appropriate equivalence relation among expansions, Kennedy-Hunt arrives at a moduli functor:

$$\mathsf{Hilb}^{\mathsf{log}}(Y|D) : \mathsf{LogSch} \to \mathsf{Sets}.$$

He proves that it satisfies the existence and uniqueness parts of the valuative criterion. He also proves a certain representability result. Precisely, he defines another functor

$$\mathsf{Supp}(X|D) : \mathsf{ConeComplexes} \to \mathsf{Sets},$$

a polyhedral geometry construction. Some readers may want to think of this as a *tropicalization* for the functor.

The fans of toric geometry are examples of functors on cone complexes, as are conical dual complexes of boundary divisors in snc compactifications. What Kennedy-Hunt constructs is slightly more general – what he calls a *piecewise linear space*. In any event, the functor $\mathsf{Supp}(X|D)$ determines a topological space just like a fan does. He then proves that there is a natural bijection:

$$\{\text{Polyhedral decompositions of } \mathsf{Supp}(X|D)\}$$
$$\updownarrow$$
$$\{\text{Subfunctors of } \mathsf{Hilb}^{\mathsf{log}}(Y|D) \text{ representable by stacks with logarithmic structures}\}.$$

The representable subfunctors still satisfy the valuative criterion. The situation is analogous to toric geometry: different fan structures give rise to a different, equivariant birational toric compactifications of a torus.

In the lecture, I described examples that demonstrate that the spaces, with some combinatorial burden, are essentially as well-behaved as a one could hope and lead to a good theory of degenerations for the Hilbert scheme:

(1) The logarithmic Hilbert scheme of hypersurfaces in a toric variety is itself a toric variety, via the “secondary fan” of Gelfand–Kapranov–Zelevinsky.
(2) The logarithmic Hilbert scheme of a surface pair (Y, D) is itself an snc pair. As a consequence, the relative logarithmic Hilbert scheme of an snc surface degeneration is itself an snc degeneration.
(3) The theory coincides with work of Maulik–R for curves in a threefold. This space had been constructed using different methods earlier; the space has a virtual fundamental class and leads to *logarithmic DT invariants*.

Finally, I note that in the special case when D is smooth, the construction above reduces to a theory developed by Li–Wu [2]

References

[1] P. Kennedy-Hunt, *The logarithmic Quot scheme and its tropicalisation*, available online.
[2] J. Li and B. Wu, *Good degeneration of Quot-schemes and coherent systems*, Communications in Analysis and Geometry, 23(4), 841-921.
[3] D. Maulik and D. Ranganathan, *Logarithmic Donaldson–Thomas theory*, arXiv preprint.
[4] J. Tevelev, *Compactifications of subvarieties of tori.*, American Journal of Mathematics 129.4 (2007): 1087-1104.

Reporter: