Realizability of Tropical Curves via Hurwitz Theory

Mia Lam

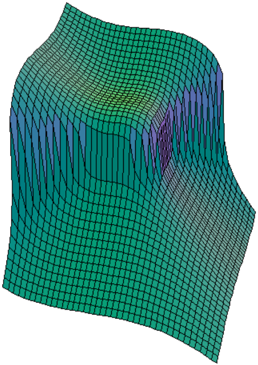
joint work with Chi Kin Ng
supervised by Dhruv Ranganathan

CMP Presentation Day, 23 Aug 2023

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION

Algebraic geometry

Algebraic geometry studies the geometry of solution sets of multivariate polynomial systems.



The above is the Fermat cubic, defined by $x^3 + y^3 + z^3 = 1$.

Motivation and general principle

But algebraic geometry is difficult, so:

Tropical strategy

Transform geometric objects into combinatorial ones (such as graphs or polyhedral complexes).

We can then use discrete methods on these objects.

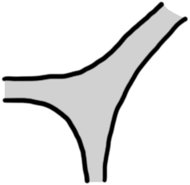
Prototypical tropicalization

Suppose we have a curve $C = \{f(x, y) = 0\} \subset \mathbb{C}^2$.

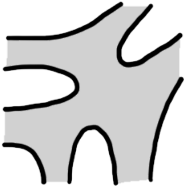
1. Apply the map

$$- \operatorname{Log}_t : (\mathbb{C}^*)^2 \to \mathbb{R}^2$$
$$(z_1, z_2) \mapsto (-\log_t |z_1|, -\log_t |z_2|)$$

for small $t \in \mathbb{R}$.



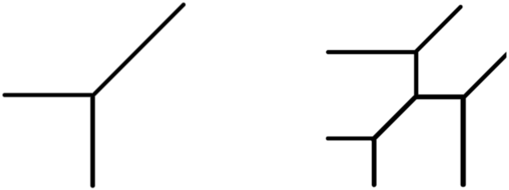
(a) $C = \{z : z_1 + z_2 = 1\}$



(b) $C =$ a generic conic

Prototypical tropicalization

2. Study the limit as $t \to 0$.



(a) $C = \{z : z_1 + z_2 = 1\}$ (b) $C =$ a generic conic

We call this the tropical curve determined by C.

Our setup

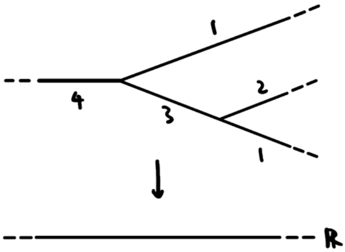
Input:

Compact Riemann surface C and meromorphic function $\varphi : C \to \mathbb{C}$

Output:

Enhanced metric graph Γ and PL function $f : \Gamma \to \mathbb{R}$

E.g.,



Numbers decorating edges and legs indicate the slopes along them.

Realizability

When does (Γ, f) *arise as a result of this process?*

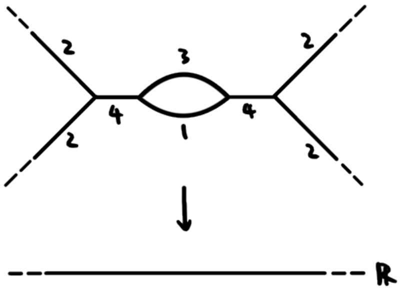
Some necessary conditions:

- The slopes are integers.
- f is balanced.

In fact, these are the **only** conditions when the genus of Γ is zero.
(Cheung, Fantini, Park and Ulirsch, 2016)

Results: Genus 1

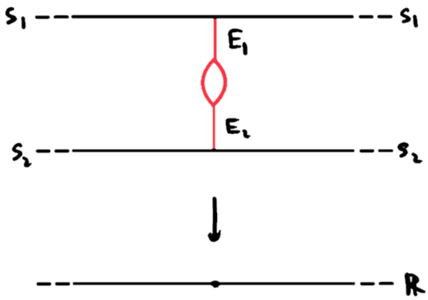
Here is an example of a genus 1 tropical curve which is realizable.



Note that f is non-constant on every edge.

Results: Genus 1

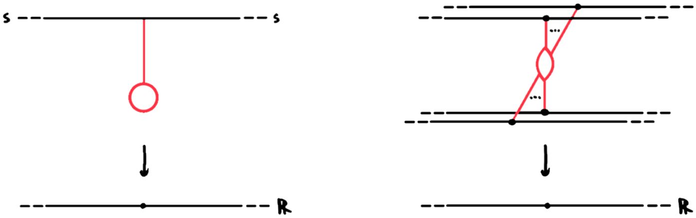
However, if Γ has genus 1 and f is constant on the cycle, then (Γ, f) might not be realizable. (Speyer, 2005)



The pair (Γ, f) above is realizable if and only if E_1 and E_2 have the same length.

Results: Genus 1

Moreover, the case here on the left is not realizable regardless of the length of the edge connected to the cycle.



The case on the right is realizable if and only if the minumum of the lengths of the edges occurs at least twice.

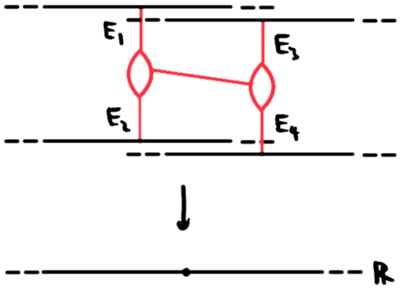
Why is this important/interesting?

Just this genus 1 result of Speyer's has produced lots of applications in classical algebraic geometry (Brill-Noether theory).

But there has no progress in genus 2 until now.

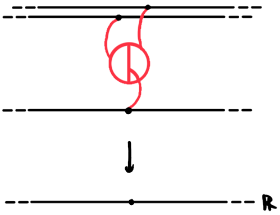
Results: Genus 2 (Dumbbell)

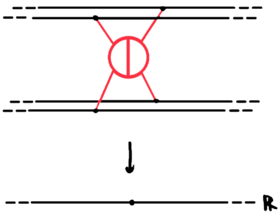
The pair (Γ, f) below is realizable if E_1 and E_2 have the same length, and E_3 and E_4 have the same length.



Results: Genus 2 (Theta)

The tropical curve depicted on the left is realizable if the edges connected to the Theta all have the same length.





The tropical curve on the right is realizable under some conditions.

Acknowledgements

Thank you to my supervisor for giving me the opportunity to work on this project and for all the support throughout the project.

And thank you to the Cantab Capital Institute for the Mathematics of Information (CCIMI) for funding this project under the Philippa Fawcett Internship Programme.

CCIMI
CANTAB CAPITAL INSTITUTE FOR THE MATHEMATICS OF INFORMATION