PIECEWISE POLYNOMIALS
&
the MODULI SPACE OF CURVES

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LAGARTOS

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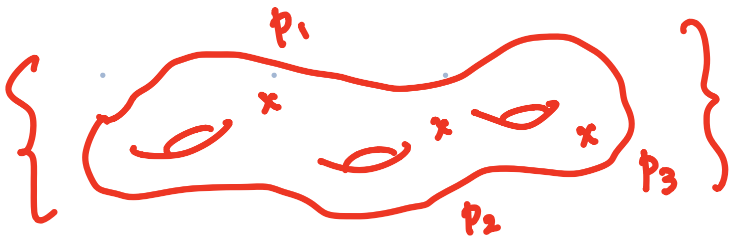
LINE BUNDLES ON CURVES & $\mathcal{M}_{g,n}$

$\mathrm{Pic}_{g,n} \leftarrow \mathcal{M}_{g,n}$



$(C, p_1, \ldots, p_n)$

On each curve <u>two</u> types of line bundles

$C \longmapsto \omega_C^{\otimes k}$ $\quad k \in \mathbb{Z}$

$C \longmapsto \mathcal{O}_C(\Sigma a_i p_i)$ $\quad a_i \in \mathbb{Z}$

DOUBLE RAMIFICATIONS

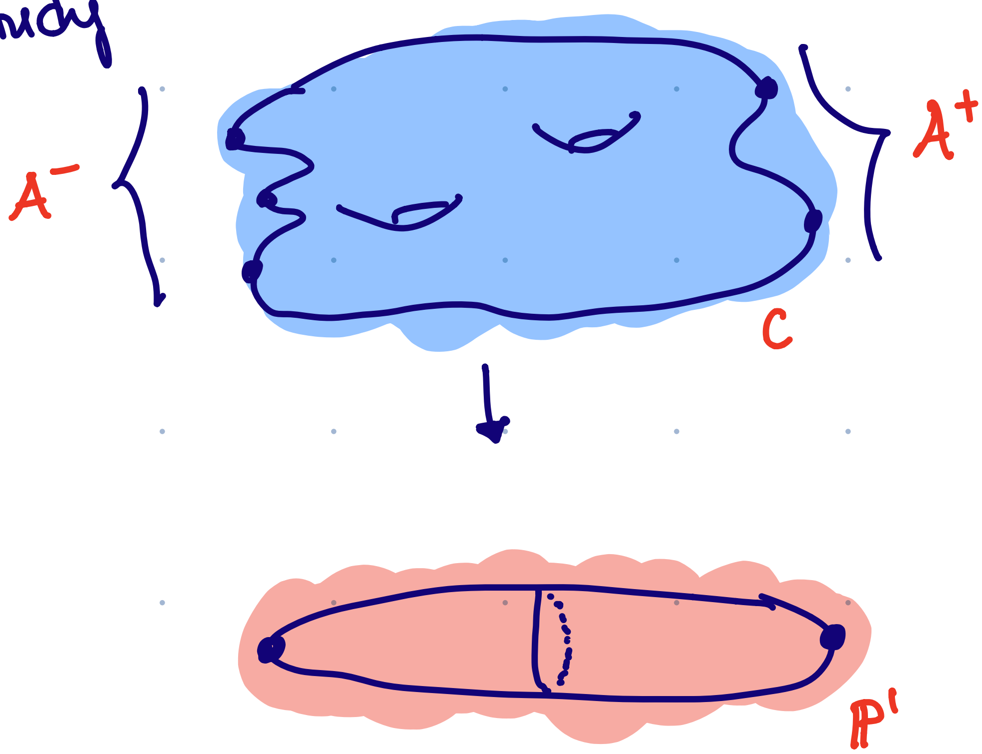
Fix $A = (a_1, \dots, a_n)$ with $\sum a_i = 0$

$g=1$; $A = (1,1,-1,-1)$

$$\boxed{DR_g^\circ(A) \hookrightarrow \mathcal{M}_{g,n}}$$

Curves C with $\mathcal{O}_C(\sum a_i p_i) \simeq \mathcal{O}_C$

GW: study



There are "twisted" versions.

Compactifications:

We have

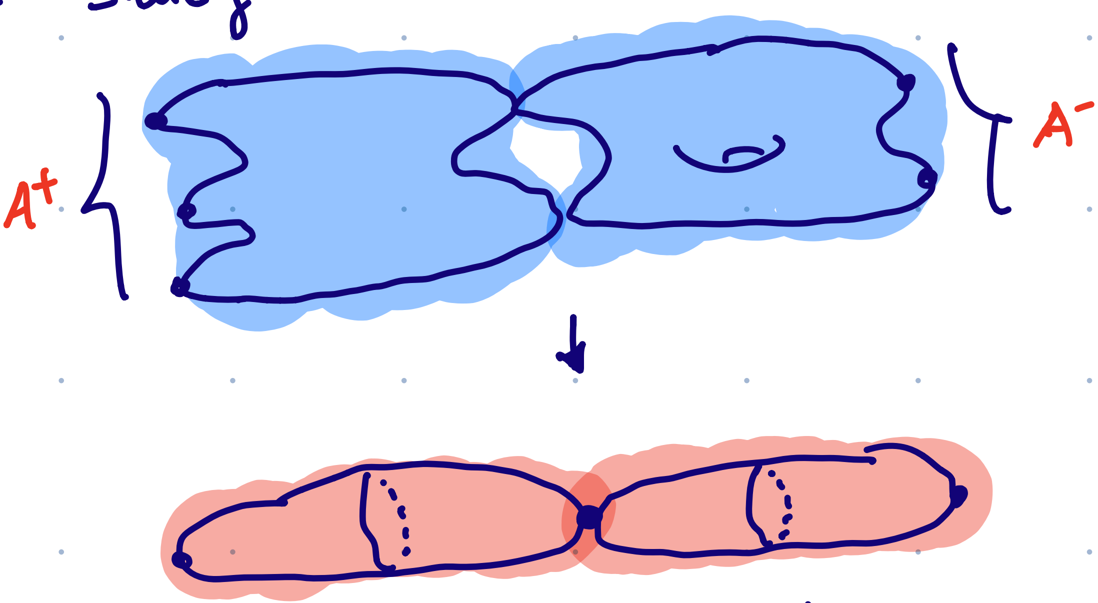
$$\begin{tikzcd}
 & DR_g^{\circ}(A) \arrow[r, hook] \arrow[d] & \mathcal{M}_{g,n} \arrow[d] \\
\text{a codim } g \text{ cycle class in } CH^*(\overline{\mathcal{M}}_{g,n}). & DR_g(A) \arrow[l] \arrow[r, hook] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

via Gromov-Witten theory

or

Abel-Jacobi resolutions

GW: Study



$\mathbb{P}^1 \cup \mathbb{P}^1$

A TIMELINE: $[DR_g(A)] \in CH^g(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$.

2000 - 2010

- DEF'n OF CLASS (J. Li)
- $[DR_g(A)]$ is TAUTOLOGICAL in Chow (Faber - Pandharipande)
- Applications to $CH^*(\overline{\mathcal{M}}_{g,n})$; THEOREM ★ (Graber - Vakil)
- Partial formulas (Hain, Grushevsky Zakharov).

2010 - Now

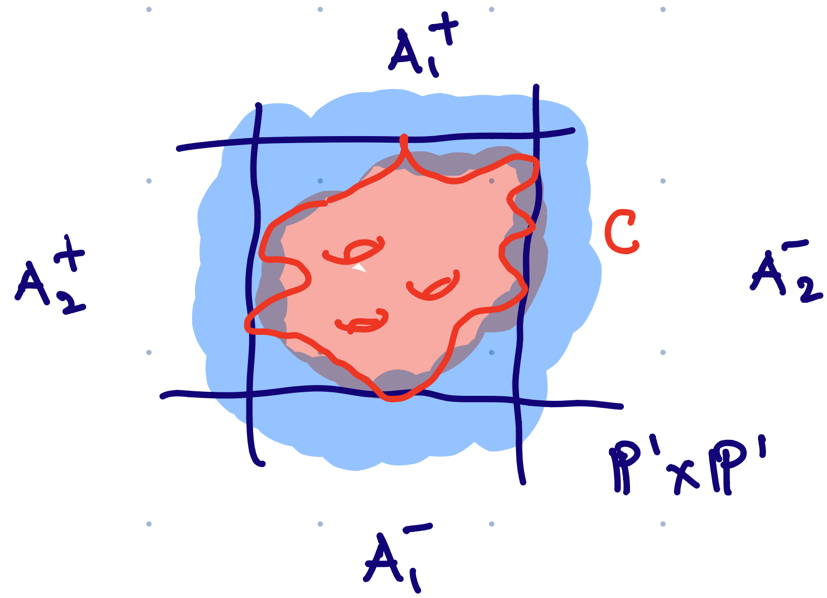
- Full formula for [DR(A)] (Janda - Pandharipande - Pixton - Zvonkine)
- Logarithmic Abel-Jacobi theory [Marcus, Molcho - Wise, Holmes, Guéré Abreu - Pacini, ...]

SIGNS OF AN INCOMPLETE STORY

- How about maps to $\mathbb{P}^1 \times \mathbb{P}^1$? or a TORIC VARIETY?

Fix A_1 & A_2

length n:



TORIC CONTACT CYCLE

$$TC_g^\circ(A_1 | A_2) \subseteq \mathcal{M}_{g,n}$$

CODIM $2g$

$$\| $$

$\{(C, p_1, \ldots, p_n) \mid C \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with specified contact orders$\}$

PRODUCT FORMULAS

INTERIOR: $$TC^{\circ}_g(A_1|A_2) = DR^{\circ}_g(A_1) \cap DR^{\circ}_g(A_2)$$

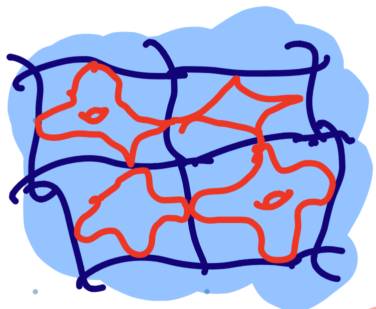
in $CH^*(\mathcal{M}_{g,n})$.

FAILS ON $\overline{\mathcal{M}}_{g,n}$!

$$TC_g(A_1|A_2) \neq DR_g(A_1) \cap DR_g(A_2).$$

in $CH^*(\overline{\mathcal{M}}_{g,n})$.

DEFINED via log GW theory: $\longrightarrow$



maps to broken toric varieties.

Abel–Jacobi: Holmes–Pixton–Schmitt

Gromov-Witten: R

BASIC QUESTIONS about $[TC_g(A_1|A_2)]$ in $CH^*(\overline{\mathcal{M}}_{g,n})$

- Does it lie in the TAUTOLOGICAL ring in Chow?
- Why & how does the product rule fail?
- Can we compute top intersections against it?
- Can we find a formula for $TC_g(A_1|A_2)$?

Note: I didn't define $TC_g(A_1|A_2)$: if this bothers you ask now!

A FEW TAUTOLOGIES:

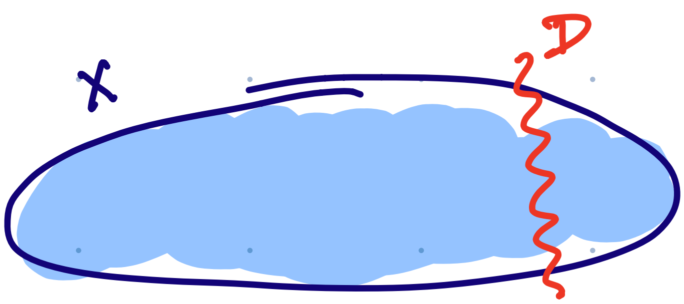
- $D \subseteq X$ a Cartier divisor

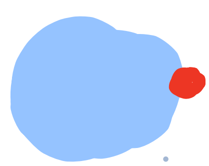
$$\rightsquigarrow \quad X \longrightarrow [\mathbb{A}^1/\mathbb{G}_m]$$

"Artin fan"

- $D = D_1 + \cdots + D_k \subseteq X$

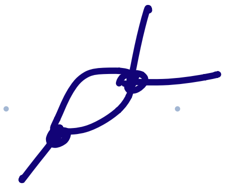
$$\rightsquigarrow \quad X \longrightarrow [\mathbb{A}^k/\mathbb{G}_m^k]$$



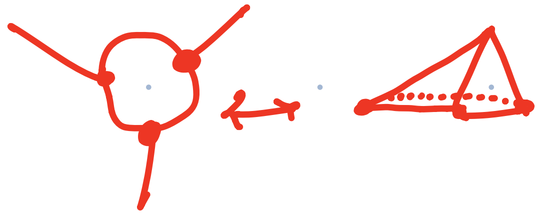


$[\mathbb{A}^1/\mathbb{G}_m]$

Moduli of Tropical Curves



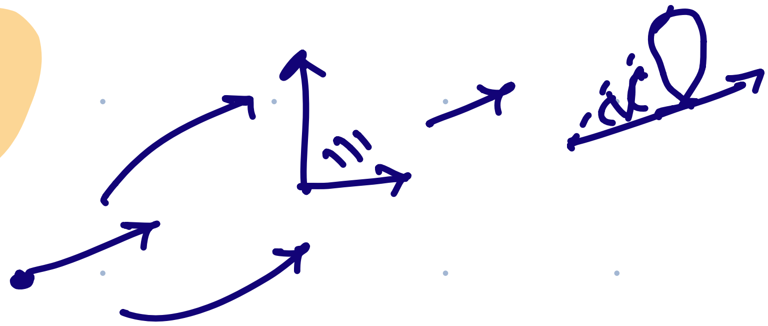
$M^{trop}_{g,n} \approx$ Glued from cones σ_G

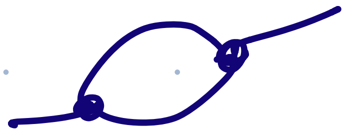


"Fan" of $\overline{M}_{g,n}$

$G \longleftrightarrow \sigma_G = \mathbb{R}^n_{\geq 0}$

$\equiv \varinjlim_{G} \sigma_G$





Algebraic stacks

$\varinjlim \mathcal{A}_G$

If $\sigma_G = \mathbb{R}^n_{\geq 0}$

$\mathcal{A}_G = [\mathbb{A}^n / \mathbb{G}_m^n]$

Abramovich-Wise; Caporaso-Payne; Ulirsch; Olsson; Chan-Cavalieri

$\overline{M}_{g,n} \longrightarrow \mathcal{A}_{g,n}$

The Artin fan.

THE TAUTOLOGICAL RING (Mumford, ...)

$$R^*(\overline{\mathcal{M}}_{g,n}) \subseteq CH^*(\overline{\mathcal{M}}_{g,n}) \text{ or } H^*(-)$$

SUBRING

$$\overline{\mathcal{M}}_{g,n} \xrightarrow{\pi} \mathcal{A}_{g,n} \simeq M^{trop}_{g,n}$$

$$\pi^*: CH^*(\mathcal{A}_{g,n}) \longrightarrow CH^*(\overline{\mathcal{M}}_{g,n}).$$

and

cotangent line classes $\psi_1, \ldots, \psi_n$

$$\overline{\mathcal{M}}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g,n+1}.$$

FACT: (Molcho-Pandharipande-Schmitt)

$\mathrm{im}(\pi^*)$ & $\psi_1, \ldots, \psi_n$ define the tautological ring.

What is $CH^*(\mathcal{A}_{g,n})$?

THEOREM (Molcho–R; Molcho-Pandharipande-Schmitt)

$$CH^*(\mathcal{A}_{g,n}) = PP^*(M^{trop}_{g,n})$$

RING OF PIECEWISE POLYNOMIALS

More generally, if $\mathcal{A}$ is an ARTIN FAN

$$CH^*(\mathcal{A}) = PP^*(trop(\mathcal{A}))$$

[After Payne, Brion, ...]

[Uses Kresch, Kimura, Bae-Park]

TAUTOLOGICAL CLASSES ARE COMBINATORIAL NOW!

THEOREM (Molcho – R '21 ; Holmes–Schwarz '21)

- The classes $TC_g(A_1|A_2)$ lie in $R^\star(\overline{M}_{g,n})$

TAUTOLOGICAL

what is the geometry ?

why does product rule fail ?

If $\tilde{X} \xrightarrow{\pi} X$ is a blowup

then $\pi_\star(\alpha \cdot \beta) \neq \pi_\star(\alpha) \cdot \pi_\star(\beta)$

But if $\tilde{X} \longrightarrow X$ is a "tropical blowup"

the failure is controlled via $PP^\star(X^{trop})$.

TROPICAL DOUBLE RAMIFICATION :

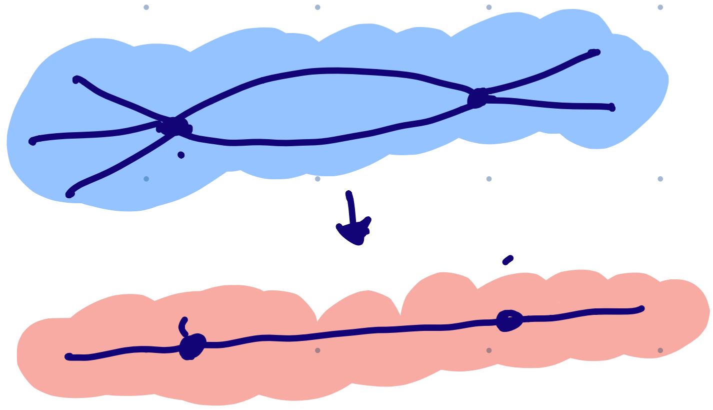
$$DR_g^{trop}(A) \subseteq M_{g,n}^{trop}$$

$$\{ \Gamma \mid \Sigma a_i p_i \sim 0 \}$$

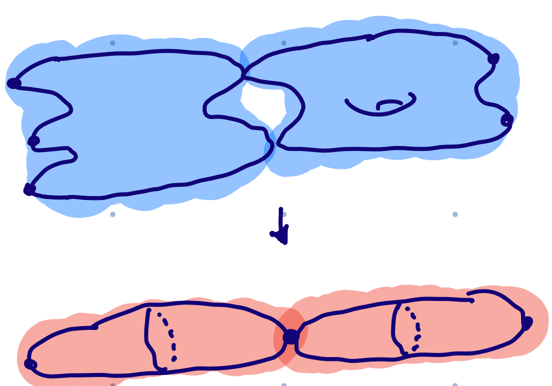
Ulirsch - Zakharov

Cavalieri - Markwig - R

BALANCED & CTS.

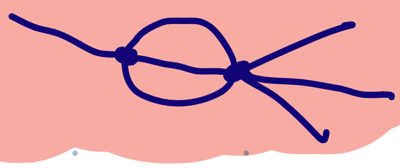


$A = (2,2,-4)$

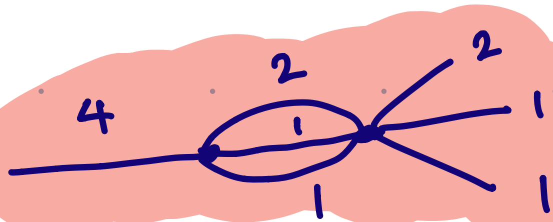


$\mathbb{P}^1 \cup \mathbb{P}^1$

WHAT DOES $DR^{trop}(A)$ look like?

I. $T =$  $A = (4,-2,-1,-1)$

II. Assign slopes by balancing



(not necessarily unique)

III. Solve for edge lengths that allow continuity

$$2\ell_1 = \ell_2 = \ell_3 \subseteq M^{trop}_{2,4}$$

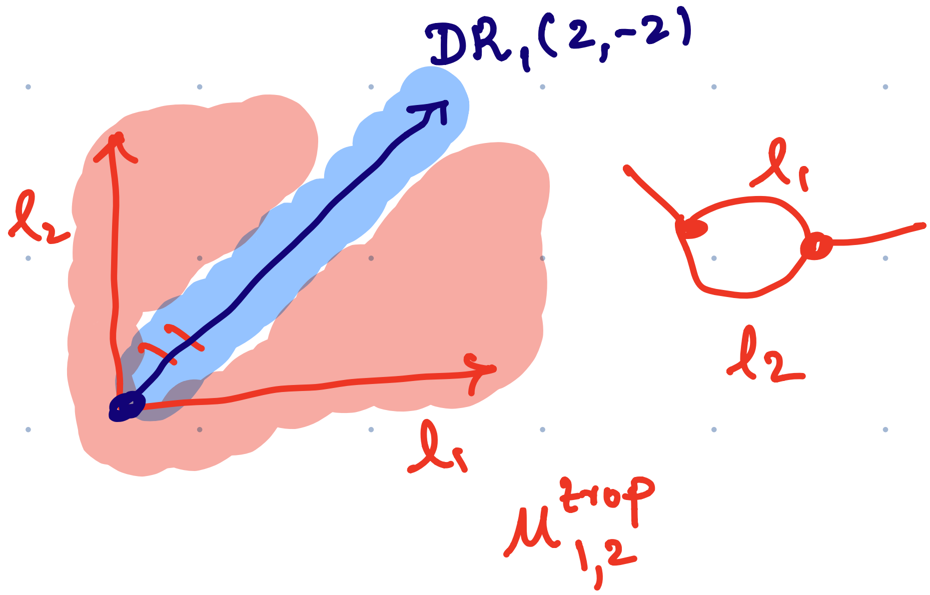
Just like toric geometry:

$$DR_g^{trop}(A) \subseteq M_{g,n}^{trop}$$

subfan

$\updownarrow$

(open subsets in) blowups of $\overline{M}_{g,n}$



RECIPE

~~THEOREM~~ (Molcho – R '21): To calculate

$TC_g(A_1|A_2)$:

1. Intersect $DR_g^{trop}(A_1) \cap DR_g^{trop}(A_2) \subseteq M_{g,n}^{trop}$

2. Blowup accordingly

$$\widetilde{M}_{g,n} \longrightarrow \overline{M}_{g,n}$$

3. Calculate the (VIRTUAL) strict transform

$\widetilde{DR}_g(A_1)$ & $\widetilde{DR}_g(A_2)$.

4. Intersect there, pushforward.

THE VIRTUAL STRICT TRANSFORM:

Molcho – R '21:

We explain what this is using:

- Fulton's blowup formula.
- Aluffi's formulas for Segre classes.
- Picewise polynomials.

No formula yet...

w/ Cavalieri – Markwig: Explain how to compute (combinatorially) calculate integrals.

$$\int_{\widetilde{\mathcal{M}}_{g,n}} \widetilde{DR}_g(A) \cap [p] \quad \text{for } p \in PP^*(\widetilde{\mathcal{M}}^{trop}_{g,n}).$$

Includes tropical correspondence theorems & generalizations.

PLEASE SEE WORK OF

Molcho - Pandharipande - Schmitt

Holmes - Pixton - Schmitt

Holmes - Schwarz

Nabijou - R

& hopefully in a few weeks Molcho - R

+ Cavalieri - Gross - Markwig

THANKS !