GROMOV-WITTEN THEORY & INVARIANTS OF MATROIDS

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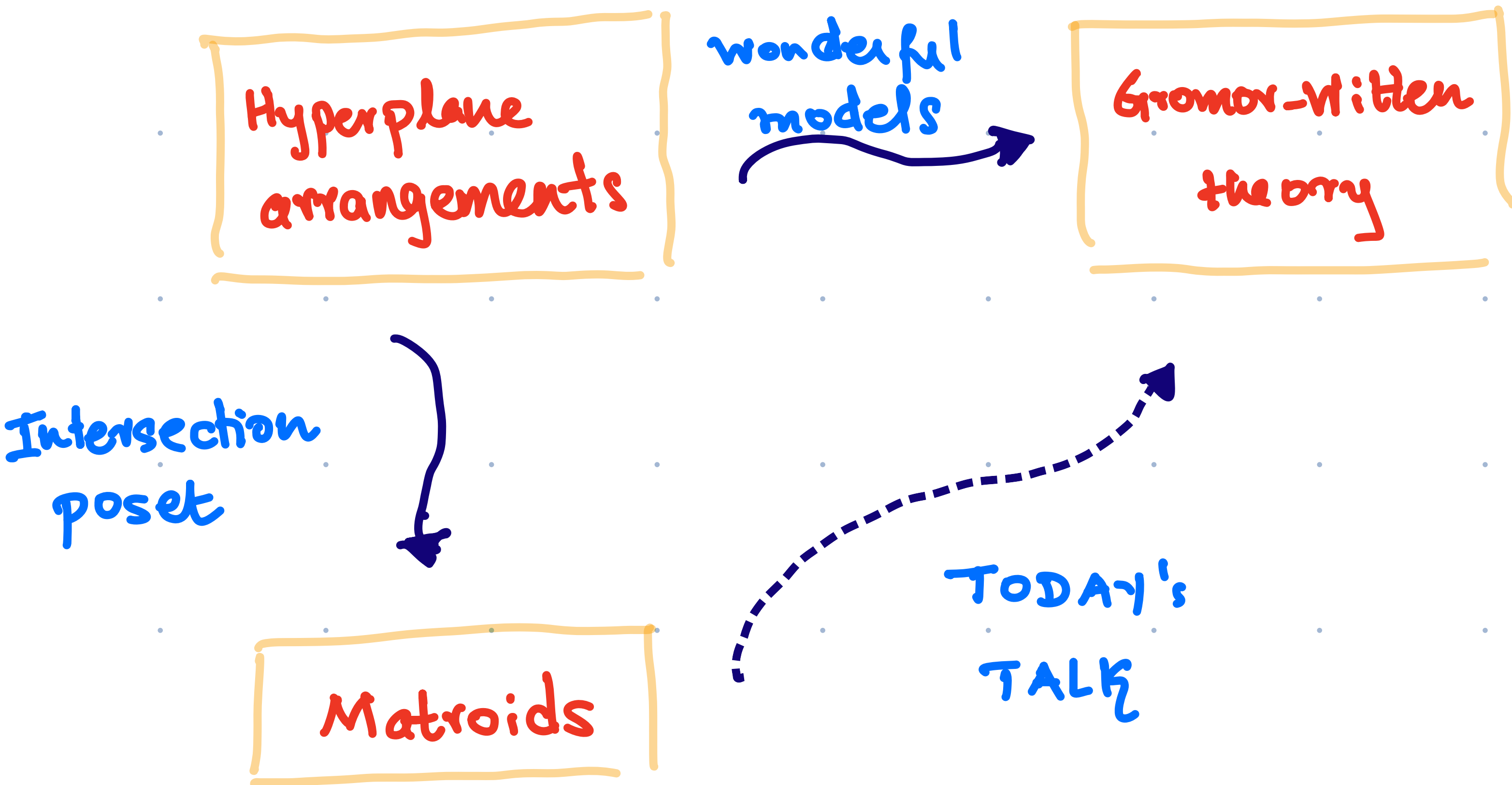
24th September '21

@ KIAS

The Plan:



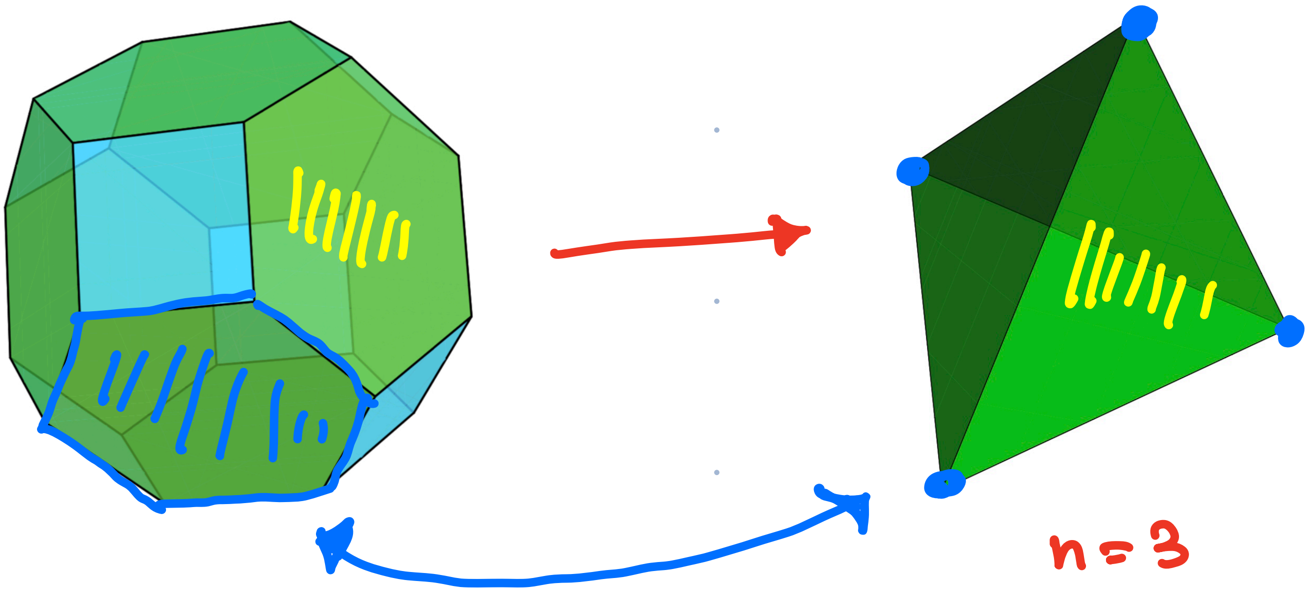
Long story short: We will $\boxed{\text{conjecturally}}$ define the genus 0 GW theory of any matroid.

THE PERMUTOHEDRON:

$$X(\Pi_n) \longrightarrow \mathbb{P}^n$$

BLOWUP ALL
COORDINATE STRATA

THE TORIC POLYTOPES:



$n=3$

A remarkable variety!

$\chi_{top} = (n+1)!$ Betti numbers given in terms of Stirling numbers

see: Losev-Manin; Hurwitz theory; Quasimaps; ...

WHAT IS A MATROID?

A matroid of rank $d+1$ on $n+1$ elements

is a Chow/homology class:

$$\alpha \in A_d(X(\Pi_n))$$

such that

$$\alpha \cdot H^{n-d} = 1$$ "Degree 1".

Note: the classes generate $A^*(X(\Pi_n))$.

Sources of Matroids:

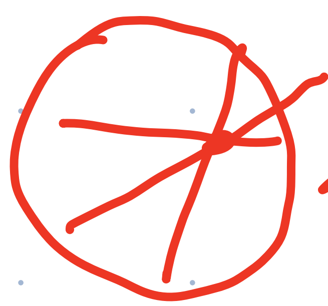
Let

$$\begin{tikzcd} \mathbb{P}^d \arrow[r, hook] & \mathbb{P}^n \\ W \arrow[u] \arrow[r] & X(\Pi_n) \arrow[u] \end{tikzcd}$$

be a linear inclusion NOT CONTAINED IN COORDINATE HYPERPLANE

STRICT TRANSFORM.

Easy: $[W] \in A_d(X(\Pi_n))$ is a matroid.



NOT SNC

A FEW NOTES:

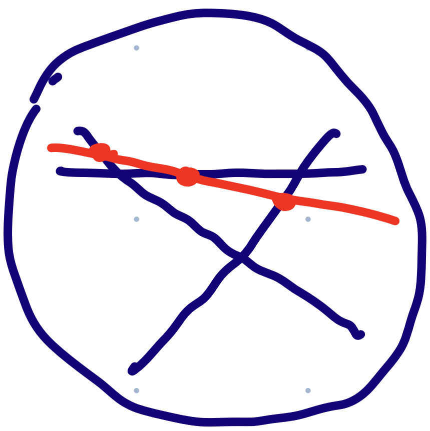
- Most matroids DO NOT arise in this way
- The space W is the WONDERFUL MODEL of $\mathbb{P}^d \cap (\mathbb{C}^*)^n$ [dCP '95]

TOY EXAMPLES

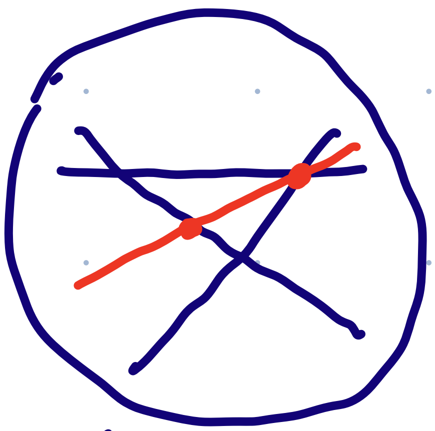
A. $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as $\{X+Y+Z=0\}$

B. $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as $\{X+Y=0\}$

A:



B:



Different classes in

$A^*(X(\pi_2))$!

REALIZATION SPACES [GGMS]

Fix a matroid $\alpha \in A_d(X(\Pi_n))$.

$$Gr_\alpha = \{ [\mathbb{P}^d \hookrightarrow \mathbb{P}^n] \in \mathbb{G}(d,n) : \text{the class of } [\widetilde{\mathbb{P}^d}] \text{ in } A_d(X(\Pi_n)) \text{ is } \alpha \}$$

What is Gr_α?

If Gr_α is nonempty then

$$Gr_\alpha = \mathbb{G}(d,n) \cap L^\circ$$

Plücker coordinate subspace

Gr_α: Typically DISCONNECTED & SINGULAR

Mnev's; Vakil's Murphy

Law

INTERSECTION THEORY: $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$

SMALLEST OPEN TORIC

$$W \xhookrightarrow[i]{} U \xhookrightarrow[j]{} X(\Pi_n)$$

WONDERFUL TORIC

THEOREM (de Concini - Procesi ; Feichtner - Yuzvinsky)

Not true in H^*

$$A^*(U) \xrightarrow[i^*]{\text{ISOMORPHISM}} A^*(W)$$

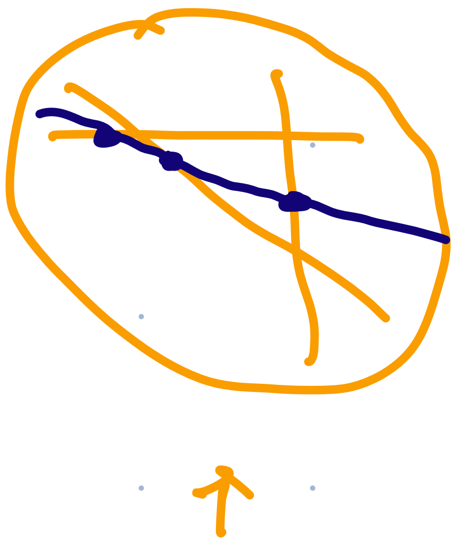
n-dim d-dim'l

COROLLARY: $A^*(W)$ depends only on the matroid $[W] \in A^*(X(\Pi_n))$.

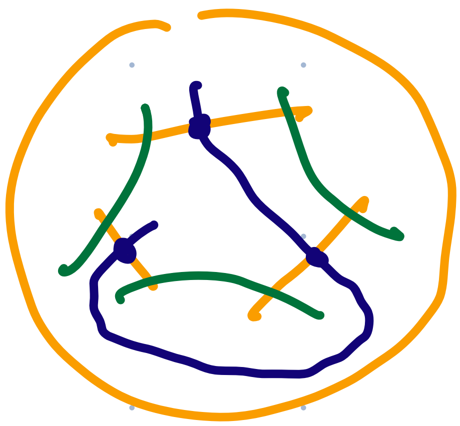
SURPRISING? Given $[\mathbb{P}^d \hookrightarrow \mathbb{P}^n]$ let $V = \mathbb{P}^d \cap (\mathbb{C}^*)^n$. Then $\pi_1(V)$ is not combinatorial.

Example:

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as $\{x+y+z=0\}$



$\mathbb{P}^2$



$\mathbb{P}^2 \setminus \{p_1, p_2, p_3\} =$

$X(\Pi_2)$

MAIN RESULTS:

Quantum cohomology is combinatorial:

THEOREM (RU'21) Let $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$ be linear and $W \hookrightarrow X(\Pi_n)$ be the strict transform.

Then

$QH^*(W)$ depends only on $[W]$ in $A^*(X(\Pi_n))$.

CONSEQUENCE: This defines QH^* for this class of non-compact toric varieties!

LOGARITHMIC VERSIONS:

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & X(\Pi_n) \arrow[d, "\pi"] \\ \mathbb{P}^d \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

de Concini–Procesi: The divisor

$\partial W = W \cap \pi^{-1}(\partial \mathbb{P}^n)$ is

SIMPLE NORMAL CROSSINGS.

THEOREM (RU '21) The logarithmic GW theory in $g=0$ of W with any log structure subordinate to ∂W is combinatorial.

How to access rational curves? $\mathrm{Maps}(W)$

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & X(\Pi_n) \arrow[d] \\ \mathbb{P}^d \arrow[r, hook] & \mathbb{P}^n \end{tikzcd} \rightsquigarrow \quad j : \overline{M}(W) \hookrightarrow \overline{M}(X(\Pi_n))$$

$j_* [\overline{M}(W)]^{vir}$ in $A_*(\overline{M}(X(\Pi_n)))$.

$\to$ Maps from rat'l curves to W

STRATEGY (OLD TRICKS!).

1. Express $W \hookrightarrow X(\Pi_n)$ as $V(s)$ where $s \in H^0(\mathbb{E})$

2. Express $j_* [\overline{M}(W)]^{vir}$ via Chern classes of $\mathbb{E}$

3. Prove $c_i(\mathbb{E})$ are depend only on $[W]$ in $A_*(X(\Pi_n))$.

A VECTOR BUNDLE CONSTRUCTION:

Berget - Eur - Spink - Tseng,
Tevelev, Speyer, ...

We want:

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & X(\Pi_n) \arrow[d] \\ \mathbb{P}^d \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

Now: $[\mathbb{P}^d \hookrightarrow \mathbb{P}^n] = p \in G(d,n)$

Beautiful Geometric Fact:

The $(\mathbb{C}^*)^n$ - orbit of p is a toric

$\overline{W} \hookrightarrow Y \hookrightarrow G(d,n)$

Now $\overline{W}$ is cut out by universal quotient and W is as well.

ROUTE TO GW theory:

Katz-Cox-Lee

Kim-Kresch-Pantev

Manolache.

If $W \hookrightarrow X(\Pi_n)$ with $W = \mathbb{V}(s)$, s in $H^0(\mathbb{E})$

and $\mathbb{E}$ is convex

the moduli spaces reflect this:

$$\begin{tikzcd} & & \mathbb{E} \arrow[d] \\ \mathcal{C} \arrow[rr, bend left, "F"] \arrow[d, "\pi"'] & & X(\Pi_n) \\ \mathrm{Maps}(X(\Pi_n)) & & \end{tikzcd}$$

Now $\mathbb{F} = \pi_* F^* \mathbb{E}$ has a section

cutting out Maps(W)

NON-REALIZABLE CASES:

Fix a matroid $\alpha \in A\delta(X(\Pi_n))$

There is a smallest toric open

$$U \hookrightarrow X(\Pi_n)$$ as before

Now define

$$A^*(\alpha) := A^*(U)$$

Adiprasito–Huh–Katz: The ring $A^*(\alpha)$ behaves like a realization exists from the view of HODGE THEORY $\longrightarrow$ Rota's conjecture.

CONJECTURES: Fix $\alpha \in Ad(X(\Pi_n))$ a matroid.

RU '21: We define a class

$$[M(\alpha)]^{vir} \quad \text{in} \quad A_*(\overline{M}(X(\Pi_n)))$$

the pushforward of a mythical virtual class on a mythical mapping space.

CONJECTURE: These determine GW invariants with insertions in $A^*(\alpha)$

FURTHER RESULTS:

If $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$ is the identity [n=d]

the $GW^{\log}(X(\Pi_n) \mid \partial X(\Pi_n))$ is

TORIC BOUNDARY

TROPICAL [Mikhalkin; Nishinou & Siebert]

we generalize this

Each matroid α determines an open

$$U \hookrightarrow X(\Pi_n).$$

The tropicalization $\mathrm{trop}(\alpha) := \Sigma_U$

Then $GW^{\log}(\alpha)$ [if defined] is determined

by counts of tropical curves on $\mathrm{trop}(\alpha)$.

Further Questions:

Higher Genus; $QH^*(\alpha)$; Deletion – Truncation operations;

degeneration & product formulae.

- Is $QH^*(W)$ semisimple? cf Bayer

Maybe this is easy..?

- Deletion/contraction operations are natural in GW theory

Inversion of degeneration formulae

THANKS !