Quantum geometry for matroids

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@ Brown

CAGTR @ icts
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THE PLAN:

$$
\begin{tikzcd}
\boxed{\text{HYPERPLANE ARRANGEMENTS}} \arrow[r, "\text{wonderful}", "\text{model}"'] \arrow[d, "\text{tropicalization}"'] & \boxed{\text{GROMOV-WITTEN THEORY}} \\
\boxed{\text{MATROIDS}} \arrow[ur, dashed, "\text{TODAY'S TALK}"'] &
\end{tikzcd}
$$

LONG STORY SHORT: . We will conjecturally define the genus 0 GW theory of any matroid.

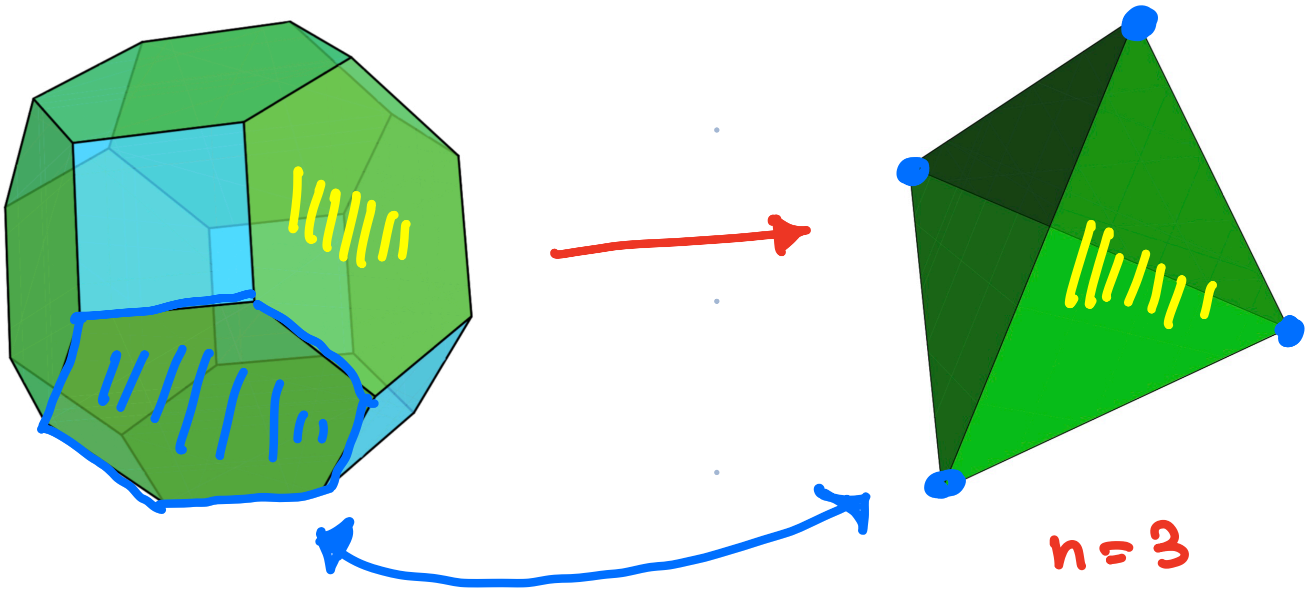
THE PERMUTOHEDRON

... the hero of our story

$$X(\pi_n) \longrightarrow \mathbb{P}^n$$

BLOWUP ALL
COORDINATE STRATA

THE TORIC POLYTOPES :



$n=3$

A remarkable space!

$\chi_{top} = (n+1)!$

Betti numbers given in terms of Stirling numbers

WHAT IS A MATROID?

A matroid of rank $\underset{=}{d+1}$ on $n+1$ elements

is a (Chow) homology class:

$$\boxed{\alpha \in A_d(X(\Pi_n))}$$

such that

$$\alpha \cdot H^{n-d} = 1 \quad \text{“ Degree } 1\text{''}.$$

[The result is due to Fink '13. I learned it from Eric Katz]

Hampe '16 ~~shows~~ interprets the product structure in terms of matroids

SOURCES OF MATROIDS:

Arrangement of n+1 hyperplanes.

Let
$$\begin{tikzcd} \mathbb{P}^d \arrow[r, hook] & \mathbb{P}^n \\ W \arrow[u] \arrow[r] & X(\Pi_n) \arrow[u] \end{tikzcd}$$
be a linear inclusion

NOT CONTAINED IN COORDINATE HYPERPLANE

STRICT TRANSFORM.

$[W] \in A_d(X(\Pi_n))$ is a matroid.

NOTES:

- Most matroids DO NOT arise in this way
- The space W is the WONDERFUL MODEL of $\mathbb{P}^d \cap (\mathbb{C}^*)^n$ [dCP '95]

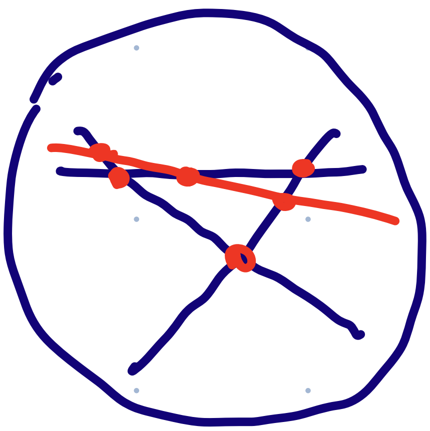
TOY EXAMPLES

A. $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as $\{x+y+z=0\}$

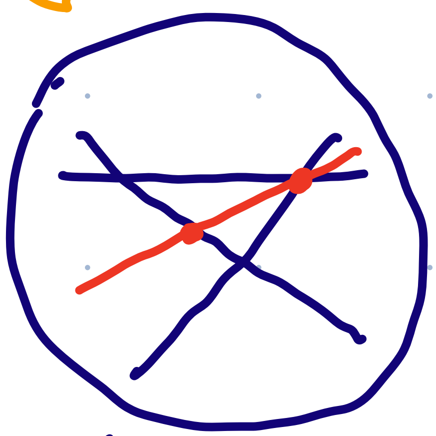
B. $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as $\{x+y=0\}$

$X(\Pi_2)$

A:



B:



Different classes in $A^*(X(\Pi_2))$!

REALIZATION SPACES

Fix a matroid $\alpha \in A_d(X(\Pi_n))$.

$$\mathrm{Gr}_\alpha = \{ [\mathbb{P}^d \hookrightarrow \mathbb{P}^n] \in G(d,n) : \text{the class of } [\widetilde{\mathbb{P}^d}] \text{ in } A_d(X(\Pi_n)) \text{ is } \alpha \}$$

Locally closed in $G(d+1, n+1)$

What is Gr_α?

If Gr_α is nonempty then

$\mathrm{Gr}_\alpha = G(d,n) \cap L^\circ$

Plücker coordinate subspace

A big if

Gr_α : Typically DISCONNECTED & SINGULAR

cf Corey-Luber '22

Mnev's ; Vakil's Murphy Law

INTERSECTION THEORY: $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$

SMALLEST OPEN TORIC

$$\begin{tikzcd} W \arrow[r, hook, "i"'] & U \arrow[r, hook, "j"'] & X(\Pi_n) \end{tikzcd}$$

WONDERFUL TORIC

THEOREM (de Concini–Procesi; Feichtner–Yuzvinsky)

$$\begin{tikzcd} A^*(U) \arrow[r, "\text{ISOMORPHISM}", "i^*"'] & A^*(W) \end{tikzcd}$$

n-dim d-dim'l

False if we work with H^* rather than A^*

COROLLARY: $A^*(W)$ depends only on the matroid $[W] \in A^*(X(\Pi_n))$.

SURPRISING? Given $[\mathbb{P}^d \hookrightarrow \mathbb{P}^n]$ let

$V = \mathbb{P}^d \cap (\mathbb{C}^*)^n$. Then $\pi_1(V)$ is **NOT** combinatorial.

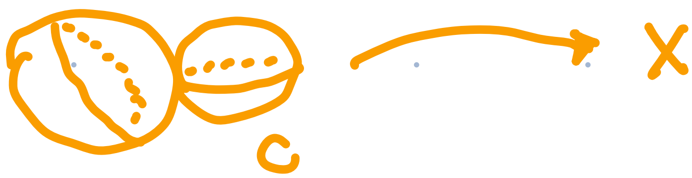
QUANTUM GEOMETRY:

PREMISE:

$$\underset{\text{space}}{X} \rightsquigarrow \underset{\text{Moduli}}{\mathcal{M}[X]} \rightsquigarrow \text{Virtual Enumerative Invariants.}$$

GW theory: $\boxed{\text{rational curves}}$

- $\overline{\mathcal{M}}_{0,n}(X,\beta) = \left\{ (C, p_1, \ldots, p_n) \xrightarrow{f} X : C \text{ pointed nodal rational with } f_*[C] = \beta \right\}$



- $\overline{\mathcal{M}}_{0,n}(X,\beta)$ has a distinguished VIRTUAL FUNDAMENTAL (homology) CLASS

Invariants:

via intersection theory and ...

$$\overline{\mathcal{M}}_{0,n}(X,\beta) \xrightarrow{ev} \underbrace{X \times \cdots \times X}_{n\text{-times}}$$

QUANTUM COHOMOLOGY: (Kontsevich).

Deformation of the usual cohomology ring

- Create $H^*(X)[[q]]$ & form a RING

Quantum product: $\alpha \star \beta = \sum \underbrace{(\ldots)}_{\text{number}} q^{\beta}$

Constructed from counts of rational curves of class β.

- PRECISELY: structure constants $C_{\alpha\beta}^{\gamma}$ have q^{β} via $\overline{\mathcal{M}}_{0,3}(X,\beta) \xrightarrow{ev} X \times X \times X$

$(\alpha, \beta, \gamma^{\vee})$

GAME OF THE DAY:

$[\mathbb{P}^d \hookrightarrow \mathbb{P}^n]$

Hyperplane arrangement

$[W \hookrightarrow X(\Pi_n)]$

Wonderful model

Gromov–Witten theory

&

Quantum cohomology

of W

Ring-valued invariant for arrangements extending the cohomology ring.

MAIN RESULTS :

Quantum cohomology is combinatorial :

THEOREM (RU'21) Let $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$ be linear and $W \hookrightarrow X(\Pi_n)$ be the strict transform.

Then $QH^*(W)$ depends only on $[W]$ in $A^*(X(\Pi_n))$.

In fact, all $g=0$ GW invariants are combinatorial

CONSEQUENCE : This defines QH^* for this class of non-compact toric varieties !

MAIN RESULTS: tropical correspondence

THEOREM (RU '21): There is a equality of $GW(W)$ with counts of tropical rational curves in W^{trop}.

wonderful model

⚠ The virtual fundamental class is [highly] nontrivial here.

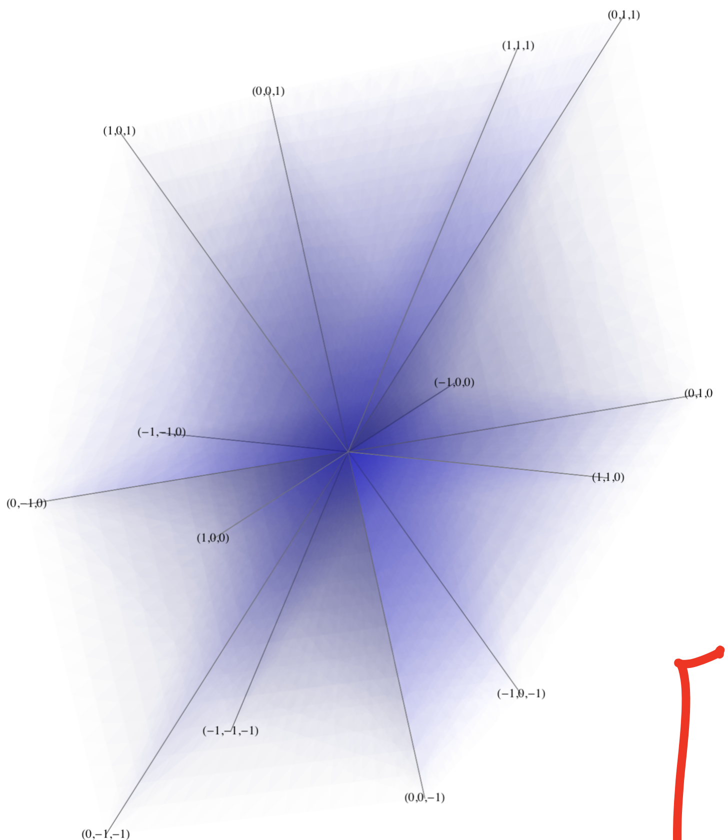
Extends: Mikhalkin, Nishinou-Siebert, Mandel-Ruddat, A. Gross, Tyomkin, R, ...

MORE ON CORRESPONDENCE...

If $\alpha \in A_*(X(\Pi_n))$ is a matroid, there is a TROPICALIZATION:

$$\mathrm{trop}(\alpha) = \text{Union of } \sigma \in \mathrm{Fan}(X(\Pi_n)) \text{ st } \alpha \cap [\sigma] = 1.$$

ORBIT CLOSURE



There is a way to count genus 0 tropical curves in trop(α) to calculate $GW(W_\alpha)$.

[+ virtual fundamental class].

LOGARITHMIC VERSIONS (I should be quick here).

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & X(\Pi_n) \arrow[d, "\pi"] \\ \mathbb{P}^d \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

de Concini–Procesi: The divisor

$$\partial W = W \cap \pi^{-1}(\partial \mathbb{P}^n)$$ is

SIMPLE NORMAL CROSSINGS.

THEOREM (RU '21) The logarithmic GW theory in $g=0$ of W with any log structure subordinate to ∂W is combinatorial.

A VECTOR BUNDLE CONSTRUCTION:

Berget - Eur - Spink - Tseng,
Tevelev, Speyer, Kapranov, ...

We want:

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d] & X(\Pi_n) \arrow[d] \\ \mathbb{P}^d \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

Now: $[\mathbb{P}^d \hookrightarrow \mathbb{P}^n] = p \in G(d,n)$

Beautiful Geometric Fact:

The $(\mathbb{C}^*)^n$ - orbit of p is a toric

$\overline{W} \hookrightarrow Y \hookrightarrow G(d,n)$

Now $\overline{W}$ is cut out by universal quotient and W is as well.

Route to GW theory:

Katz - Cox - Lee

Kim - Kresch - Pantev

Manolache.

If $W \hookrightarrow X(\Pi_n)$ with $W = \mathbb{V}(s)$, s in $H^0(\mathbb{E})$

and $\mathbb{E}$ is convex

the moduli spaces reflect this:

$$\begin{tikzcd} & & \mathbb{E} \arrow[d] \\ e \arrow[rr, bend left, "F"] \arrow[d, "\pi"'] & & X(\Pi_n) \\ \mathrm{Maps}(X(\Pi_n)) & & \end{tikzcd}$$

Now $\mathbb{F} = \pi_* F^* \mathbb{E}$ has a section

cutting out $\mathrm{Maps}(W)$

NON-REALIZABLE CASES:

Fix a matroid $\alpha \in A_\delta(X(\Pi_n))$

There is a smallest toric open

$$U \hookrightarrow X(\Pi_n)$$ as before

Now define

$$A^*(\alpha) := A^*(U)$$

Adiprasito–Huh–Katz: The ring $A^*(\alpha)$ behaves like a realization exists from the view of HODGE THEORY $\rightarrow$ Rota's conjecture.

CONJECTURES: Fix $\alpha \in \mathrm{Ad}(X(\Pi_n))$ a matroid.

RU '21: We define a class

$$[M(\alpha)]^{vir} \quad \text{in} \quad A_*(\overline{M}(X(\Pi_n)))$$

the pushforward of a mythical virtual class on a mythical mapping space.

CONJECTURE: These determine GW invariants with insertions in $A^*(\alpha)$

FURTHER RESULTS:

If $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$ is the identity [n=d]

the $GW^{\log}(X(\Pi_n) \mid \partial X(\Pi_n))$ is

TORIC BOUNDARY

TROPICAL [Mikhalkin; Nishinou & Siebert]

we generalize this

Each matroid α determines an open

$$\mathcal{U} \hookrightarrow X(\Pi_n).$$

The tropicalization $\mathrm{trop}(\alpha) := \Sigma_{\mathcal{U}}$

Then $GW^{\log}(\alpha)$ [if defined] is determined

by counts of tropical curves on $\mathrm{trop}(\alpha)$.

Further Questions:

The hardest question is CALCULATION - only GRR for now...

- Is $QH^*(W)$ semisimple? cf Bayer

Maybe this is easy..?

- Deletion/contraction operations are natural in GW theory

Inversion of degeneration formulae

- Finally, assuming higher genus conjecture, this produces some [undoubtedly beautiful] cycles in $CH^\bullet(\overline{M}_{g,n})$.

[double ramification cycles]

THANKS !