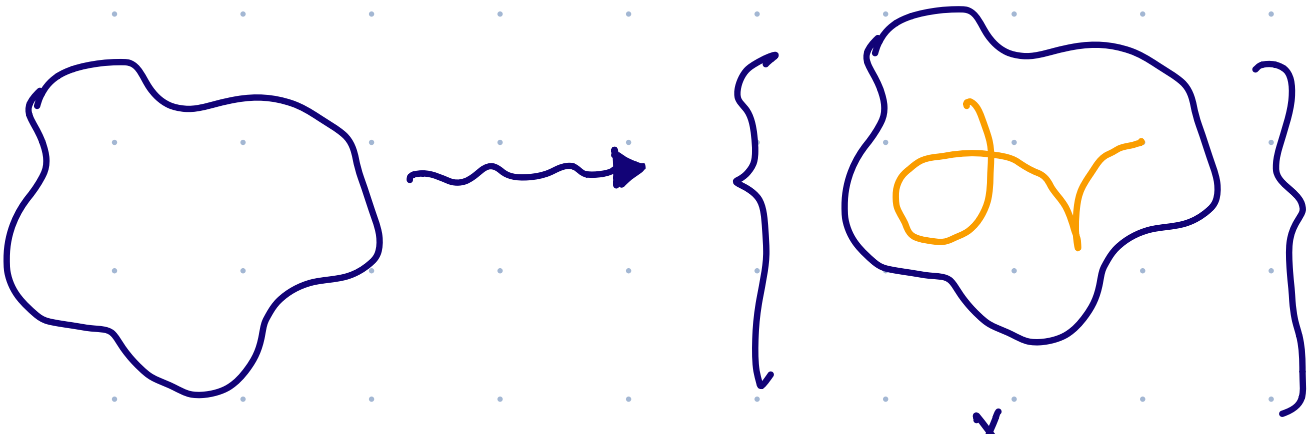
$\frac{4}{2}$ WAYS OF COUNTING CURVES IN A PAIR

"CURVE COUNTING" INVARIANTS



X PROJECTIVE MANIFOLD.

X

$\mathcal{M}(X)$ CURVES IN X

INVARIANTS OF X

Geometry & topology of $\mathcal{M}(X)$

e.g. $\chi_{top}(-)$, $\int_{\mathcal{M}(X)} \gamma$, ...

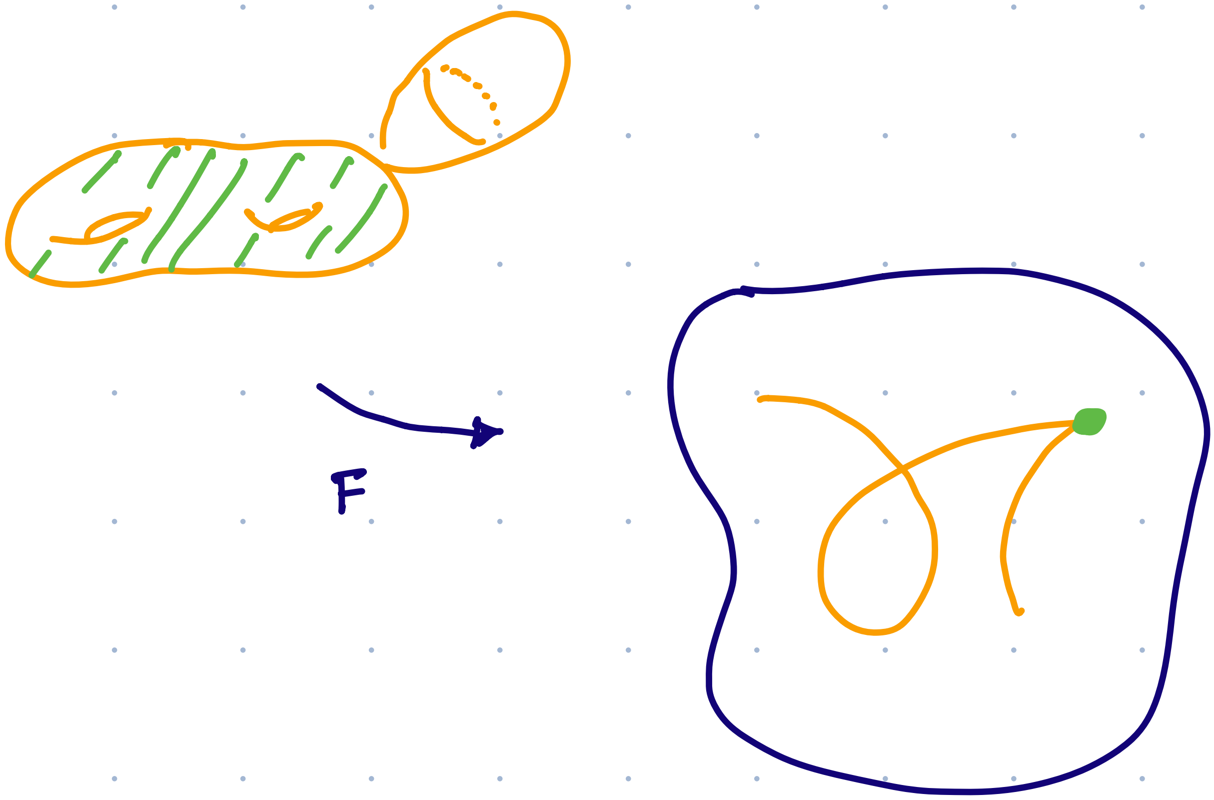
WANT: . $\mathcal{M}(X)$ has some natural system of integrals.

. Useful to have $\mathcal{M}(X)$ compact (calculus).

STUDY $\mathcal{M}(X)$ to study X or to study curves.

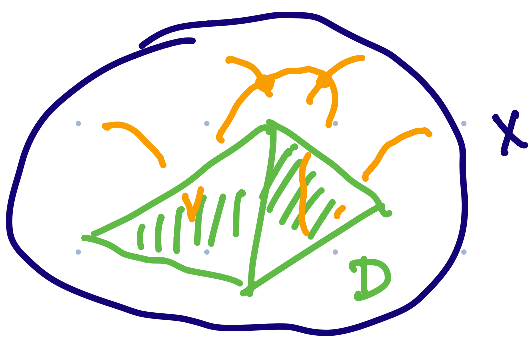
there are CHOICES! For us: ("GROMOV-WITTEN THEORY")

$$\mathcal{M}(X) = \{ C \to X \mid C \text{ nodal} \}$$ Fix degree & genus.



TODAY WILL BE ABOUT PAIRS (mostly in genus 0) X

$(X|D)$ with $D \subseteq X$ a SNC divisor



CURVES in X with TANGENCY ALONG D.

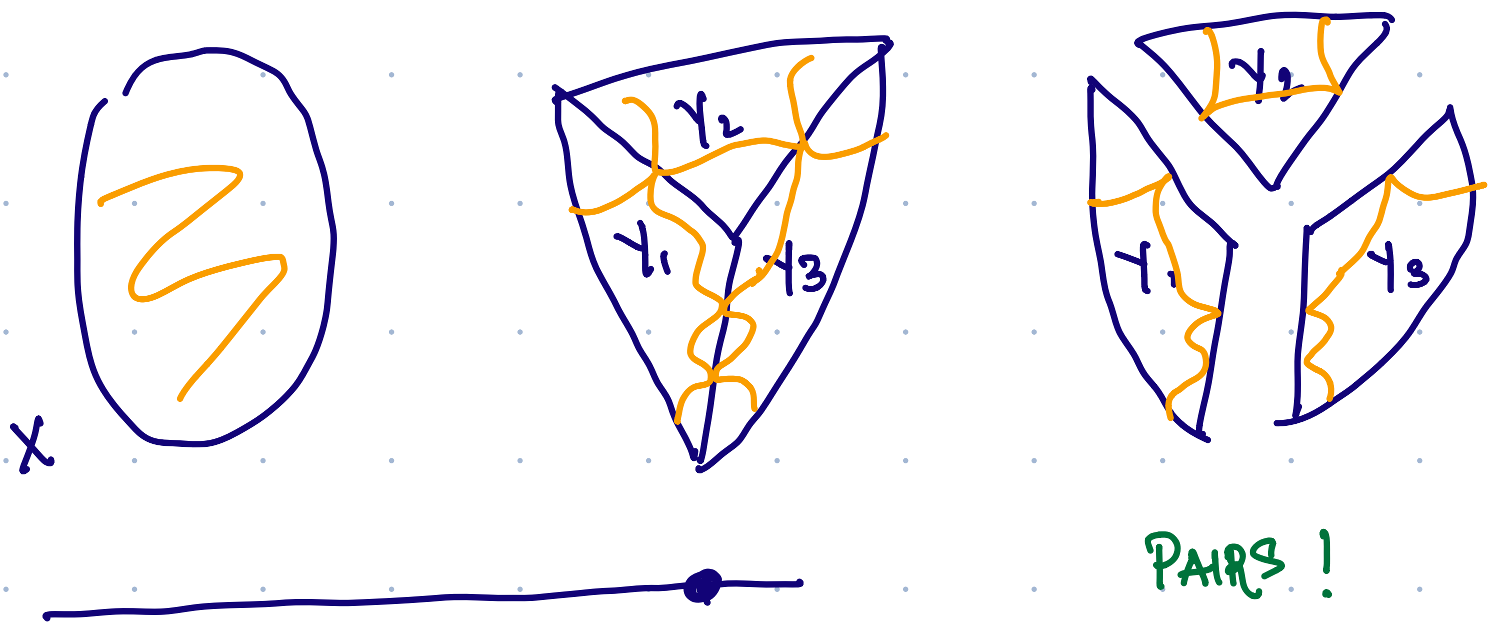
WHY WOULD ___ YOU STUDY PAIRS?

* ACCESS TO "OPEN" GEOMETRIES, i.e.

  $(X \supset D) \leftrightsquigarrow X \setminus D$ w/ "FRAMING"

* DEGENERATE FROM SMOOTH TO SINGULAR VARIETIES

* "APPLICATIONS": MIRROR SYMMETRY, REPRESENTATION THEORY, THINGS...



FRONTIERS OF WHAT WE UNDERSTAND:

- $(X|D)$ with D smooth —— "EVERYTHING" in all genus

  Vakil, Gathmann, Faber, Pandharipande, Okounkov, Maulik, Ranganathan, Pixton, Zvonkine, Tseng, You, ...

- $(X|D)$ toric pair in $g=0$ via tropical correspondence theorems.

- Lots of work on log CY surfaces (the next talk), blowups of toric varieties

  Nishinou, Siebert, Gross, Argüz, Bousseau, Brini, van Garrel, Mandel, Ruddat, Gathmann, Markwig, ...

PERSONAL GOAL: $(\mathbb{P}^n, D)$ with D SNC

$(\mathbb{P}^3, K3)$ & $(\mathbb{P}^2, E)$ are rather beautiful!

THE MODEL GEOMETRY:

: $(\mathbb{P}^2 \mid L_1 + L_2)$

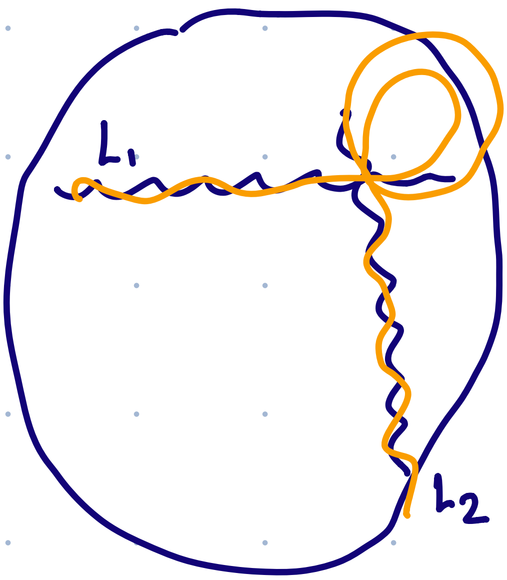
LINES

Study maps:

$$\left\{ (C, p_1, \ldots, p_n) \longrightarrow (\mathbb{P}^2 \mid L_1 + L_2) \right\}$$

WITH PRESCRIBED TANGENCY FOR EACH p_i

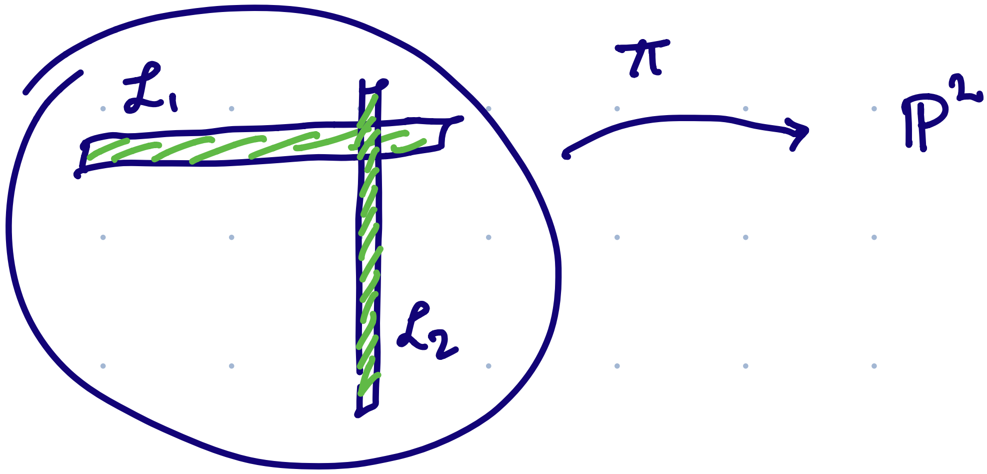
(GENUS ZERO FOR TODAY)



(OR WORSE).

4/2 MODELS FOR THE PROBLEM

- CADMAN & ABRAMOVICH–VISTOLI: Use root stacks



π is a "birational modification"

SLOGAN: If $C \subseteq \mathbb{P}^2$ is tangent to L then it is transverse to $\sqrt{L}$.

RECENTLY: TSENG–YOU have revisited these ideas and connected them to lots of more modern ideas.

DR cycles, mirror constructions, "log" quantum cohomology...

ABRAMOVICH – CHEN
GROSS – SIEBERT } USE "LOGARITHMIC STRUCTURES"

IDEALIZED SITUATION:

get nice bundles $\mathcal{O}_{\mathbb{P}^2}(L_i)\big|_C$

SLOGAN: For $C \xrightarrow{F} (\mathbb{P}^2 | L_1 + L_2)$, demand that both $F^* \mathcal{O}_{\mathbb{P}^2}(L_i)$ smooth to nice bundles.

[we usually dress this up using phrases like "sheaf of monoid" or "Artin fan"].

Relationship between approaches is mysterious at first sight (but known to coincide if D smooth).

NABIJOU says: there's something much simpler!

Defined by Fulton's intersection theory.

$$\begin{tikzcd}
 & \overline{\mathcal{M}}(\mathbb{P}^2\{L_1+L_2) \arrow[dl] \arrow[dr] & \\
\overline{\mathcal{M}}(\mathbb{P}^2\{L_2) \arrow[dr] & & \overline{\mathcal{M}}(\mathbb{P}^2\{L_2) \arrow[dl] \\
 & \overline{\mathcal{M}}(\mathbb{P}^2) &
\end{tikzcd}$$

Completely computable.

Completely computable!

$\overline{\mathcal{M}}(\mathbb{P}^2) \longrightarrow$ Nice orbifold.

REMARK: Log & orbifold approach works for any $(X|D)$

The above works for $(X|D)$ in genus 0.

$(X|D) \longrightarrow$ Ample components

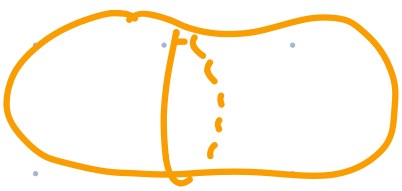
But it is completely computable!

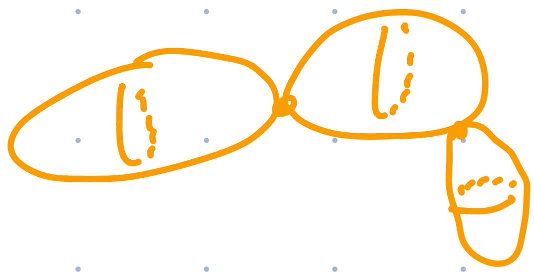
JOINT WORK WITH NAVID:

$$\begin{tikzcd}
\mathring{\mathcal{M}}(\mathbb{P}^2|L_1) \arrow[dr] & \mathcal{M}^{\circ}(\mathbb{P}^2|L_2) \arrow[d] & \mathcal{M}(\mathbb{P}^2|L_1) \arrow[d] & \mathcal{M}(\mathbb{P}^2|L_2) \arrow[dl] \\
& \mathring{\mathcal{M}}(\mathbb{P}^2) \arrow[r, hook] & \mathcal{M}(\mathbb{P}^2) &
\end{tikzcd}$$

MAPS FROM SMOOTH DOMAINS

STABLE MAPS





THM

- ACGS's LOG THEORY is the closure of the intersection of the $\mathcal{M}^{\circ}(\mathbb{P}^2|L_i)$'s.

- CADMAN / ABR. VISTOLI's ORBIFOLD THEORY is the intersection of $\mathcal{M}(\mathbb{P}^2|L_i)$'s.

{ Dhruv, please say something about generality

COROLLARIES OF DIRECT CALCULATION:

- The orbifold theory $\neq$ logarithmic theory.
- The higher rank local/logarithmic conjecture of vGGR fails (see next TALK for more).

Also gives instances where the correspondence holds and a complete cycle level expression for the orbifold side.

SO HOW TO CALCULATE THE CLOSURE OF AN INTERSECTION?

TROPICAL GEOMETRY:

- $\mathcal{M}(\mathbb{P}^2)$ is not a toric variety but it has a FAN keeping track of its boundary stratification.

INTERSECT

STRICT TRANSFORMS.

$$\begin{tikzcd}
\mathcal{M}(\mathbb{P}^2\{L_1\})^+ \arrow[dr, bend right] & \mathcal{M}(\mathbb{P}^2\{L_2\})^+ \arrow[d] & & \mathcal{M}(\mathbb{P}^2\{L_1\}) \arrow[d] & \mathcal{M}(\mathbb{P}^2\{L_2\}) \arrow[dl] \\
& \mathcal{M}(\mathbb{P}^2)^+ \arrow[r] & \cdots \arrow[r] & \mathcal{M}(\mathbb{P}^2) &
\end{tikzcd}$$

SEQUENCE OF "TORIC" BLOWUPS

STRICT TRANSFORMS ARE CONTROLLED & "COMPUTABLE IN PRINCIPLE"

INGREDIENTS:
- Tropical moduli spaces
- Fulton's blowup formula
- Aluffi's work on normal cones of monomial subschemes.

Amazing!

THIS RAISES A LOT OF QUESTIONS:

Efficiently capture the difference between log & orbifold theories (In "Maximal Contact" we do this).

Promising ingredient: Logarithmic quasimap theory &

wall-crossing

· Battistella–Nabijou '17

ONGOING $\begin{cases} \text{Qaasim Shafi} \\ \text{Battistella–Nabijou} \end{cases}$

- Understanding strict transforms using tropical methods

Molcho–Pandharipande–Schmitt "λ_g..." '21

Molcho–R '21

- Connection to local curve counting

[BBvG], Battistella–Nabijou–Tseng–You '21.