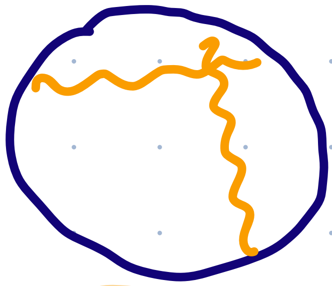
GROMOV-WITTEN THEORY via ROOTS & LOGARITHMS

w/ Battistella, Nabijou.

Geometric setup: fix $(X|D)$ a SNC pair



$D = D_1 + \dots + D_k$.

Study "GW$(X|D)$": Why? Access to open geometries, holomorphic anomaly, GW/DT, mirror symmetry, DR cycles

$$\left\{ (C, p_1, \dots, p_n) \xrightarrow{f} (X|D) \;\middle|\; \begin{array}{l} \cdot\ c_{ij} \text{ is contact order of } p_i \text{ with } D_j \\ \cdot\ \text{genus, curve class fixed.} \end{array} \right\}$$

For C smooth & $f(C) \not\subseteq D$ this is a good problem $\longrightarrow$ $\mathcal{M}^\circ(X|D)$

Compactness: tangency condition weak for contracted comp

For a compact moduli problem with virtual class:

(i) Orbifold GW theory (Abramovich, Cadman, Graber, Vistoli). ≈ 2000

(ii) Logarithmic GW theory (Abramovich, Chen, Gross, Siebert, Li). ≈ 2010

I will spell out both, but first I will situate the problem

Qualitative comparisons:

Orbifold: $QH^*_{orb}(\cdot)$, Virosoro, torus localization etc.

Logarithm: Degeneration formula for $X \to \mathbb{A}^1$ SNC, tropical curve counting, mirror algebras, symplectic cohomology...

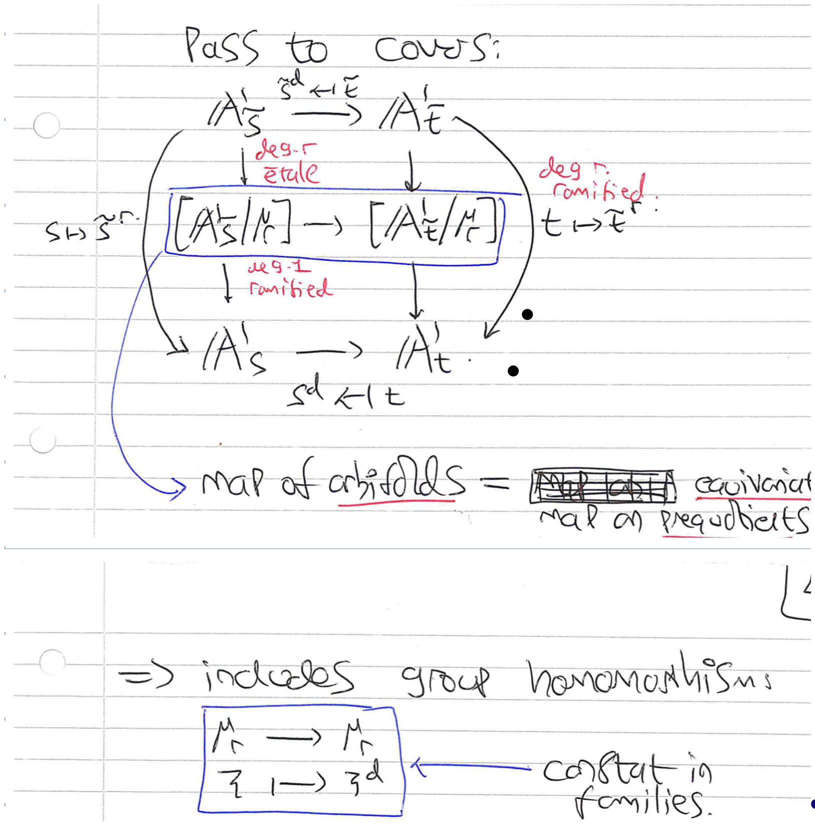
Do the theories agree? If not can we compare?

rank \ genus	0	positive
1	Abramovich Cadman Wise '10	Maulik — counterex JPPZ — corrected form
large	Nabijou-R '20 counterex BNR '22 corrected form	Mysterious ~ Holmes Molcho Pandharipande Pixton Schmitt.

↳ Main result of today

Orbifold theory:

re-examine tangency: $(\mathbb{P}^1, p) \to (X|D)$



locally given by

$$\left\{ \begin{array}{l} \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \\ s^d \leftarrow t \end{array} \right.$$

run an rth root construction.

→ Nabijou's diagram

$\rightsquigarrow$ $\mathrm{Orb}(X|D, r) \supseteq \mathcal{M}^\circ(X|D)$ from before

Basic question: what is the dependence on r? Subtle!

Logarithmic theory: An expansion of X along D is obtained by def to the normal cone of a monomial (wrt D) subscheme of X that is also reduced.

Ex :

A map $C \to X$ is logarithmic if it factors through

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & X \\ & \mathcal{Y} \arrow[u] \end{tikzcd}$$

(i) where $\mathcal{Y} \to X$ is an expansion

(ii) dimensionally transverse predeformable.

(iii) tangency orders c_{ij} ← wrt strict transform of D_i.

If feeling frisky about how to do this in families, see Maulik-R '20 or look at what a log structure is

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THEOREMS & EXAMPLES

i. If D is smooth & $g=0$, then

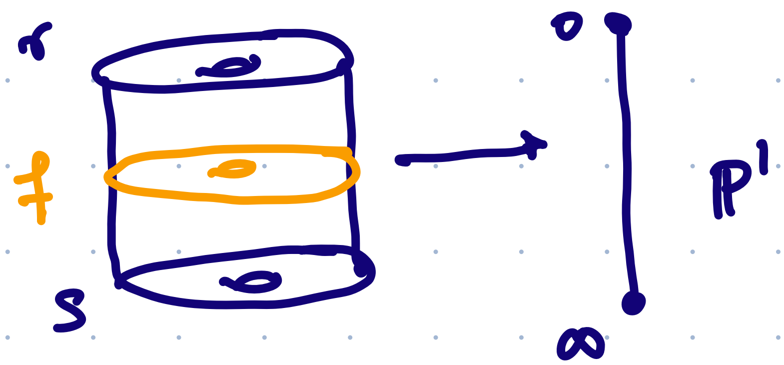
[Abramovich-Cadman-Wise + ε] There is an equality of spaces & obstruction theories:

$$\boxed{\mathrm{Orb}(X|D,r)^c = \mathrm{Log}(X|D)^c}$$

taking relative coarse moduli space over $\overline{M}(X)$.

Notice the strength of the theorem!

ii. Example (Maulik): $X = \mathbb{P}^1 \times E$; $D = 0 + \infty$



Invariants differ in the fibre class in $g=0$ since the orbifold theory depends on r & s.

iii. Theorem (JPPZ + Tseng–You) If D is smooth, then $[\mathrm{Orb}(X|D,r)]^{vir}$ is a polynomial in r in $\overline{M}(X)$. The constant term is the class of

$[\mathrm{Log}(X|D)]^{vir}$

Notice – not as strong a result!

iv. Example (Nabijou-R '21)

If $X = \mathbb{P}^2$ $D = L_1 + L_2$, and $g=0$, then

for $d=2$, invariant with floating ψ. class

differs.

v. Theorem (BNR '22) Fix discrete data & $g=0$

After sufficient blowup

$$(\tilde{X}|\tilde{D}) \longrightarrow (X|D),$$ there is an equality

of spaces & virtual classes:

$$\mathrm{Orb}(\tilde{X},\tilde{D})^c = \mathrm{Log}(\tilde{X}|\tilde{D})$$

taking rel. coarse space over $\overline{\mathcal{M}}(\tilde{X})$.

Number of blowups grows like (degree)$^{\text{log rank} - 1}$.

Corollary: Log determined by absolute (TVGWT)

vi. Conjecture: In all genus, for fixed data,

after $(\tilde{X}|\tilde{D}) \longrightarrow (X|D)$ the class

$[\mathrm{Orb}(\tilde{X}|\tilde{D}, \underline{r})]$ is polynomial in $\underline{r}$

in $\overline{\mathcal{M}}(\tilde{X})$ & the constant term is $\log(\tilde{X}|\tilde{D})$.

Missing: Higher double ramification cycles.

vii. Key fact: Product formulas fail in log GW $\begin{cases}\text{HPS} \\ \text{R}\end{cases}$

A sketch of proof: [after you guess the answer]

. $(X|D) \longrightarrow [\mathbb{A}^k/\mathbb{G}_m^k]$ tautological map

by virtual intersection theory smoke & mirrors

we reduce to comparing

Product formulas.

$$\boxed{\mathrm{Orb}(\mathbb{A}^k/\mathbb{G}_m^k) \ \& \ \mathrm{Log}(\mathbb{A}^k/\mathbb{G}_m^k).}$$

Fact: $C \to \mathbb{A}^k/\mathbb{G}_m^k$ induces a map on posets of strata.

. If $C \longrightarrow \mathbb{A}^k/\mathbb{G}_m^k$ is logarithmic, then there exists [nearly has a converse]

- A metrization Γ of the dual graph $G(C)$

TROPICAL GEOMETRY

- A piecewise linear map

$\Gamma \to \mathbb{R}^k_{\geq 0}$ compatible with the poset

Here one sees the issue: lifts to the factors are not compatible.

★: A toy model via toric morphisms

Negative tangency: [Assume $k=1$; complexity is orthogonal]

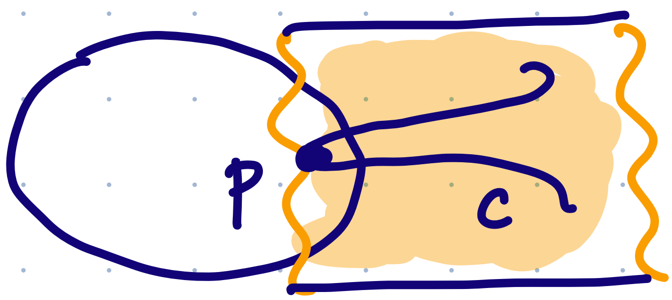
Logarithmic: $(C,p) \longrightarrow (X|D)$ has tangency

$-k$ if it factors as:

$$(C,p) \to X \cup_D \mathbb{P}(\mathcal{O}\oplus N_D) \to X$$

$$(C,p) \longrightarrow X$$

like



tangency k

Orbifold: the isotropy is given by

$$\mu_r \longrightarrow \mu_r$$
$$\zeta \longmapsto \zeta^{r-k/r}$$

Make r large, divide out by $\gamma^{\text{[illegible]}}$ and extract coefficient.

Constructions: log side:

$k=1$ Fan-Wu-You — a virtual class on $\mathrm{Log}(X|D)$ by localization

arbitrary rank | Battistella-Nab.-R — virtual class by Fulton-Manolache.

THM (BNR '22, in writing progress) On a blowup

$$\mathrm{Log}(\tilde{X}|\tilde{D}) = \mathrm{Orb}(\tilde{X}|\tilde{D})$$

why? Allows a definition of $QH^*_{\log}(-)$

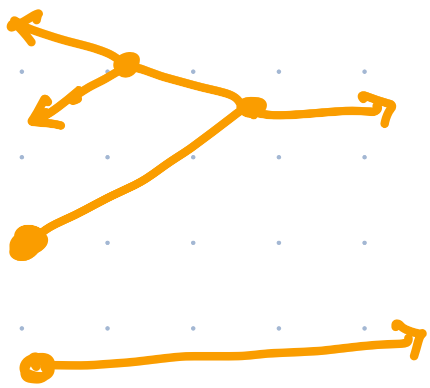
In particular (by Johnston), mirror algebra is the degree 0 part of orbi QH^*.

Also why? Splitting axioms

In my view, the most interesting part of the newer developments is the virtual class:

Negative tangency markings prevent components from moving to the interior.

This means that the relations between node smoothing bundles are complicated; encoded by a certain monomial ideal in a toric stack



$\rightsquigarrow$ Segre class / normal cone of this gets involved