LOGARITHMIC & TROPICAL MODULI

GAeL 2022

§1 INTRODUCTION & MOTIVATION

work over $\mathbb{C}$

Basic premise: study non-compact algebraic varieties X° by a combination of

(i) Geometry of a projective variety $X \supseteq X^\circ$, where X is "NICE"

(ii) Combinatorics of the "tropicalization" or "fan" of X.

Logarithmic geometry is the combination.

Examples: Non-compactness arises in nature.

- Moduli theory: $\mathcal{M}_g$, $\mathcal{A}_g$ are moduli of smooth proper objects but are themselves non-proper
- Degenerations: Give $X / \mathbb{C}((t))$ the model $\mathcal{X} \to \operatorname{Spec} \mathbb{C}[[t]]$ is non-compact
- "Secondary" geometries: Powerful move:

$X \rightsquigarrow \mathcal{M}(X) \rightsquigarrow$ "Invariants" | GW theory, DT theory, etc.

Eg: $\mathrm{Hilb}(X)$, $\mathrm{Quot}(X)$, $\overline{M}_g(X)$, etc.

Works well when X proper.

Goal of the lectures: Sketch the construction of some of these moduli spaces and give a sense for what they look like.

/$\mathbb{C}$

§1 TORIC GEOMETRY: basic notions

Let $T \cong \mathbb{G}_m^n$ be an n-dimensional torus.

Let $\begin{cases} M = \mathrm{Hom}(T, \mathbb{G}_m) \\ N = \mathrm{Hom}(\mathbb{G}_m, T) \end{cases}$ be the character and cocharacter lattices.

Let X be a T-variety containing T as a dense

X is a way of systematically setting coordinates to $0/\infty$.

Given $v \in N$, get $\varphi_v : \mathbb{G}_m \longrightarrow X$. Ask:

- Does limit $\lim_{t\to 0} \varphi_v(t)$ exist?
- If the limit for v, v' exists, when are they the same?

Ex: Calculate this for $\mathbb{A}^2$, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$.

DEF: A cone in $N_{\mathbb{R}}$ is an $\mathbb{R}_{\geq 0}$-span of a finite collection of vectors in N - contains no lines.

A cone has natural faces*. A fan is a collection of cones that meet NICELY - intersection of two cones is a face of each.

Basic Correspondence:

$$\{\text{Fans in } N_{\mathbb{R}}\} \longleftrightarrow \{\text{Normal } T\text{-equiv. comp. } X \text{ of } T\}.$$

$$\Sigma(X) \longleftarrow X \quad \text{by } *$$

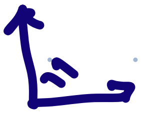
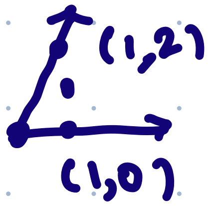
$$\Sigma \longmapsto X(\Sigma) \quad \text{as follows:}$$

Non-negative functions

Relations

Given $\sigma \in \Sigma$, $X(\sigma) = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$

gluing along open immersions $X_\tau \hookrightarrow X_\sigma$.

Ex:  $\rightsquigarrow \mathbb{A}^2$; 

$\rightsquigarrow \mathbb{V}(y^2 - xz) \subseteq \mathbb{A}^3$

Quadric Cone.

Toric Dictionary:

- X is smooth $\longleftrightarrow$ $\sigma \in \Sigma$ has $\mathbb{Z}$-independent primitive generators.
- X is proper $\longleftrightarrow$ Every $v \in N$ lies in some $\sigma \in \Sigma$.
- Equivariant morphisms: $\longleftrightarrow$ Integer maps

$X \longrightarrow Y$ $\qquad$ $N_{\mathbb{R}} \longrightarrow N'_{\mathbb{R}}$ sending cones to cones.

TORI : T $\quad T'$

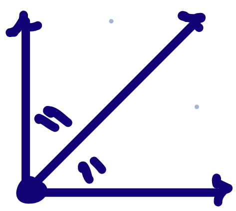
LATTICES : N $\quad N'$

- Picard group, (equivariant) Chow groups, K-theory, equivariant vector bundles, etc.

The theory is extremely effective but not the topic of these lectures – we want to push this very effective theory to study X°

smooth & non-compact.

A FEW BASIC EXERCISES:

1. Take the fan Σ  with rays generated by $(1,0)$, $(0,1)$, $(1,1)$. Identify $X(\Sigma)$ with the blowup of $\mathbb{A}^2$ at 0

2. Take $\sigma \subseteq \mathbb{R}^3$ to be the positive span of $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ & $(1,1,1)$ CONE over a SQUARE. Identify $X(\sigma)$ with $\mathbb{V}(xz-wy)$.

3. Prove that if X is toric then the class group of X is generated by invariant divisors for the T-action.

4. Let σ be a cone in $N_{\mathbb{R}}$ w/ linear span $N_{\mathbb{R}}$. Prove that $X(\sigma)$ is contractible (in the complex Euclidean topology).

§2 LOGARITHMIC STRUCTURES

- The key aspect about toric varieties is the fact that $T \subseteq X$ comes with MONOMIAL FUNCTIONS.

- We multiply but do not add monomials

DEF: A logarithmic structure on a scheme X, is a sheaf* of monoids M_X with

$$\varepsilon : M_X \longrightarrow \mathcal{O}_X$$ that preserves

units: $\varepsilon^{-1}\mathcal{O}_X^* \xrightarrow{\sim} \mathcal{O}_X^*$.

M_X : the sheaf of monomials

$M_X/\mathcal{O}_X^* =: \overline{M}_X$: the sheaf of pure monomials.

TRIVIAL EXAMPLES : $M_X = \mathcal{O}_X^*$ & $\mathcal{O}_X$.

Examples: i. Let X be toric, then M_X is the sheaf that locally picks out monomials: $M_X(U) = \{ \alpha \cdot f \mid \alpha \in \mathcal{O}_X^*(U),\ f \text{ is monomial} \} \subseteq \mathcal{O}_X(U)$.

*DAY 2

ii. $(X|D)$ a simple normal crossing pair

then take $M_X(U) = \{ f \mid f|_{U \setminus D} \text{ is invertible} \}$.

DIVISORIAL STRUCTURE

iii. Given X toric and $Y \xrightarrow{f} X$ any morphism

then we can get monomials by pullback

Formally:
$$\begin{tikzcd} f^{-1}M_X \arrow[r] \arrow[rr, bend right, "\varepsilon"'] & f^{-1}\mathcal{O}_X \arrow[r] & \mathcal{O}_Y \end{tikzcd}$$

It may not preserve units; fix it universally:

$$\begin{tikzcd} \alpha^{-1}\mathcal{O}_Y^* \arrow[r] \arrow[d] & f^{-1}M_X \arrow[d] \\ \mathcal{O}_Y^* \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & f^*M_X . \end{tikzcd}$$

Remark: You can view this as an adjoint construction.

iv. (More Later) Every reasonable logarithmic structures arises locally from iii via Artin fans.

DEF: If X & Y are logarithmic schemes then a morphism is a scheme morphism

$X \xrightarrow{f} Y$, a pullback for monomials

$$f^\flat : f^* \mathcal{M}_Y \longrightarrow \mathcal{M}_X$$ compatible

with the maps to $\mathcal{O}_X$.

Ex: If (X,D) & (Y,E) are pairs, then a logarithmic map for the divisorial structure is a scheme map

$$\begin{cases} f : X \longrightarrow Y & \text{st} \\ f^{-1}(E) \subseteq D \end{cases}$$

Ex: If X & Y are toric, then if $f : X \to Y$ equivariant then f is equivariant.

Converse is ALMOST true (Exercise why only almost?)

UPSHOT: LogSch form a category of these "enhanced" schemes. Standard concepts — smooth, flat, tangent bundle, etc make sense.

But rather than telling you, I will convince you that you know it already.

FIBRE PRODUCTS EXIST but are more delicate.

§3 Tropical Geometry

A finitely generated monoid is toric if it is $\sigma^\vee \cap M$ for some σ a cone in $N_{\mathbb{R}}$.

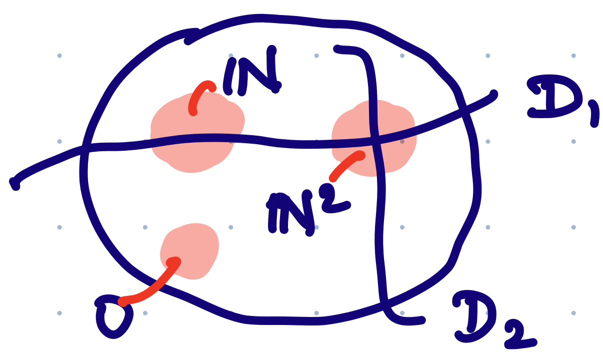
A fine & saturated* logarithmic scheme is one where for each $\bar{x} \in X$, the stalk $\overline{\mathcal{M}}_{X,x}$ of pure monomials is a toric monoid.

*: Traditional wisdom is this is necessary

We now construct the tropicalization from the sheaf $\overline{\mathcal{M}}_X$ and the stalks at points of X.

How to think: $\overline{\mathcal{M}}_{X,x}$ tracks monomials near x that vanish on a neighborhood.

If (X,D) is SNC, then



SPECIALIZATION

If $x \rightsquigarrow y$ then $\overline{\mathcal{M}}_{X,x}$ is a quotient of $\overline{\mathcal{M}}_{X,y}$ [this should be believable]

Dually: define $\Sigma(X) = \operatorname{colim}_{x \in X} (\overline{\mathcal{M}}_{X,x})^\vee$ $= \operatorname{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0})$

What is this? Basically a cone complex, but...

Useful: $\Gamma(X, \overline{\mathcal{M}}_X^{gp})$ = piecewise linear functions on Σ

Ex: If $(X|D)$ is SNC with all intersections connected, then $\Sigma_X \subseteq \mathbb{R}_{\geq 0}^{\#\text{comp of } D}$

Ex: If X is toric then $\Sigma(X)$ is as it was but without the embedding in $N_{\mathbb{R}}$ though this can be recovered (Exercise: How?)

THM (Danilov, Stepanov, Payne, Harper, ...) If $(X|D)$ is SNC then $H^*(\Sigma(X) \setminus \{0\}; \mathbb{Z})$ (in fact, the homotopy type) depends only on $X \setminus D = U$.

$\mathbb{R}_{>0}$ action, quotient: $\Delta(X)$

Powerful: $H^*(\Delta(X);\mathbb{Q}) \twoheadleftarrow H^*(U;\mathbb{Q})$. Good way to construct classes — e.g. if $U = \mathcal{M}_g$, $X = \overline{\mathcal{M}}_g$ then $\Delta(X)$ is the moduli of trop. curves; used to construct classes on $\mathcal{M}_g$ [Chan–Galatius–Payne].

This is about the topology of $X \setminus D$, but we access topology of X:

Maybe skip first time?

DEF: A <u>piecewise polynomial</u> function on $\Sigma(X)$ is a continuous map:

$$|\Sigma(X)| \longrightarrow \mathbb{R} \quad \text{such that}$$

{ COLIMIT as TOP SPACES.

for every $\sigma \to \Sigma(X)$, the restriction is polynomial. Ring of such: $PP^*(\Sigma(X))$

THM (Brion, Payne, Molcho–R, Holmes–Schwarz)

There is a ring map $PP^*(\Sigma(X)) \longrightarrow CH^*(X)$

Chow groups

[extending the association in the SNC case sending the PL function of slope 1 along the ray of D in $\Sigma(X)$ to the class [D]]

DAY 3

Useful supply of Chow classes. In particular,

$0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$ gives a map

$$\{\text{PL-functions on } \Sigma(X)\} \longrightarrow \{\text{Line bundles on } X\}$$

"BIRATIONAL" MODIFICATIONS:

In toric geometry, given a fan Σ, a SUBDIVISION $\Sigma' \to \Sigma$ determines an equiv. birational map

$$X(\Sigma') \longrightarrow X(\Sigma)$$

In order to really study these, we use the Artin fan swindle. [Olsson, Abramovich-Wise, Ulirsch]

Basic Idea: Want to map X "$\to$" $\Sigma(X)$

Let (X, M_X) be a logarithmic scheme

__Know:__ $\Sigma(X) = \operatorname*{colim}_{x \in X} (\overline{M}_{X,x})^\vee$

__Trade:__ $\overline{M}_{X,x} \rightsquigarrow \left[\operatorname{Spec} \mathbb{C}[P_x] / \underset{\substack{\nearrow \\ \text{dense torus}}}{T} \right] = A_x$

$\overline{M}_{X,x} = P_x$

ARTIN STACK; but look, whatever, everyone makes more of a fuss about this than they should and tautologically, there is a map $X \longrightarrow \operatorname*{colim}_{x} A_x = A(X)$.

Again TAUTOLOGICALLY the map

$X \longrightarrow A(X)$ is STRICT ie. the log structure on X is the pullback.

The stack $A(X)$ is not scary.

__Ex:__ Take $D \subseteq X$ smooth divisor. If D is principle, then $f : X \longrightarrow \mathbb{A}^1$ gives $D = f^{-1}(0)$. Equivalently, if

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "\varphi"'] & \mathbb{A}^1 \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

is the

composite, $D = \varphi^{-1}(0/\mathbb{G}_m)$. If D not principle, composite exists, but doesn't factorize through $\mathbb{A}^1$!

<u>Exercise</u>: Using the fibre square

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^1 \arrow[d] \\ X \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

Convince yourself of this!

<u>Ex</u>: $X \longrightarrow [\mathbb{A}^k/\mathbb{G}_m^k]$ is basically the same.

what are the fibres?

<u>Once done</u>: Subdivisions $\longleftrightarrow$ Blowups of $A(X) \rightsquigarrow$ Pullback

LOGARITHMIC TERMINOLOGY: Fix (X, M_X) & $A(X)$

- X is <u>LOG SMOOTH</u> if $X \longrightarrow A(X)$ is smooth

  <u>Eg</u>: All toric varieties

- X is <u>LOG FLAT</u> if $X \longrightarrow A(X)$ is flat.

- LOG TANGENT BUNDLE of (log smooth) X is T_π for $\pi : X \longrightarrow A(X)$.

etc...

§4 LOGARITHMIC MODULI — prototype

The fundamental example is $\overline{\mathcal{M}}_g$; it is equipped with divisorial structure $\partial\overline{\mathcal{M}}_g$

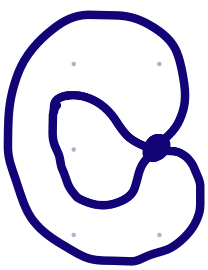
$$\overline{\mathcal{M}}_g = \{ \text{Nodal curves of arith. genus } g \underset{+\text{stability}}{} \}$$

$$\partial\overline{\mathcal{M}}_g = \{ \text{singular curves} \}$$

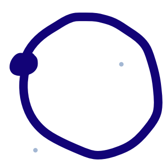
Formally: $\overline{\mathcal{M}}_g \longleftrightarrow$ Flat families of nodal genus g curves (+stability)

Basic facts:

- Smooth DM stack of dim $3g-3$
- Stratified by topological type; encoded by dual graph — $\Gamma(C)$: vertices components, edges are nodes



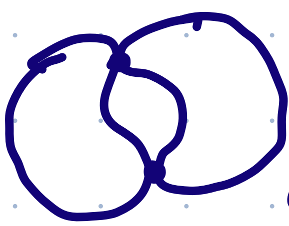
$\rightsquigarrow$



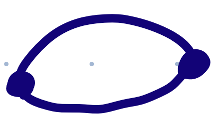
$\Gamma(C)$

C

[technically not stable but ok]



$\rightsquigarrow$



$\Gamma(C)$

C

Another perspective:

Consider a new functor/category:

$$L_g : \text{logSch} \to \text{Groupoids}$$

$$(S, M_S) \to \left\{ \begin{array}{l} \text{log smooth + flat \& reduced fibres} \\ C \text{ a log scheme}/(S, M_S) \end{array} \right\}$$

Fibre products are taken in "fine & saturated" world.

A priori this has no obvious relation to $\overline{M}_g$.

OR STACK or WHATEVER

Observe: If X is a logarithmic scheme then can take $F_X : \text{logSch} \to \text{Set}$ by

$$(S, M_S) \to \text{Hom}(S, X)$$

THM (F. Kato) There is a natural equivalence

$$L_g \cong F_{\overline{M}_g}$$

w/ above log structure

the most important thing here is to realize this is not a tautology!

Note: fs fibre products are delicate; toric fibre products are key.

See references

TROPICAL MODULI

We have $\overline{\mathcal{M}}_g$ and consider:

$$\Sigma(\overline{\mathcal{M}}_g) := \mathcal{M}_g^{trop}$$

$$A(\overline{\mathcal{M}}_g) \quad \text{Artin fan}$$

Essentially equivalent data; the space $A(\overline{\mathcal{M}}_g)$ is the "poset" of faces of $\Sigma(\overline{\mathcal{M}}_g)$

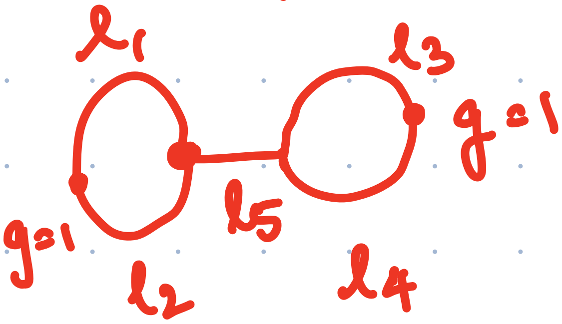
There is a map $\overline{\mathcal{M}}_g \longrightarrow A(\overline{\mathcal{M}}_g)$

smooth morphism

<u>THM</u> (Abramovich-Caporaso-Payne) The space $\Sigma(\overline{\mathcal{M}}_g)$ is a moduli space of genus g tropical curves

[They prove much more]

Dual graph with a metric.



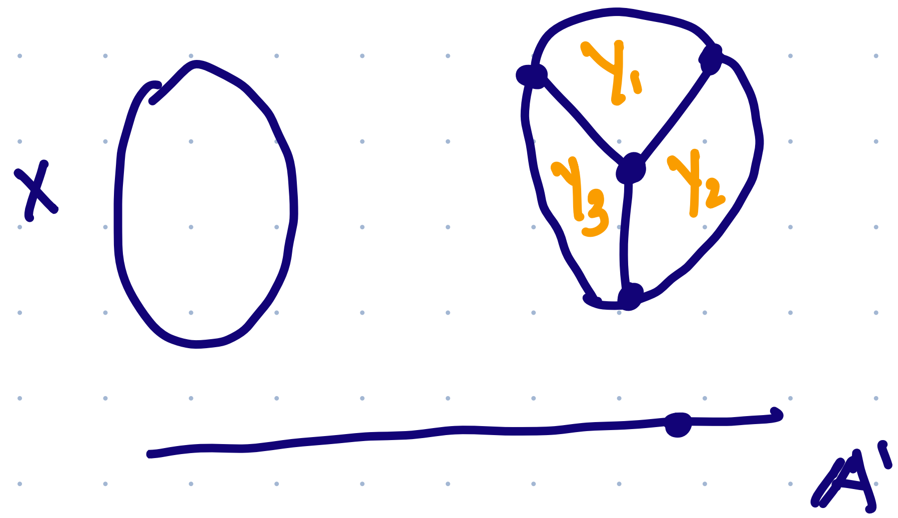
Tropical <u>moduli</u>: a point in $\mathbb{R}^5_{\geq 0}$

<u>Formally</u>: Category fibred in groupoids over cones

§5. MODULI under DEGERATIONS

With this language and technology, we can examine a fundamental problem in AG: the behaviour of moduli spaces under degenerations

Let $\mathcal{X} \to \mathbb{A}^1$ be a simple normal crossing degeneration:



Basic question: If $M(X)$ is a moduli space of [...] how does it degenerate as X degenerates?

Ex: Take $X = K3$ & $M(X) = \mathrm{Hilb}^n(K3)$

X = Anything & $M(X) = \mathrm{Hilb}_P(X)$ or $\mathrm{Quot}(X)$

X = Anything $M(X)$ = Stable Maps to X

There is often a good answer via some kind of $\mathcal{M}^{\log}(\mathcal{X}_0)$. Given this, one can ask for DEGENERATION FORMULAE i.e.

$$\overline{\mathcal{M}}^{\log}(\mathcal{X}_0) = \ast \; \overline{\mathcal{M}}^{\log}(Y_i|D_i)$$

mysterious gluing process

components

intersection with other comp.

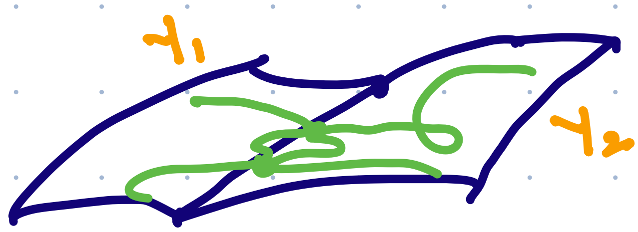
We will examine this for Hilbert scheme of X

Restrict to subschemes of dimension 1.

Hilbert polynomial (β, n)

We will build $\mathrm{Hilb}^{\log}(Y_i|D_i)$

Best possible situation:



Subschemes $Z_i \subseteq Y_i$ meeting $D = Y_1 \cap Y_2$

TRANSVERSELY

Right notion of transversality:

Z_i is logarithmically flat with restricted log structure from (Y_i, D).

$\Longrightarrow$ the intersection $Z_i \cap D$ is zero dim of length $\beta \cdot D$.

How to build a logarithmic moduli space:

i. Open moduli problem – $\mathrm{Hilb}^{\circ}(X)$ locus where subschemes are flat

ii. Examine 1-parameter limits – how to degenerate?

iii. Divide up 1-parameter subgroups into equivalence classes.

LIMITS after TEVELEV

THM If $Z \hookrightarrow X$ is a closed subscheme of a toric variety, then there exists a blowup
$$\tilde{X} \to X$$
st the strict transform of Z is flat over $[X/T]$.

Proof: We can assume X is proper and that Z is not preserved by a subtorus of T. Now take the T-orbit of $[Z]$ in $\mathrm{Hilb}(X)$ $\square$

Smoke & mirrors $\rightsquigarrow$ valuative criterion we need.

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & X \arrow[d] \\ & \operatorname{Spec} \mathbb{C}((t)) \end{tikzcd} \rightsquigarrow$$

A family

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[dr] & \mathcal{X} \arrow[d] \\ & \operatorname{Spec} \mathbb{C}[[t]] \end{tikzcd}$$

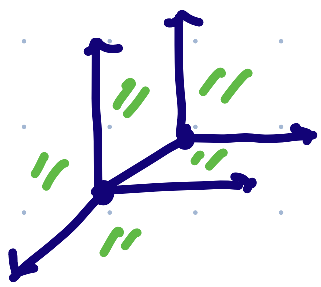
transverse limit.

where $\mathcal{X}$ is an expansion of X.

TORIC DICTIONARY:

Expansions of X $\leftrightarrow$ Polyhedral subdivisions of X

Eg: $Bl_{pt}\, \mathbb{P}^2 \times \mathbb{A}^1$ Deformation to the normal cone



Polyhedral subdivisions can be parametrized by a topological space $\mathcal{T}$ [a word about its nature]

Treat it like the cocharacter vector space $N_{\mathbb{R}}$.

Choose a fan structure $\Sigma \to \mathcal{T}$

Apply Artin fan swindle:

$$\mathrm{Exp} = \mathrm{ArtinFan}(\Sigma).$$

Now examine logarithmic subschemes

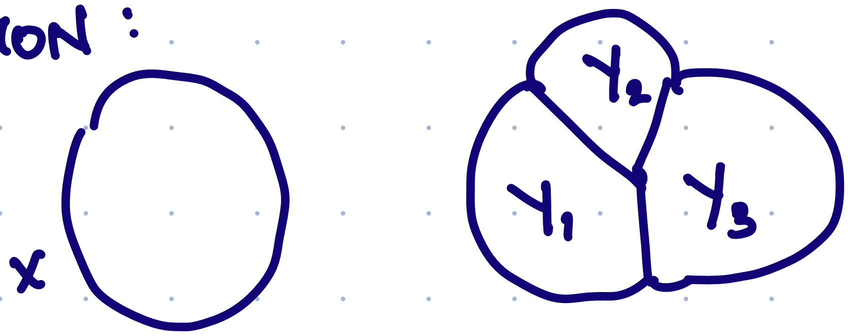
$$\left\{\begin{aligned} &\mathrm{Hilb}^{\log}_{\Sigma}(X|D) : \mathrm{LogSch} \longrightarrow \mathrm{Sets} \\ &(S, M_S) \mapsto \{S \to \mathrm{Exp} \text{ log map; } \mathcal{Z} \hookrightarrow \mathcal{X}_S \text{ family of subschemes}\} \end{aligned}\right.$$

+ stability.

THM: $\mathrm{Hilb}^{\log}_{\Sigma}(X|D)$ is representable & proper.

& satisfies a degeneration formula

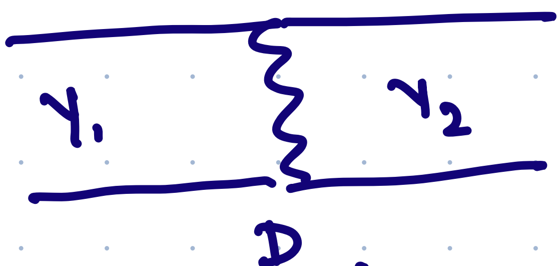
DEGENERATION:



$\mathbb{A}^1$

Nice degeneration of $\mathrm{Hilb}(X)$, $\mathrm{Hilb}^{\log}(\mathcal{X})/\mathbb{A}^1$

Reconstruct 0-fibre as fibre products:



$\mathcal{X}_0$

$$\mathrm{Hilb}^{\log}(\mathcal{X}_0) = \mathrm{Hilb}^{\log}(Y_1|D) \times_{\mathrm{Hilb}(D)} \mathrm{Hilb}^{\log}(Y_2|D).$$

Equally applicable: · stable maps, Hilbert scheme of points, etc.

References:

· **Log structures:** K. Kato's "Log structures of Fontaine & Illusie", Abramovich-Chen-Gillam-Huang Olsson-Satriano-Sun "Log geometry & moduli", F. Kato's "Log smooth deformation theory & moduli of log smooth curves"

· **Tropical geometry:** Abramovich-Caporaso-Payne's "Tropicalization of moduli space of curves",* M. Ulirsch's "Functorial tropicalization of logarithmic schemes", Tevelev's "Compactifications of subvarieties of tori"

Expansions: R "Log GW with expansions"
Maulik-R "Log Donaldson-Thomas theory".

*: Beautiful non-technical introduction without any logarithmic language. Instead uses Berkovich spaces.