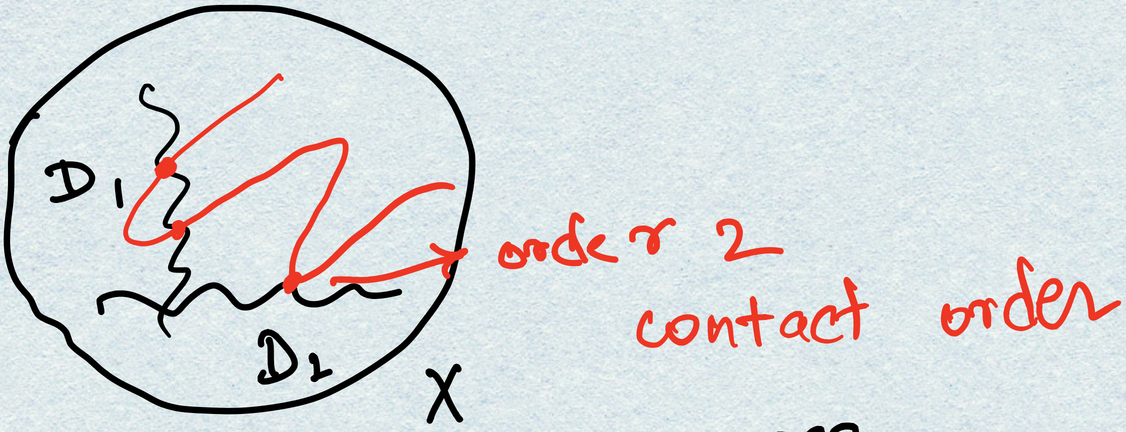
GROMOV-WITTEN THEORY & LOGARITHMIC INTERSECTIONS

$(X|D)$ snc pair

$$\mathcal{M}(X|D) = \left\{ (C, p_1, \ldots, p_n) \xrightarrow{f} (X|D) \;\middle|\; \begin{array}{l} * \text{ fix degree, genus,} \\ * \text{ contact order of } f \\ \quad \text{at each } p_i \text{ along } D \\ \quad \text{also fixed} \end{array} \right\}$$



BASIC STRUCTURES

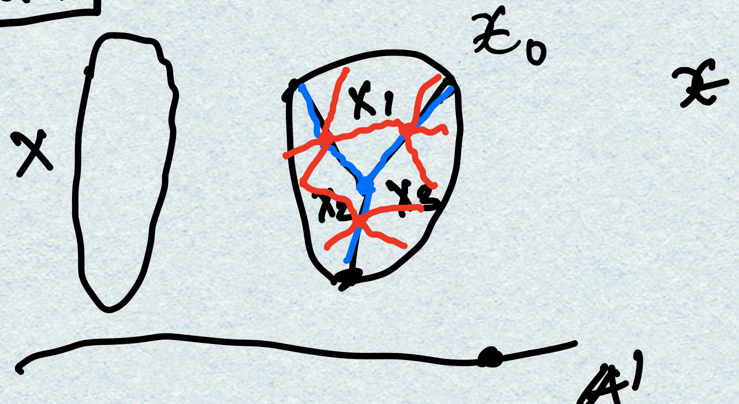
PRODUCTS $\quad (X|D), (Y|E) \rightsquigarrow (\underline{X \times Y} \mid D+E)$

$$\begin{tikzcd}
\mathcal{M}(X \times Y | D+E) \arrow[r, "\mu"'] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}(X|D) \times \mathcal{M}(Y|E) \arrow[d] \\
& \overline{\mathcal{M}}_{g,n} \arrow[r, "\Delta"'] & \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

Expect:

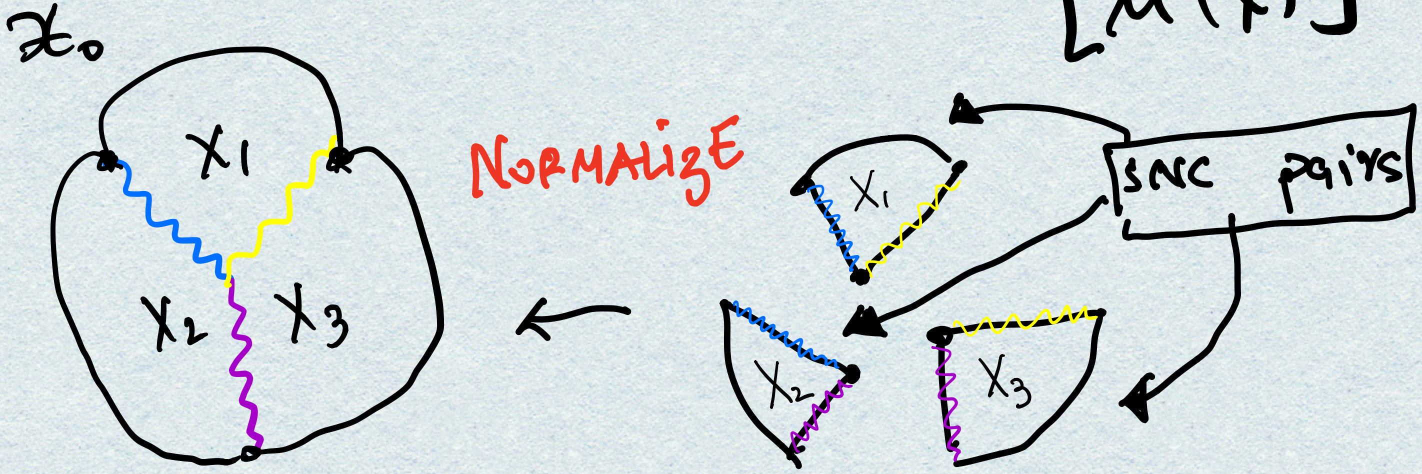
$$\mu_* [\mathcal{M}(X \times Y | D+E)]^{vir} = \Delta^! [\,- \times -\,]^{vir}$$

DEGENERATION



GEOMETRY

Glue curves that "match" along the double divisors & recover $[\overline{M}(X)]^{vir}$



"Matching" ⟶ schematic gluing + matching contact orders

DEGENERATION FORMULA

(1) $$[M(X)]^{vir} = \sum_{\gamma \,:\, \text{decorated graphs}} [M_\gamma(\mathfrak{X}_0)]^{vir}$$

(EASY)

γ : decorated graphs → TROPICAL CURVE

RECORDS ⟶
- Topological type of curves in $\mathfrak{X}_0$
- Contact orders
- Degrees in each X_i.

(2) For fixed γ we can write

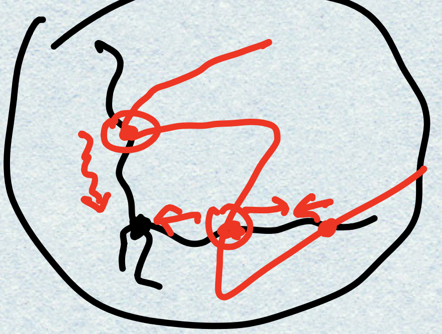
(HARD)

$$\mathcal{M}_\gamma(\mathcal{X}_0) = \Big(\prod_{v:\text{ vertices of }\gamma} \mathcal{M}_v(X_v|D_v)\Big) \cap \text{"}\Delta\text{"}$$

matches the curves via evaluations

→ Other things: Localization, DT variants, relative/orbifold correspondence } SNC

Compactifying $\mathcal{M}(X|D)$: (why hard?)



Hard to find limit

Approach 1: (Logarithmic stable maps) ≈ 2010
Abramovich-Chen; Gross-Siebert
Proper; has virtual class; basically contained in usual $\overline{\mathcal{M}}(X)$

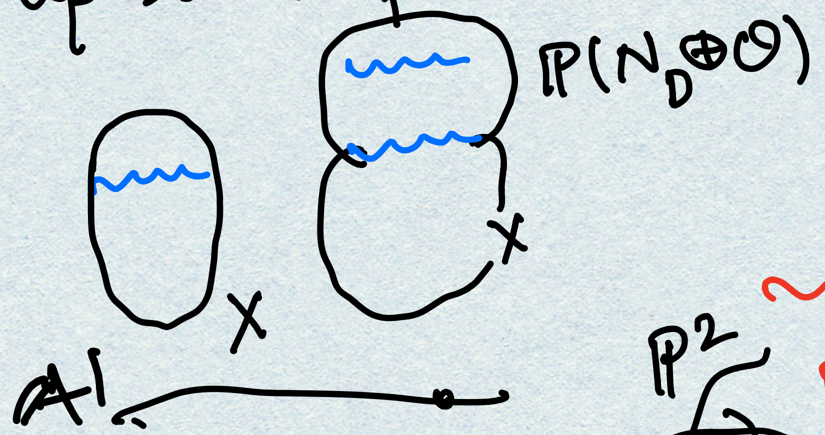
(Ignoring this for today)

Approach 2: when the contact orders try to jump, change the $\boxed{\text{target}}$ by expanding

Study stable maps to $\boxed{\text{expansions}}$ of X along D.

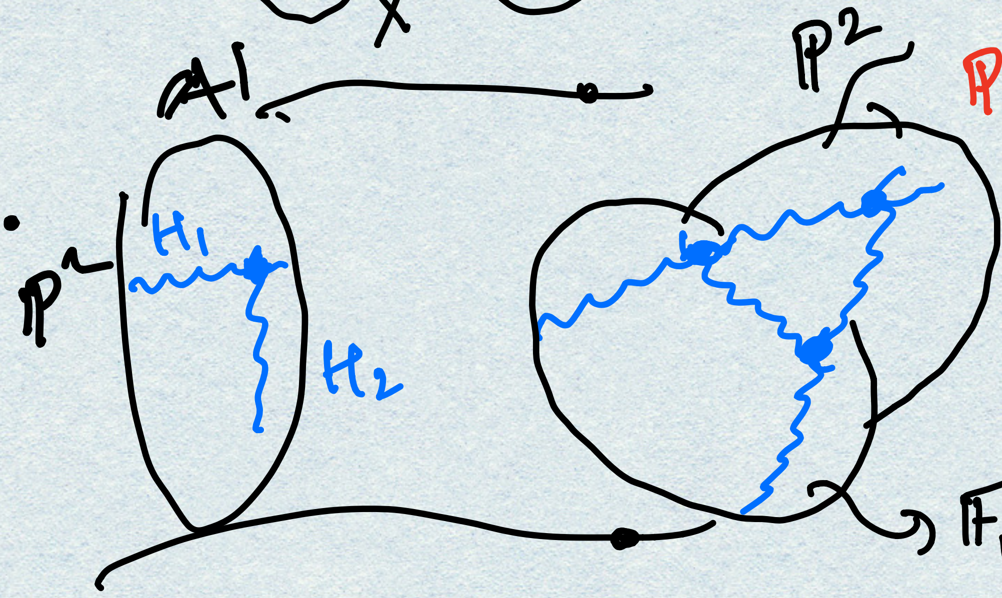
An *expansion* is obtained by taking X, $\rightsquigarrow X \times \mathbb{A}^1$ & iteratively blowing up strata of D in the 0 fiber.

Ex I.



Ex II.

• $\mathbb{P}^2$



$\mathbb{F}_1 = Bl_{pt}\,\mathbb{P}^2$

$\rightsquigarrow$ Really $\boxed{\text{TORIC}}$ $\mathbb{P}^2$ as a pair, access to TONNES of blowups.

$\boxed{\text{THEOREM}}$ There exist moduli spaces of maps to expansions of $(X|D)$; universal structures

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{X} \arrow[r] \arrow[d] & X \\ & \overline{\mathcal{M}}(X|D) & \end{tikzcd}$$

• Proper • Virtually smooth & $\mathcal{C}$ is dimensionally transverse to the strata of the expansion.

Remarks: * Jun Li for D smooth

* R '19 for D general SNC.

* ⚠ The moduli spaces appearing above are not unique! They form an inverse system $\{\overline{\mathcal{M}}^{\bullet}(X|D)\}_{\bullet\in\Lambda}$

Is this a feature or a bug? Let's explore

• Virtual class "doesn't care":

arrows: $\overline{\mathcal{M}}^{\bullet}(X|D) \xrightarrow[\pi]{} \overline{\mathcal{M}}^{+}(X|D)$

π_* identifies the virtual classes.

Example to keep in mind:

the system of all toric compactifications of a fixed torus.

(Aside for Tom's question)

$(\mathbb{P}^2, 3\text{ lines}) \leftarrow$ target

What are the birational models?

$\beta = 1$ (lines), transverse contact.

Universal diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathbb{P}^L \times (\widetilde{\mathbb{P}^2})^\vee \arrow[d] \\ & (\widetilde{\mathbb{P}^2})^\vee \end{tikzcd}$$

$\mathcal{C}$ has an SNC divisor

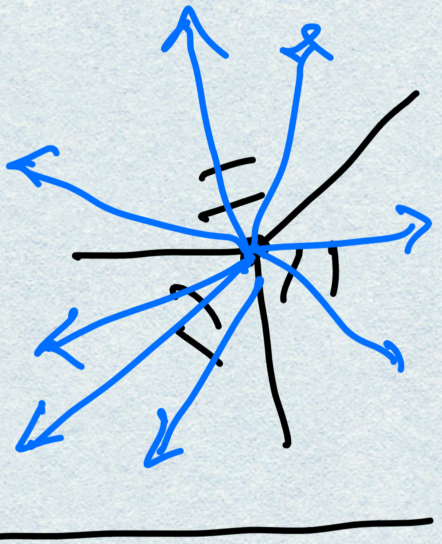
$$\begin{tikzcd} \Delta(\mathcal{C}) \arrow[r, hook] \arrow[d] & \mathbb{R}^2 \times \mathbb{R}^2 \arrow[dl] \\ \mathbb{R}^L & \end{tikzcd}$$

simplicialize all maps to toroidal the moduli

$(\mathbb{P}^2)^\vee$

$\uparrow$

$(\widetilde{\mathbb{P}^2})^\vee$



Formulas:

$$\begin{tikzcd} \overline{\mathcal{M}}(X \times Y | D+E) \arrow[r, "\mu"] & \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}(X|D) \times \overline{\mathcal{M}}(Y|E) \arrow[d] \\ & \overline{\mathcal{M}}_{g,n} \arrow[r, hook, "\Delta"] & \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

Hope: $\Delta^! [- \times -]^{vir} \neq \mu_* [\overline{\mathcal{M}}(X \times Y | D+E)]^{vir}$

THM (Herr, Holmes-Pixton-Schmitt, R '19)

There exists a blowup of $\overline{\mathcal{M}}_{g,n}$ along its strata $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ such that the product formula holds after replacing

$\boxed{\overline{\mathcal{M}}_{g,n} \text{ w/ } \widetilde{\mathcal{M}}_{g,n}}$

Degeneration formula: story is very similar

blowups, blowups, everywhere

$$\overbrace{\coprod_{\substack{\gamma \\ \text{decorated}}} \overline{\mathcal{M}}_v(X_v | D_v)} \cap \text{``}\widetilde{\Delta}\text{''}$$

$\boxed{\text{Back to products:}}$ want to intersect two GW cycles over $\overline{\mathcal{M}}_{g,n}$ & we get the wrong answer unless we blowup.

$$\begin{tikzcd} \boxed{?} \arrow[r] \arrow[d] & Z_i \arrow[d] \\ \widetilde{\mathcal{M}}_{g,n} \arrow[r, "\text{Blowup}"'] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

(want to intersect the Z_i's)

what is $\boxed{?}$?

In practice: • If $Z_1 \hookrightarrow \overline{\mathcal{M}}_{g,n}$ was regularly embedded then $\boxed{?}$ = strict (or lci) transform

(but we want virtual class too)

If $Z_1 = \overline{M}(X|D)$ then we need the virtual class, so this doesn't work.

<u>However</u>: "virtual" strict transform b/c

$$\overline{M}(X|D) \longrightarrow \boxed{\overline{M}(\mathcal{A}_X)} \longrightarrow \overline{M}_{g,n}$$

space of maps to an Artin stack.

=

$$(X|D) \longrightarrow [\mathbb{A}^k / \mathbb{G}_m^k]$$

<u>Turns out</u>: $M([\mathbb{A}^k/\mathbb{G}_m^k])$ is a smooth Artin stack, birational to $\overline{M}_{g,n}$.

Dan & Jonathan

$$\overline{M}(X|D) \longrightarrow \boxed{\overline{M}(\mathcal{A}_X)} \longrightarrow \overline{M}_{g,n}$$

virtually smooth

birational

$$\begin{tikzcd}
\overline{M}^{\bullet}(X|D) \arrow[r, "\sim \text{smooth}"'] & \underset{\text{STRICT}}{\widetilde{M}(A_X)} \arrow[r, "\text{bir.}"'] & \widetilde{M}_{g,n} \\
& \underset{\text{TOTAL}}{\left(\overline{M}(A_X) \times_{\overline{M}_{g,n}} \widehat{M}_{g,n}\right)} \arrow[u] &
\end{tikzcd}$$

Fulton: to compare the strict & total transform, there is a formula:

(usual Fulton type formula: expected part of polynomial in Chern classes. Key is:

→ Ingredients: • Excess bundle of the blowup
• Normal bundle in Z of the stratum that gets blown up.

Z (cycle we're interested in)

Basic strategy: ~~All~~ Most the bundles we care about arise in the following fashion:

take $\overline{M}(X|D) \longrightarrow [\mathbb{A}^N / \mathbb{G}_m^N]$ given by virtual strata (this is a tautology) and pullback (treat like we treat $\mathbb{L}_i$, $\mathbb{E}_g$ etc)

For moduli of curves:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n} \arrow[r] & \mathcal{A} \\
\widetilde{\mathcal{M}}_{g,n} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & {[\mathbb{A}^N / \mathbb{G}_m^N]} \arrow[u]
\end{tikzcd}$$

$\leftarrow$ zero dim'l

= tropical moduli stack

Pass to inverse limit; want $\varinjlim CH^*(\mathbb{A}^N/\mathbb{G}_m^N)$

Let X be a toric variety. The limit of CH^* over blowups already has a beautiful answer.

(Non-equivariant case): $\lim\limits_{\substack{X' \to X \\ \text{(iterated blowup)}}} CH^*(X') =$ POLYTOPE ALGEBRA [Fulton & Sturmfels]

(Equivariant case, more relevant): $\lim\limits_{X' \to X} CH_T(X')$

$= PP^*(\Sigma_X)$ [Ring of piecewise polynomial functions on $|\Sigma|$] [Payne; Brion]

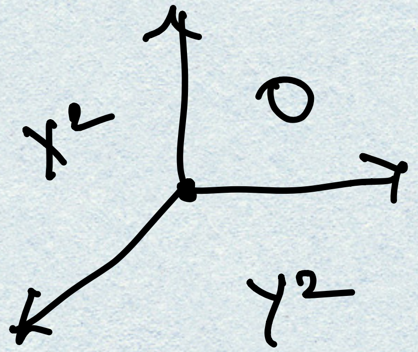
<u>Remark</u>: The map $PP^*(\Sigma_X) \to$ Polytope Algebra <u>is</u> equivariant localization

Piecewise polynomials:

continuous polynomial on each cone

[as we refine, the "support" condition disappears]

Σ



$\to$ In $CH^*(\overline{\mathcal{M}}_{g,n})$ we have an additive basis for $R^*(\overline{\mathcal{M}}_{g,n})$. [Tom, Rahul, friends]

$\to$ we want to "marry" these two algebras.

$\overline{\mathcal{M}}_{g,n} \to \mathcal{A}$ TORIC STACK

$\rightsquigarrow$ easy to replace w/ global toric quotient $[X/T]$

$$\varinjlim_{X' \to X} CH^*_T(X') = PP^*(\Sigma_X).$$

THM (discussions w/ Dawesh, Rahul, Tom, David, ...) Let X be a toric variety $D = \partial X_{\text{toric}}$, then consider the "toric contact" cycles, i.e.

take $\pi : \overline{\mathcal{M}}_\Gamma(X|D)^{\text{rubber}} \longrightarrow \overline{\mathcal{M}}_{g,n}$

& consider $\pi_* [\overline{\mathcal{M}}_\Gamma(X|D)^{\text{rubber}}]$.

This class lies in $R^*(\overline{\mathcal{M}}_{g,n})$

w/ Holmes: (we think) same w/ a target manifold.

(harder than you might think!)

Application: (w/ Danesh) The GW cycles of a $\boxed{\text{large class}}$ of surfaces lie in $R^*(\overline{M}_{g,n})$

(In curve case, see work of Felix.)

$[n] = p_1, \ldots, p_r, q_{r+1}, \ldots, q_n$ (The DDR's)

$$\sum_{i=1}^{r} a_i = 0 \;;\; \sum_{i=r+1}^{q_n} b_i = 0$$

- Study (in $M_{g,n}$) the locus of curves where

$$\mathcal{O}(\Sigma a_i p_i) = \mathcal{O}(\Sigma b_i q_i) = \mathcal{O}$$

Instance of product formula for rubber.

$$(X|D) = (Y|E) = (\mathbb{P}|0,\infty)$$

- How to prove that the Toric contact cycle is in R^*; apply the product formula.

$\leadsto$ Use prod. of 2 DR cycles as the "wrong" answer & correct using Fulton's formula

In Fulton's formula for the strict transform

we need
- piecewise polynomials $\boxed{\checkmark}$
- Strata of the DR cycle to be tautological (application of the splitting axiom).

$$\underline{\text{Relative}}\ \mathrm{Maps}(\mathbb{P}^1) \longrightarrow [\mathbb{A}^k / \mathbb{G}_m^k].$$