Logarithmic DT theory

w/ Davesh Maulik

arXiv : 2006: ...

&

work in progress

ETH SEMINAR

MAY '21

where are we playing...

GROMOV-WITTEN THEORY

MNOP

DONALDSON-THOMAS THEORY

LI
GROSS SIEBERT
ABRAMOVICH-CHEN
[& the SYMPLECTOMORPHS]

Li-Wu
Maulik-R '20
[maybe some of you!]

LOGARITHMIC GW THEORY

NEW CONJECTURES

LOGARITHMIC DT THEORY

SMOOTH PAIR VERSIONS: Li, Li-Wu

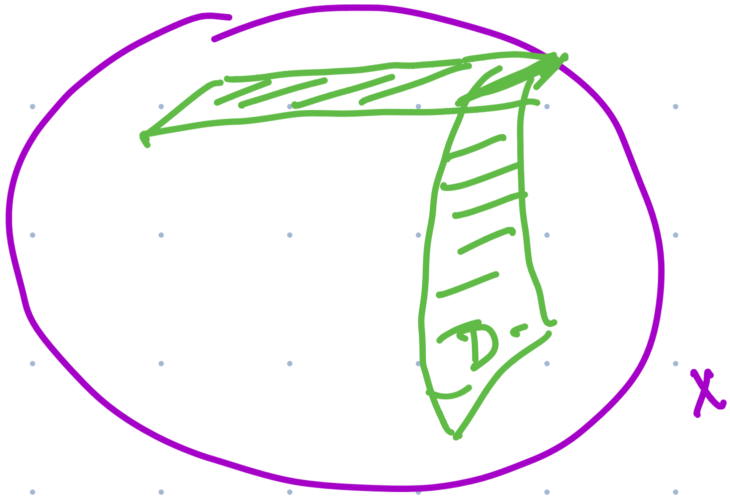
LOGARITHMIC DT PACKAGE : a summary

- $(X|D)$ an SNC threefold pair.
  - $\mathrm{Hilb}^{\log}_{\text{curves}}(X|D)$ a logarithmic moduli space of subschemes of X expanded along D
- A version for points on surfaces

for users... virtual class, invariants, log deformation invariance, log birational invariance, PT, degeneration formulas, GW/DT conjectures

for developers... a logarithmic Hilbert scheme (of curves), a new blueprint for building log moduli, new approach to $GW^{\log}$

Enumerative Geometry of SNC pairs ...



... via curves $(C, p_1, \dots, p_r) \xrightarrow{f} (X|D)$

& fix the tangency conditions

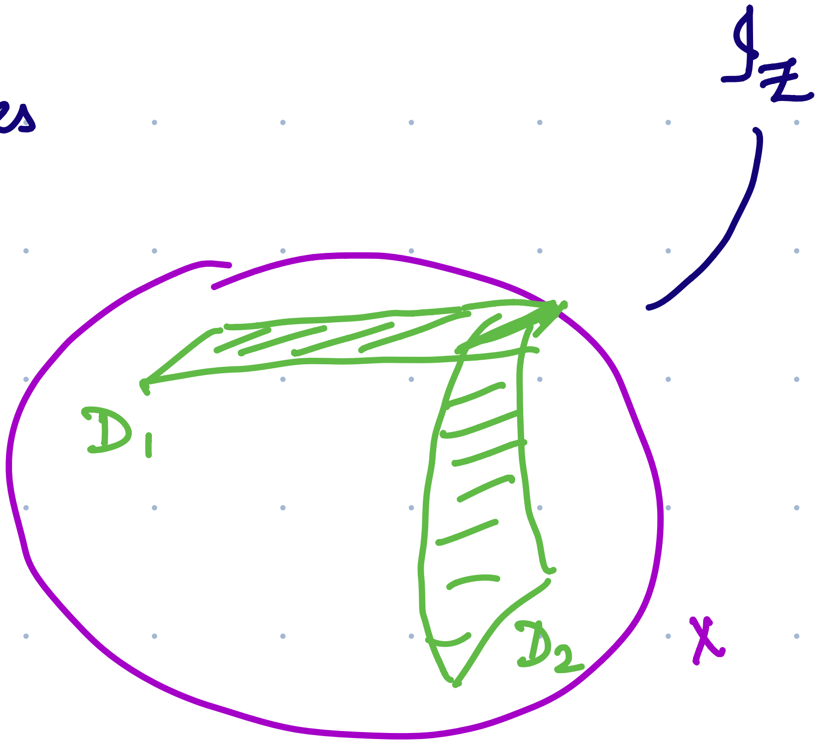
$\longrightarrow \overline{M}^{\log}(X|D)$

RICH SUBJECT!

{ Abramovich - Chen
Gross - Siebert.

... mirror symmetry, Picard stack, Cluster stuff, ...

... via sheaves



ideal sheaves on X

Solve the equations of $\mathcal{I}_Z$ on D_1 & D_2:

$$\begin{array}{ccc} \mathrm{Hilb}_{\text{curves}}(X) & \dashrightarrow & \mathrm{Hilb}_{\text{pts}}(D_i). \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathrm{Hilb}^{\circ}_{\text{curves}}(X) & \longrightarrow & \mathrm{Hilb}^{\circ}_{\text{pts}}(D_i) \end{array}$$

KEY PIECE OF STRUCTURE !

On the interior...

$$\mathrm{Hilb}^{\circ}_{\text{curves}}(X) \xrightarrow{\varepsilon_{D_i}} \mathrm{Hilb}^{\circ}_{\text{pts}}(D_i)$$

In an ideal world, define

$$DT^{\log} \equiv \int_{\mathrm{Hilb}^{\circ}_{\text{curves}}(X)} \prod_i \varepsilon^*_{D_i}(-) \cdot \text{Classes from } X$$

Globally we want a compactification:

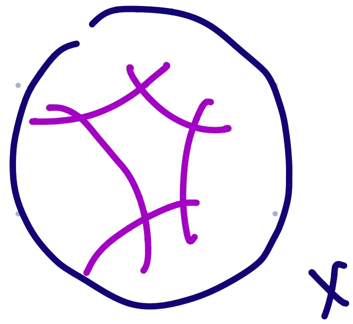
DT theory lives here

$$\mathrm{Hilb}^{\log}_{\text{curves}}(X|D) \longrightarrow \mathrm{Hilb}^{\log}_{\text{pts}}(D_i|D_i \cap D)$$

COMPACT & VIRTUALLY SMOOTH.

NOTIONS OF TRANSVERSALITY:

Fix $(X|D)$ with D SNC

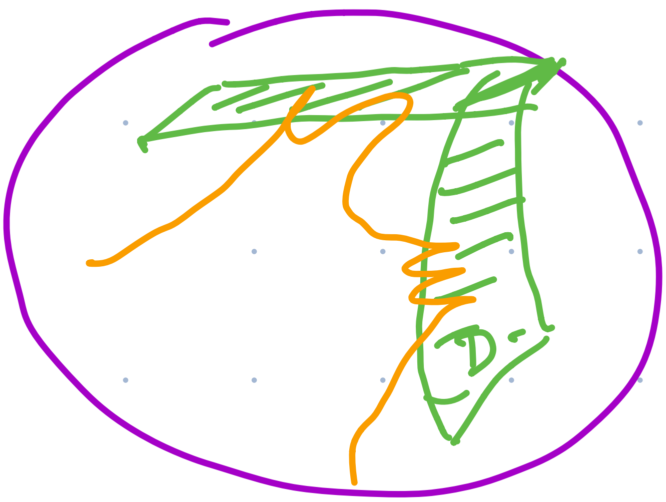


$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & X \times \mathrm{Hilb}(X) \arrow[r] \arrow[dl] & X \arrow[r] & {[\mathbb{A}^r / \mathbb{G}_m^r]} \\ \mathrm{Hilb}(X) & & & \end{tikzcd}$$

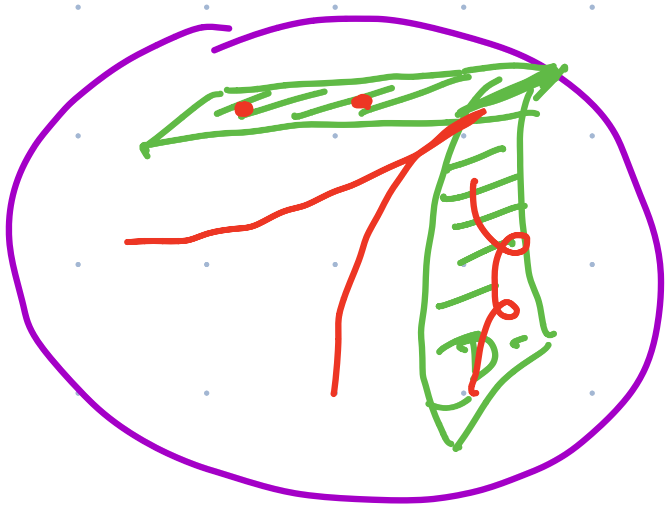
$$\mathrm{Hilb}(X)$$

$$\cup\!\!\mid$$

$$\left\{ Z \;\middle|\; Z \to [\mathbb{A}^r/\mathbb{G}_m^r] \text{ is } \underline{\text{FLAT}} \right\} =: \mathrm{Hilb}^{\circ}(X|D)$$



GOOD



NO GOOD

$\boxed{\text{How to compactify?}}$

INSIGHTS OF Kapranov & Tevelev:

* not preserved by a subtorus.

Say* $Z^\circ \hookrightarrow \mathbb{G}_m^r$

what is a good toric compactification $X \supseteq \mathbb{G}_m^r$ in which to compactify Z°?

Use Z° to map

$$\mathbb{G}_m^r \longrightarrow \mathrm{Hilb}(\mathbb{P}^r) \rightsquigarrow X \supseteq \mathbb{G}_m^r$$

THEOREM (Tevelev & Kapranov)

$$\begin{tikzcd} \overline{Z^\circ} \arrow[r, hook] \arrow[rr, bend right, "\text{FLAT}"'] & X \arrow[r] & {[X/T]} \end{tikzcd}$$

w/ strong uniqueness properties.

Generalized by Ulirsch, Vogiannou, ...

Nearby Mathematics...
Gröbner theory, resolution of singularities,

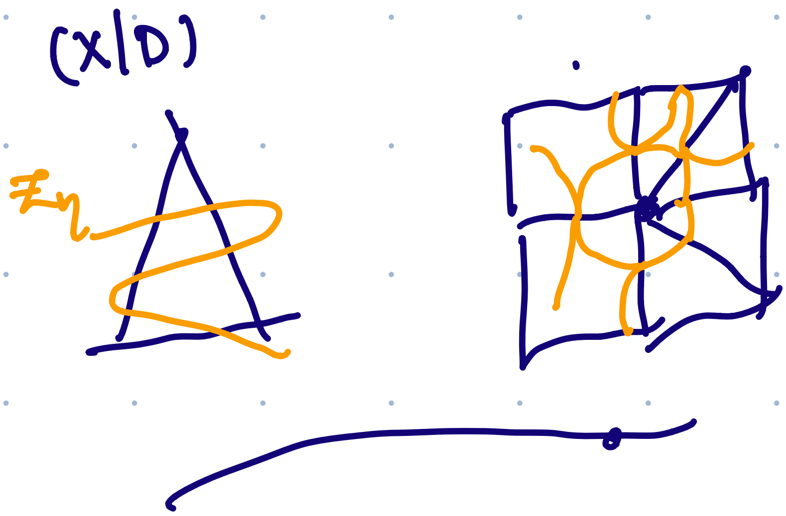
We can apply this in the following context...

$$\begin{tikzcd} Z_{\Delta^\circ} \arrow[rr, hook] \arrow[dr] & & (X|D) \arrow[dl] \\ & \Delta^\circ & \end{tikzcd}$$

... a family of transverse subschemes

PUNCTURED - - -

Output is a completion of the family:



$$\begin{tikzcd} \mathfrak{X} \arrow[d] \\ \Delta \end{tikzcd}$$

$\mathfrak{X}$ is determined by the general fiber via the tropicalization of Z_η

IDEA: Build a moduli space whose **PROPERNESS** is guaranteed by the above construction.

THEOREM: There exist moduli spaces

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, hook] \arrow[dr] & \mathcal{X} \arrow[r] \arrow[d] & X \\
& \mathrm{Hilb}^{\log}_{\text{curves}}(X|D) \arrow[r] & \mathrm{Hilb}^{\log}_{\text{pts}}(D_i|D_i \cap D)
\end{tikzcd}$$

with $\mathcal{Z}_p \hookrightarrow \mathcal{X}_p$ transverse. It is proper, has a virtual class. Expected properties hold.

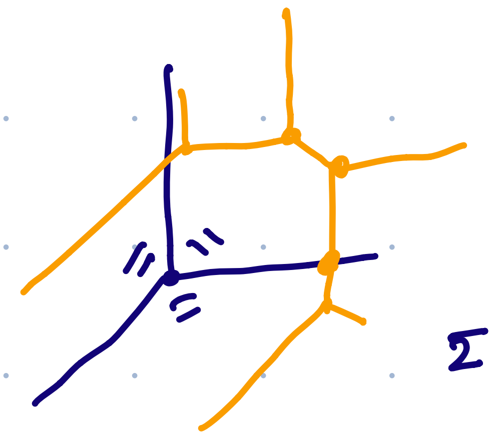
NEW **ABNORMAL**: These guys are not unique, but there is a unique moduli stack on logarithmic schemes.

nevertheless $\longrightarrow$ DT^{LOG} invariants are well defined.

KEY STEP:

$$(X|D) \overset{\text{TROPICALIZATION}}{\rightsquigarrow} \Sigma \rightsquigarrow T(\Sigma)$$

EMBEDDED TROPICAL CURVES in Σ.



TORIC (STACK) DICTIONARY:

TROPICAL CURVE Γ $\longrightarrow$ EXPANSION

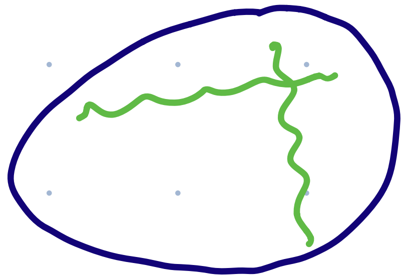
PRACTICALLY: Write down a tropical curve $\Gamma \subseteq \Sigma$ and then a subscheme of the expansion $\mathfrak{X}_\Gamma$

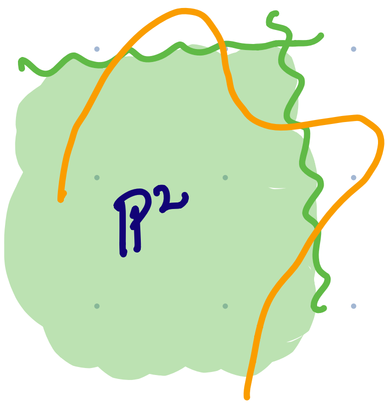
CONSTRUCTION: Build a polyhedral space parametrizing tropical curves. Then use the toric dictionary.

{Olsson, Abramovich-Wise, Cavalieri-Chan-Ulirsch-Wise

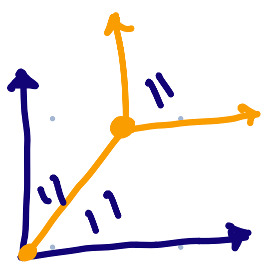
FOR EXAMPLE

$(\mathbb{P}^2 | L_1 + L_2)$

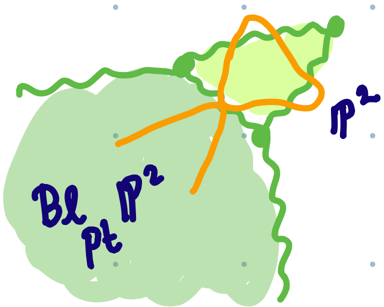




DEGENERATES

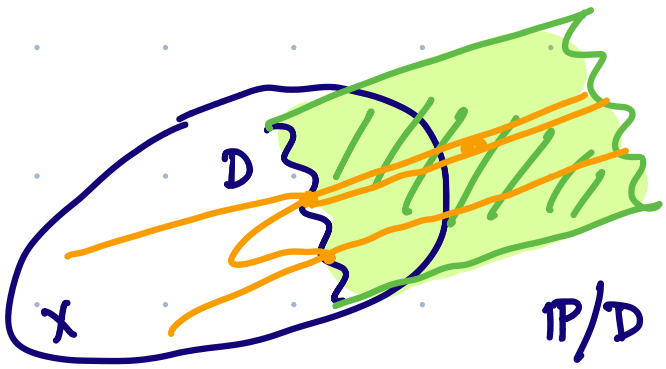


DEFORMATION TO THE NORMAL CONE OF $L_1 \cap L_2$



CONNECTIONS:

If D is smooth $\longrightarrow$ Li-Wu '11



Perfectly parallel to logarithmic GW theory with EXPANSIONS (R '19) but not [ACGS].

$$\begin{array}{ccc} \mathcal{M}^{\log}(X|D) & & \mathrm{Hilb}^{\log}(X|D). \\ \searrow & & \swarrow \\ & \mathrm{Exp}(X|D) & \end{array}$$

$\leftarrow$ Pick any one of an inverse system.

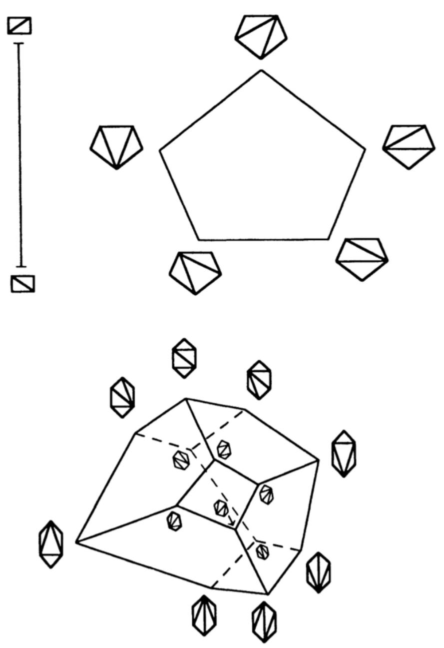
also close to work of Rahul & Aaron and Pierrick

ANOTHER CONNECTION:

If we study divisors on a toric surface X in the linear system L (minimal Euler characteristic)

$$DT^{\log}_L(X, \partial X) \longrightarrow \mathbb{P}_L$$

TORIC MODIFICATION



SECONDARY POLYTOPES

GELFAND KAPRANOV ZELEVINSKY.

also Mikhalkin's theorems & generalizations [forthcoming work of Kennedy-Hunt]

Conjectures & "Evidence"

Degree zero

$$Z_{DT}(X|D;q)_{\beta=0} = M(-q)^{\int_X c_3(T^{\log}\otimes K^{\log})}$$

$$M(q) = \prod \frac{1}{(1-q^n)^n}$$

For $(X|D)$ toric pair this checks out.

Rationality

$$Z_{PT}(q) = Z_{DT}(q) / Z(q)_{\beta=0}$$

&

$Z_{PT}(q)$ is rational.

For $(X|D)$ toric pair Bousseau & (Brett) Parker prove something very close.

GW/DT

there is one, but stating it is subtle; see Maulik–R '21.

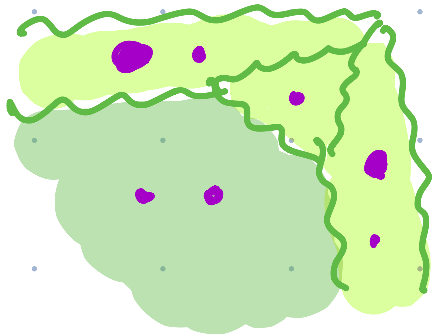
Tangency condition matching

Points:

SNC

$\mathrm{Hilb}^{PTS}(S|E)$

SURFACE



SNC compactification of $\mathrm{Hilb}^{pts}(S\setminus E)$

Evaluations from $\mathrm{Hilb}^{curves}(X|D)$ arrive here.

$CH^\bullet_{log}(\mathrm{Hilb}^{pts}(S\setminus E))$ gives boundary insertions

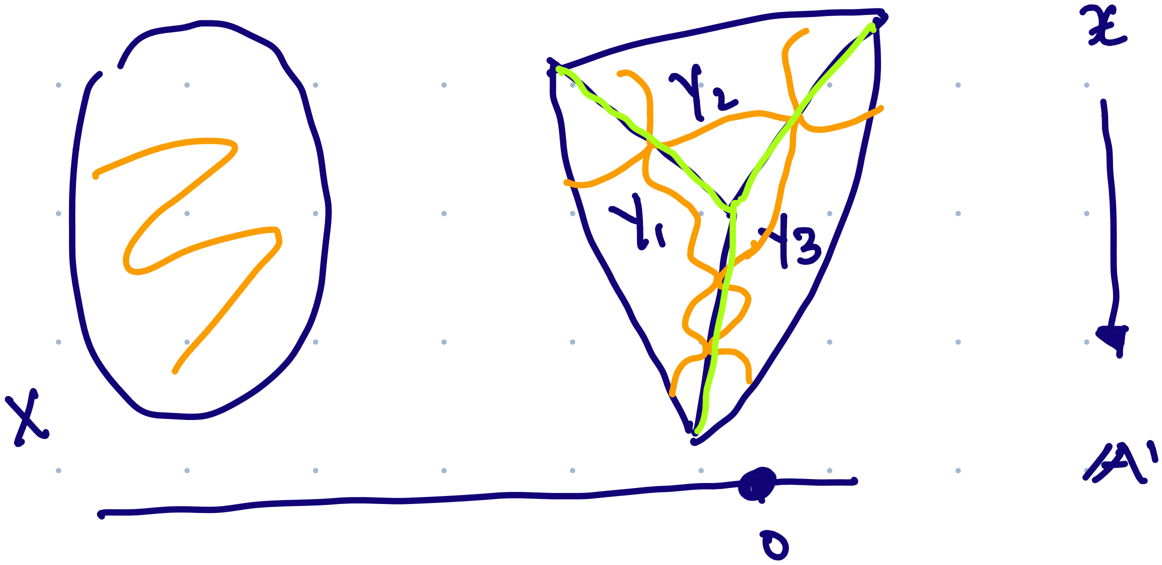
Matching:

$$\begin{tikzcd}
& \mathrm{Hilb}^\bullet(S|E) \arrow[dr] \arrow[rr, leftrightsquigarrow] & & (S|E)\times_{Exp}\cdots\times_{Exp}(S|E) \arrow[dl] \\
\mathrm{Chow}^{log} \arrow[rr, dash] & & \mathrm{Sym}^\bullet(S|E) \arrow[d] & \\
& & \mathrm{Exp}(S|E) &
\end{tikzcd}$$

zero cycles on the fibers of the expansion family

[c.f. Setayesh]

DEGENERATION PACKAGE [IN PROGRESS]



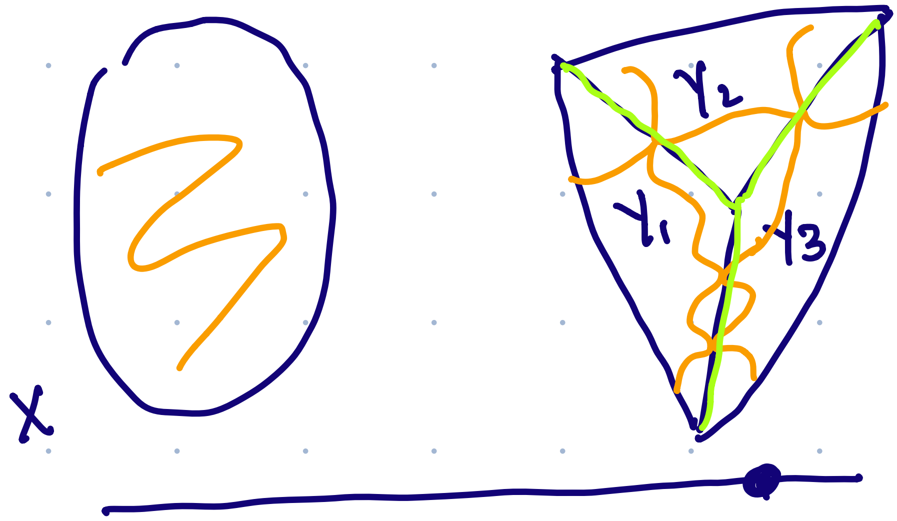
DEFORMATION INVARIANCE: $\quad DT^{LOG}(\mathcal{X}_\eta) = DT^{LOG}(\mathcal{X}_0)$

DECOMPOSITION: $\quad DT^{LOG}(\mathcal{X}_0) = \sum\limits_{\substack{\text{tropical} \\ \text{curves} \\ \gamma \subseteq \Sigma}} DT^{LOG}_{\gamma}(\mathcal{X}_0)$

GLUING: $\quad DT^{LOG}_{\gamma}(\mathcal{X}_0) = \mathop{\ast}\limits_{\substack{\text{vertices} \\ v \text{ of } \gamma}} DT^{LOG}_{v}(Y_v).$

REMARK: This decomposition is numerical!

WE formulate GW/DT & prove it is compatible with...



GOAL: $GW^X_\beta(u) \overset{q=-e^{iu}}{\longleftrightarrow} DT^X_\beta(q)$

SUM OVER TROP's: $\sum_\gamma GW_\gamma(u) \longleftrightarrow \sum_\gamma DT_\gamma(q)$

FOR EACH γ: $GW_\gamma(u) \longleftrightarrow DT_\gamma(q)$

INSIDE THE VERTICES: $GW_v(u) \longleftrightarrow DT_v(q)$

vertices v of γ

vertices v of γ

THANKS !