Cohomology of toric graph associahedra

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Homology and cohomology

- Topology is hard, but algebra is easy $\to$ homology is invented
- topological space $X \to$ sequence of groups $H_0(X), H_1(X), \dots, H_n(X), \dots$
- Loosely, $H_n(X)$ tells us how many n- dimensional holes X has; e.g.

$$H_0(X) \cong \mathbb{Z}^{\#\text{connected components}}$$

- For nice X, homology $H^*(X)$ is obtained by putting a ring structure on

$$H_0(X) \oplus H_1(X) \oplus \cdots \oplus H_n(X) \oplus \cdots$$

Homology example

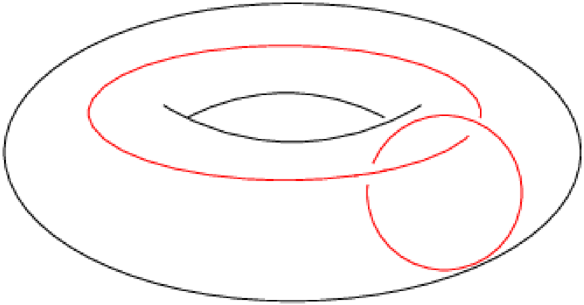


Figure: $H_0(T) = \mathbb{Z}, H_1(T) = \mathbb{Z}^2, H_2(T) = \mathbb{Z}$ and the others are trivial

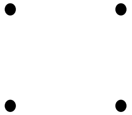
Goal of the project

- Introduce a recipe that takes a graph G on $n+1$ vertices and associates a topological space $X(G)$ of dimension $2n$ to it.

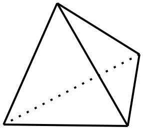
- Understand the ring $H^*(X(G))$ and how it varies based on the graph G.

Toric graph associahedra

We have a recipe for taking graphs $\rightarrow$ polytopes $\rightarrow$ spaces



disconnected graph
on $n+1$ vertices

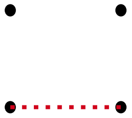


n-simplex

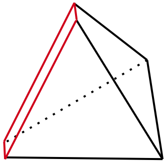
$\mathbb{CP}^3$

complex projective
space of dimension n

Toric graph associahedra



adding an edge



truncating the corresponding face

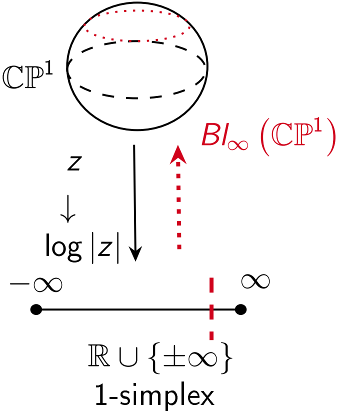
blowup of $\mathbb{CP}^3$

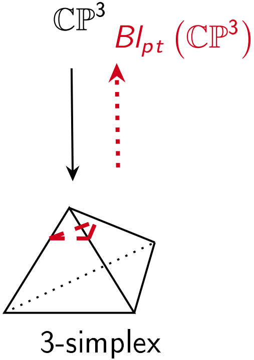
blowing up the projective space

- In general, toric geometry associates a space to each polytope; roughly, to a face of dimension k we associate $\mathbb{C}^k \times \mathbb{C}^{n-k}$ and we glue them together according to inclusion of faces.

Complex projective space and blowups

- *Complex projective space* $\mathbb{CP}^n$ is defined as $(\mathbb{C}^{n+1} - \{0\})/\sim$, where $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for any $\lambda \in \mathbb{C}^*$



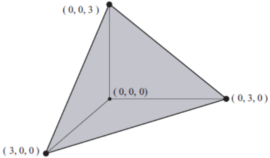


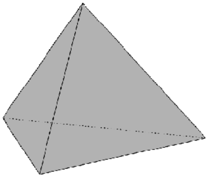
Important examples of graph associahedra

disconnected graph

⇩

simplex





complete graph

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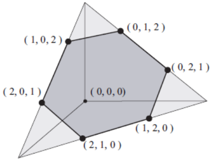
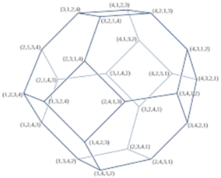
permutohedron





path graph

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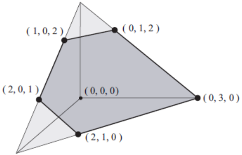
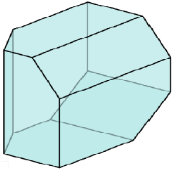
associahedron





Matroids

Definition

Matroid is a *nice* cohomology class of permutohedron - nice means simple and irreducible.

- Cohomology of permutohedron is huge! Direct sum of all $H^k(X(G))$ adds up to the number of vertices, which is $(n+1)!$

Fact

A graph on $n+1$ vertices induces a subring of cohomology of permutahedron via the previous construction.

Goal: Characterise all matroids which 'fit' into our particular graph.

Matroids in combinatorics

Motivation: Given a set of vectors, we can keep track of which subsets are independent. Axiomatising this yields the following notion of a matroid:

Definition

Matroid is defined to be $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a family of subsets of E (called the independent sets) satisfying:

(1) $\emptyset \in \mathcal{I}$;

(2) If $A' \subseteq A \in \mathcal{I}$ then $A' \in \mathcal{I}$;

(3) If $A, B \in \mathcal{I}$ and $|A| > |B|$, then there exists $x \in A - B$ s.t. $B \cup \{x\} \in \mathcal{I}$.

Note that this is **purely combinatorial**!

In fact, matroids generalize both graphs and notion of linear independence.

Goals of the project

We want an explicit **combinatorial** criterion for when a matroid is compatible with a particular graph.

In particular, we were interested in graphs of the form $K_n - K_m$, bipartite graphs...

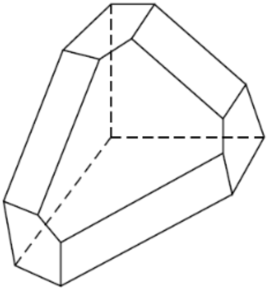


Figure: Stellahedron - graph associahedron of a star graph

Goals of the project

Question: What is the 'smallest' graph containing a subring generated by all matroids? What about all graphic matroids?
Answer: Complete graph - all blowups are necessary!

Theorem

The smallest graph containing all simple (graphic) matroids, is precisely the one obtained from K_n by deleting $\lfloor \frac{n}{2} \rfloor$ disjoint edges. ***Not all blowups are necessary!***

Goals of the project

Question: What is smallest graph containing all matroids *up to symmetry*?

Theorem

For all n large enough, graph $K_n - K_3$ is compatible with all matroids up to symmetry.

Conjecture

For all n large enough, graph $K_n - K_m$ is compatible with all (graphic) matroids up to symmetry.

Lastly...

Thank you to my supervisor for giving me an opportunity to work on such an interesting project and for all the advice and support I have gotten this summer!

Questions?