A story of roots & logarithms in Gromov–Witten theory

Abramorama

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On numerous occasions since 2013, I have observed Abramovich make the following claim:

"I don't do Gromov–Witten theory."

Despite many experts refuting this over the years, he continues to make the claim.

the claim is false for trivial reasons...

1. arXiv:1709.09864 [pdf, ps, other] math.AG

Decomposition of degenerate Gromov-Witten invariants

Authors: Dan Abramovich, Qile Chen, Mark Gross, Bernd Siebert

Abstract: We prove a decomposition formula of logarithmic Gromov-Witten invariants in a degeneration setting. A one-parameter log smooth family X->B with singular fibre over b_0 \in B yields a family M(X/B,β) -> B of moduli stacks of stable logarithmic maps. We give a virtual decomposition of the fibre of this family over b_0 in terms of rigid tropical curves. This generalizes one aspect of known results in... ▽ More

Submitted 2 June, 2020; **v1** submitted 28 September, 2017; **originally announced** September 2017.

Comments: 63 pages, major revision of v2: streamlined exposition with the core results rewritten with simplified and cleaner proofs (now in §2.4-2.6 and §3). To appear in Compositio Math. Accepted version

MSC Class: 14N35 (14D23)

2. arXiv:1306.1222 [pdf, ps, other] math.AG

Birational invariance in logarithmic Gromov-Witten theory

Authors: Dan Abramovich, Jonathan Wise

Abstract: Gromov-Witten invariants have been constructed to be deformation invariant, but their behavior under other transformations is subtle. In this note we show that logarithmic Gromov-Witten invariants are also invariant under appropriately defined logarithmic modifications.

Submitted 1 June, 2017; **v1** submitted 5 June, 2013; **originally announced** June 2013.

Comments: 26 pages; exposition expanded and several arguments simplified

MSC Class: 14H10; 14D23; 14M25; 14N35; 14T05

3. arXiv:1207.2085 [pdf, ps, other] math.AG

Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations

Authors: Dan Abramovich, Steffen Marcus, Jonathan Wise

Abstract: ...consider four approaches to relative Gromov-Witten theory and Gromov-Witten theory of degenerations: Jun Li's original approach, Bumsig Kim's logarithmic expansions, **Abramovich**-Fantechi's orbifold expansions, and a logarithmic theory without expansions due to Gross-Siebert and **Abramovich**-Chen. We exhibit mo... ▽ More

Submitted 24 May, 2013; **v1** submitted 9 July, 2012; **originally announced** July 2012.

Comments: 42 pages. Some minor changes. To appear in Annales de l'Institut Fourier

MSC Class: 14N35; 14H10; 14D23; 14D06; 14A20

4. arXiv:1004.0981 [pdf, ps, other] math.AG

Relative and orbifold Gromov-Witten invariants

Authors: Dan Abramovich, Charles Cadman, Jonathan Wise

Abstract: We prove that genus zero Gromov--Witten invariants of a smooth scheme relative to a smooth divisor coincide with genus zero orbifold Gromov--Witten invariants of an appropriate root stack construction along the divisor.

Submitted 19 April, 2015; **v1** submitted 6 April, 2010; **originally announced** April 2010.

Comments: completely rewritten, with slightly stronger results; 36 pages; comments welcome!

MSC Class: 14N35; 14D23; 14D15

5. arXiv:math/0603151 [pdf, ps, other] math.AG math.SG

Gromov-Witten theory of Deligne-Mumford stacks

Authors: Dan Abramovich, Tom Graber, Angelo Vistoli

Abstract: Long ago, in math.AG/0112004, we pledged more details on the algebraic version of Chen-Ruan's math.AG/0103156. This is it.

Submitted 13 April, 2008; **v1** submitted 6 March, 2006; **originally announced** March 2006.

Comments: 59 pages, 9 sections, 3 appendices and one figure. Several minor improvements

MSC Class: 14N35; 14A20; 14H10

6. arXiv:math/0512372 [pdf, ps, other] math.AG

Lectures on Gromov-Witten invariants of orbifolds

Authors: Dan Abramovich

Abstract: These are lecture notes of a C.I.M.E. course I gave at Cetraro, June 6-11 2005. The theory described is the version of Chen-Ruan's Gromov-Witten theory of orbifolds developed by Graber, Vistoli and me in the algebraic setting, but with introduction beginning in Kontsevich's formula on rational plane curves and through Gromov-Witten theory of algebraic manifolds. As this is not a joint paper, I... ▽ More

Submitted 4 January, 2006; **v1** submitted 15 December, 2005; **originally announced** December 2005.

Comments: A confusing typographical error fixed (thanks to Eric Katz)

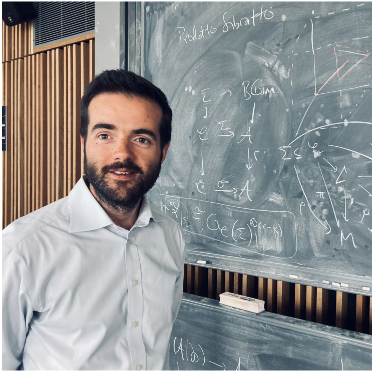
MSC Class: 14N35; 14A20; 53D45

but anyway...

Let us discuss the details of some of these counterexamples in a story of roots and logarithms in Gromov–Witten theory.

Luca & Navid

The original results presented are joint with Battistella and Nabijou. Mostly from a 2022 paper and work-in-progress.



from ICMS in Edinburgh in 2022.

Given a projective manifold X, we study X via an auxiliary space

$$X \rightsquigarrow \mathsf{M}(X):$$

a moduli of objects on X.

Virtual Enumerative Invariants. Extracted from $\mathsf{M}(X)$ by integrating natural cohomology classes against the virtual class.

Many Possible Choices! Hilbert scheme of subschemes in X. Moduli of sheaves on X. Moduli of maps from curves to X.

The choices lead to different "enumerative theories"... GW, DT, PT, GV etc.

The rest of the talk is about moduli of (stable) maps from nodal curves to X.

For today, we fix $D \subset X$ a simple normal crossings divisor and we imagine:

Model Problem. Count curves C in X with prescribed tangency orders with components of D.

Precisely, study moduli $\mathsf{M}^{\circ}_{\Lambda}(X|D)$ of maps of pairs:

$$(C, p_1, \ldots, p_n) \xrightarrow{f} (X, D), \quad D = \sum D_i$$

from smooth pointed curves to X, and fix Λ

- the genus g of C,
- the curve class β for $f_\star[C]$ in $H_2(X)$,
- matrix of tangency orders c_{ij} of p_i along D_j.

Today's Story. Two different systems of virtual enumerative invariants modeling this problem. Features, bugs, comparisons, conjectures, applications.

what are the subtleties?

Let $\mathsf{M}^\circ_\Lambda(X|D)$ be the moduli space of maps to (X, D) with fixed numerical data Λ.

For virtual enumerative geometry we need two things:

- **Compact Moduli.** A space $\mathsf{M}_\Lambda(X|D)$ compactifying $\mathsf{M}^\circ_\Lambda(X|D)$.
- **Virtual Classes.** Reasonable deformation theory for objects in $\mathsf{M}_\Lambda(X|D)$.

If we ignore tangency, then a perfect solution is provided by Kontsevich via the theory of stable maps.

The space $\mathsf{M}^\circ_\Lambda(X|D)$ is only locally closed in stable maps $\mathsf{M}_{g,n}(X, \beta)$. Closure is unworkable. Tangency not deformation open.

In 1998 Abramovich–Vistoli announced their theory of orbifold stable maps. Put orbifolds on equal footing with projective manifolds for stable maps.

Key Idea. For compact moduli of maps from nodal curves to an orbifold $\mathcal{X}$, the nodal curves should be allowed orbifold structure at nodes and markings.

We can assume everything works, but only after considerable effort. In fact:

"To prove this, Vistoli and I had to go through several chambers of hell."

(Dan in Lecture notes from Cetraro, 2005)

But what is the relevance to tangency?

In 2003 Cadman used orbifolds to study tangent curves. Interested in maps:

$$(C, p) \to (X, D),$$

with C tangent along D at p to order e. What does this look like? The model is:

$$\mathbb{A}^1_s \to \mathbb{A}^1_t, \quad \text{via} \quad s^e \leftarrow\!\shortmid t.$$

Key Idea. For large r, add r^{th} roots to t and s. Now investgate action of μ_r.

$$\begin{tikzcd}
\mathbb{A}^1_{\tilde{s}} \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\text{ramified}"'] & \mathbb{A}^1_{\tilde{t}} \arrow[d] \arrow[dd, bend left=60, "\text{ramified}"] \\
{[\mathbb{A}^1_{\tilde{s}}/\mu_r]} \arrow[r, "f"] \arrow[d] & {[\mathbb{A}^1_{\tilde{t}}/\mu_r]} \arrow[d] \\
\mathbb{A}^1_s \arrow[r] & \mathbb{A}^1_t.
\end{tikzcd}$$

Crucially the data of f includes a group homomorphism:

$$\mu_r \to \mu_r, \quad \text{via} \quad \zeta \mapsto \zeta^e \rightsquigarrow \text{locally constant!}$$

tangency via orbifolds: summing up

We have (X, D) and we've fixed discrete data Λ:

- the genus g of C and number n of markings,
- the curve class $f_\star[C]$ in $H_2(X)$,
- matrix of tangency orders c_{ij} of p_i along D_j.

Fix $\underline{r} = (r_1, \ldots, r_k)$ large rooting parameters for $D_1, \ldots, D_k$.

Vistoli teaches us to build the root stack $\mathcal{X}$, universal where D_i has a r_i^{th}-root.

Cadman teaches us to use the homomorphisms on isotropy to encode tangency.

Abramovich–Vistoli give us a moduli space $\mathrm{Orb}_\Lambda(X|D, \underline{r})$.

Abramovich–Graber–Vistoli give us virtual enumerative invariants. Modelled on curves in X tangent to D_i.

logarithmic geometry: very quickly!

In logarithmic geometry every space comes with a notion of monomial function.

Toric varieties have natural monomials. A choice of snc $D \subset X$ gives monomials.

The product of two monomials is a monomial, but the sum is not!

Basic Idea. Add to $(X, \mathcal{O}_X)$ a "sheaf of monoids" M_X that records monomials.

Warining! There can be more monomials than polynomials!

Reasonableness condition: locally monomials come from maps to toric varieties.

Logarithmic geometry is the inevitable conclusion of this line.

In 2011 Abramovich–Chen and Gross–Siebert established logarithmic stable maps completing Siebert's program from 2001.

Here is roughly how it goes:

The stack $\mathfrak{M}_{g,n}$ of prestable curves has a natural logarithmic structure. So does its universal curve. Immediately gives a notion of logarithmic curves.

Given (X, D), with natural logarithmic structure, can consider families of logarithmic maps from curves over a logarithmic base.

Key Result. Logarithmic maps from logarithmic curves to (X, D) are parameterized by an algebraic stack $\mathfrak{M}_{g,n}(X, D)$ with logarithmic structure. Stability condition gives a Deligne–Mumford stack.

tangency via logarithms

What do logarithmic structures have to do with tangency?

We are interested in maps:

$$(C, p) \to (X, D),$$

with C tangent along D at p to order e. Induced maps of "monomials up to units":

$$\mathbb{A}^1_s \to \mathbb{A}^1_t, \quad \text{via} \quad \mathbb{N}_s \xleftarrow{\cdot e} \mathbb{N}_t.$$

Key Feature. In logarithmic geometry. even when the scheme theoretic tangency is nonsensical e.g. if C maps into D, the monoid data must be specified.

Again, crucially there is a monoid homomorphism:

$$\mathbb{N}_s \xleftarrow{\cdot e} \mathbb{N}_t \rightsquigarrow \text{locally constant!}$$

Assume we work in genus zero and the pair (X, D) is

$$X = \mathbb{P}^r, \quad D = H = \text{union of } r+1 \text{ or fewer hyperplanes}.$$

My Favourite Example. The space $\mathsf{Log}_\Lambda(\mathbb{P}^r, H)$ is the normalization of the closure of

$$\mathsf{M}_\Lambda(\mathbb{P}^r|H) \subset \mathsf{M}_{0,n}(\mathbb{P}^r, d)$$

in the usual space of stable maps,

If $(Y, E) \hookrightarrow (\mathbb{P}^r, H)$ is an "snc embedding" $\mathsf{Log}_\Lambda(Y, E)$ is the locus in $\mathsf{Log}_\Lambda(\mathbb{P}^r, H)$ of curves that land in Y

tangency via logarithms: summing up

We have (X, D) and we've fixed discrete data Λ:

- the genus g of C and number n of markings,
- the curve class $f_\star[C]$ in $H_2(X)$,
- matrix of tangency orders c_{ij} of p_i along D_j.

Tangency of C at p_i with D_j always makes sense, and is locally constant. Again, more monomials than polynomials!

Abramovich–Chen and Gross–Siebert tell us we can build a space $\mathrm{Log}_\Lambda(X|D)$ of logarithmic stable maps.

They also give us virtual enumerative invariants.

The two theories have very different natures.

Orbifold theory. Packaged by quantum cohomology and CohFTs. Equivariant localization is main computational tool. Givental formalism, Virasoro constraints, etc.

Perfect parallel to traditional Gromov–Witten theory. On the other hand...

Logarithmic theory. Expected link to symplectic cohomology and mirror symmetry. Combinatorics of tropical curves and the degeneration formula gives main computational tools. Currently no localization, Givental formalism, or Virasoro constraints.

Hard to argue “better vs. worse” but for studying tangent curves, logarithmic theory is more efficient and actually counts curves more often.

comparing roots and logarithms

In 2007, Cadman and Chen calculated invariants for the geometry $(\mathbb{P}^2, E)$ in genus zero. Computations showed logarithmic and orbifold invariants coincide!

In 2010, Abramovich–Cadman–Wise proved conceptually that for (X, D) with D smooth that genus zero orbifold and logarithmic invariants always coincide.

Pivotal Insights. In doing so, and later with Fantechi and Marcus, they introduced a host of fundamental techniques, including the prototype for Artin fans.

Led to beautiful connection to tropical geometry in work of Ulirsch and others.

The theorem of Abramovich–Cadman–Wise has two hypotheses: (i) the curves have genus zero and (ii) D is smooth. Orthogonal sources of complexity.

Two Counterexamples. The theorem is sharp:

- In 2010 Maulik found examples in higher genus with D smooth, where logarithmic and orbifold invariants do not coincide.
- In 2020 Nabijou–R found examples where D has two components in genus zero, where logarithmic and orbifold invariants do not coincide.

The failures are ubiquitous, not pathological. Higher genus failure "morally" attributed to torsion in Picard group.

Combining work of Janda–Pandharipande–Pixton–Zvonkine and Tseng–You resolves the higher genus problem when D is smooth.

Higher Genus. Fix tangency data and assume D is smooth. Then for large enough r orbifold invariant is a polynomial in r. The constant term is equal to the logarithmic invariant.

Intimately tied to geometry of the double ramification cycle.

simple normal crossings

In 2013 Abramovich–Wise proved that logarithmic invariants are insensitive to strata blowups of (X, D).

With Battistella–Nabijou we prove:

Simple Normal Crossings. Fix tangency data and assume genus zero. After a sufficiently fine sequence of strata blowups

$$(X', D') \to (X, D),$$

the orbifold invariant of (X', D') becomes equal to the logarithmic invariant of (X', D') and therefore also of (X, D).

We conjecture the "obvious" combination of results deals with higher genus and snc divisors. Missing a couple of key pieces.

Proof uses one of the most powerful techniques in logarithmic geometry:

Ask yourself: what would Dan & Jonathan do?

Replace (X, D) with the universal snc pair, namely the stack $[\mathbb{A}^r/\mathbb{G}_m^r]$. The problem becomes combinatorial.

Key Aspect. Orbifold and logarithmic maps give tropical maps, i.e. maps from metric graphs Γ:

$$\Gamma \to (\mathbb{R}_{\geq 0} \cup \{\infty\})^r.$$

Metric on Γ may be singular. Tropicalization of a log map is always smoothable.

Key to the proof is showing that after blowups, same is true in orbifold world. Significant inspiration from Abramovich–Karu and Molcho on semistable reduction.

Moving from the logarithmic side to the orbifold side gives new techniques.

Reconstruction Theorem. The genus zero logarithmic invariants of (X, D) are uniquely determined by the ordinary Gromov–Witten invariants of the strata, i.e. of X, of the D_i, and all their intersections.

For D smooth, same was proved in all genus by Maulik–Pandharipande (2006).

Result has purely to do with logarithmic theory. Proof passes through the orbifold comparison.

Beautiful application is to mirror symmetry and comes from work of Johnston, giving a new proof of groundbreaking work of Gross–Siebert:

Associativity of Mirror Rings. The degree 0 symplectic cohomology ring of a logarithmic Calabi–Yau pair is associative.

Symplectic cohomology is a beautiful invariant of non-compact symplectic manifolds.

The ring $SH^0(X)$ is the ring of functions on the mirror manifold $\check{X}$.

Johnston deduces this by applying orbifold WDVV to logarithmic invariants.

Essentially uses the logarithmic/orbifold comparison to the mirror ring to orbifold quantum cohomology.

thanks!

And a very happy birthday Dan!

(Photograph in Zurich in May '23 taken by Sam Molcho, who refused to be in it!)