DONALDSON — THOMAS THEORY
&
the SECONDARY POLYTOPE

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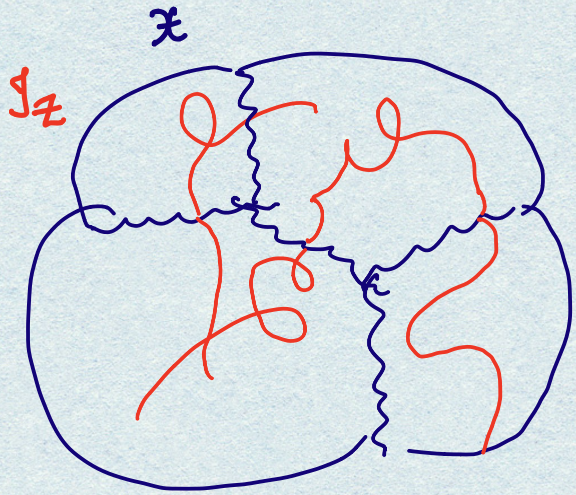
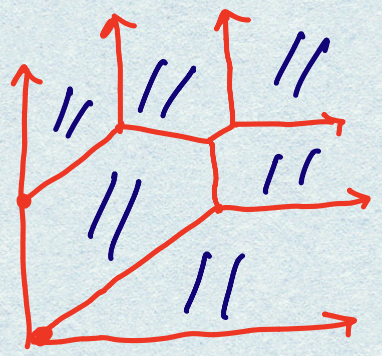
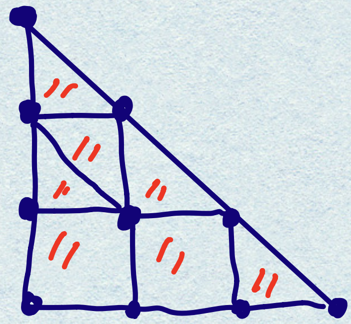
(MIT)





$\Sigma' \longrightarrow \mathbb{R}^2_{\geq 0}$



DT theory:

$$X \overset{\text{Moduli}}{\rightsquigarrow} \mathrm{Hilb}(X)_{\chi,\beta} \overset{\text{Calculus.}}{\rightsquigarrow} \text{Numbers}$$

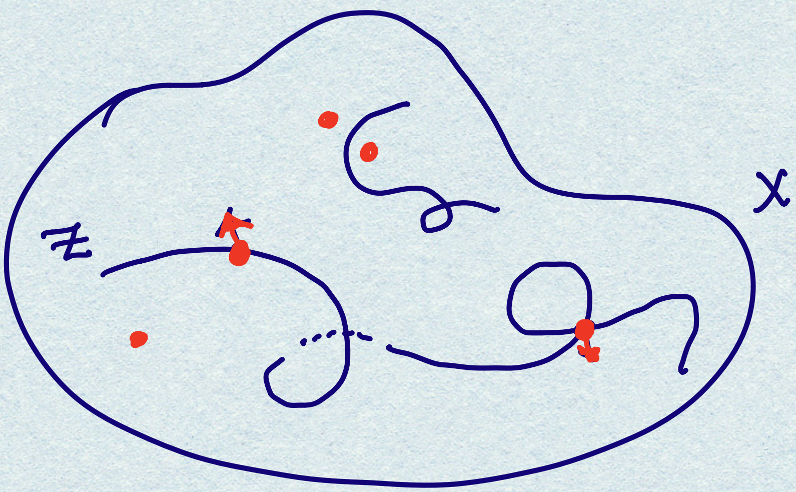
Projective surface/3-fold

"Primary"

Curves in X

"secondary"

DT invariants.



- subscheme of dimension 1 $\longleftrightarrow$ $\mathcal{I}_Z$ an ideal.

singular, non-reduced, embedded points, etc

$$\left.\begin{array}{l} \chi(\mathcal{O}_Z) = \chi \quad \text{is fixed} \\ [Z] \in H_2(X;\mathbb{Z}) \quad \text{is fixed} \end{array}\right\} \text{Genus \& Degree}$$

DT invariants: $\mathrm{Hilb}(X)_{\chi,\beta}$ should have dimension $-K_X \cdot \beta$

degree d in $\mathbb{P}^3 \longleftrightarrow$ should be $4d$.

Do calculus via "Universal structures"

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[dr, "\pi_1"'] & \mathrm{Hilb}(X)_{\chi,\beta} \times X \arrow[r, "\mu"'] \arrow[d, "\pi_2"] & X \\
& \mathrm{Hilb}(X)_{\chi,\beta} &
\end{tikzcd}$$

"Virtual fundamental class" extracts numbers from cohomology classes of degree $\underbrace{-K_X \cdot \beta}$.

Surfaces are nicer: n points on curves

$$\begin{tikzcd}
& & \mathcal{H}_d^{[n]} \arrow[r, squiggly, "\text{Calc. II}"] & \text{"DT" Invariants} \\
\mathbb{P}^2 \arrow[r, squiggly, "①"'] & \mathcal{H}_d \arrow[ur, squiggly, "②"'] \arrow[urr, squiggly, "\text{Calculus I}"'] & &
\end{tikzcd}$$

Degree d plane curves

Why is DT theory fun? The answers

- S a surface,

$$\sum_n \chi(\mathrm{Hilb}^n(S))\, q^n = \left(\prod \frac{1}{(1-q^m)}\right)^{\chi(S)}.$$

[Göttsche]

- X a 3-fold, "virtual fundamental class"

gives $\mathrm{Hilb}^n(X) \rightsquigarrow DT(X)_{n,0}$

$$\sum_{n\geq 1} DT(X)_{n,0}\, q^n = \left(\prod_{m\geq 1} \frac{1}{(1-(-q)^m)^m}\right)^{c_3(K_X \otimes T_X)}$$

["MNOP" + Levine-Pandharipande + others]

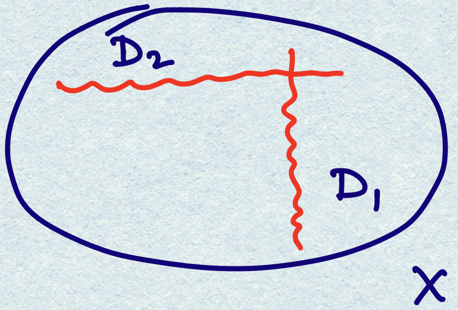
Logarithmic DT theory:

X smooth ; $D \subseteq X$ SNC

$(\mathbb{P}^3 | H)$

$(\mathbb{P}^2 | \partial \mathbb{P}^2)$

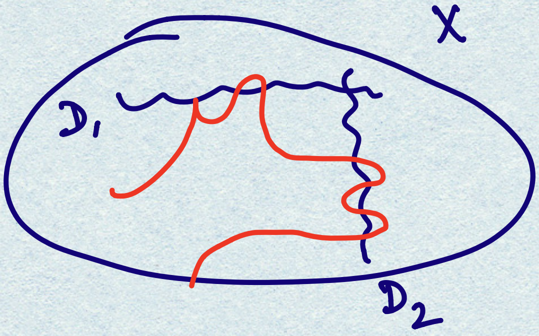
$D = D_1 \cup D_2$



$(X|D) \rightsquigarrow \mathrm{Hilb}^{\log}(X|D)_{\chi,\beta} \rightsquigarrow$ Log DT invariants.

$$\Big\downarrow \text{ open in } \mathrm{Hilb}(X|D)$$

$\mathrm{Hilb}^{\circ}(X|D)$ "nice" locus:



$$\begin{tikzcd}[row sep=small] & \mathrm{Hilb}^{pts}(D_1) \\ \mathrm{Hilb}^{\circ}(X)_{\chi,\beta} \arrow[ur, "- \cap D_1"] \arrow[dr, "- \cap D_2"'] & \\ & \mathrm{Hilb}^{pts}(D_2). \end{tikzcd}$$

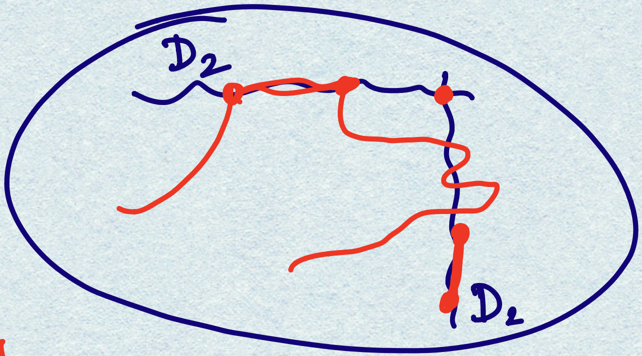
[★]

We use this to make sense of "tangency" to D_1 & D_2

Log DT theory: • Construct a compactification of this nice locus extending [★]

What is the basic geometric difficulty?

Curves fall into the boundary. In the usual Hilbert scheme



Need

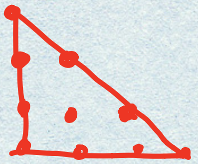
$$\boxed{\mathrm{Hilb}(X)} \leftarrow \mathrm{Hilb}^{\circ}(X|D) \hookrightarrow \boxed{\mathrm{Hilb}^{\log}(X|D)_{\chi,\beta}}$$

Not good

No chance of a map to Hilbert scheme of points

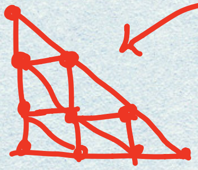
The solution to this problem involves a lot of pretty tropical combinatorics.

Where's the secondary polytope?

- P a lattice polytope 

- Q = Secondary(P)

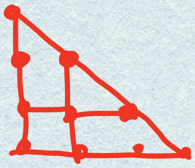
  $\underbrace{\text{(Coherent)}}$ subdivisions of the primary polytope



Vertices ⟷ Triangulations

Faces ⟷ More general subdivision

{ + a bit more data }

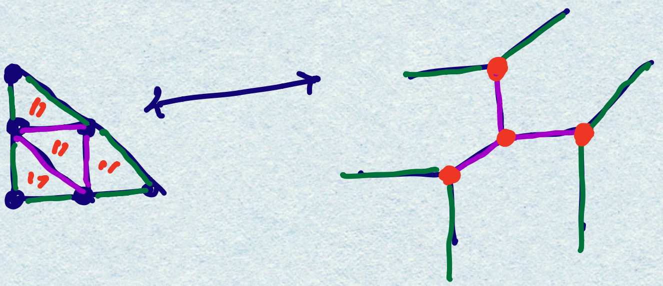


THM (GKZ) The space of subdivisions of P is also a lattice polytope.

cf. The space of subschemes is a scheme.

Tropical geometry

Coherent subdivisions of $P \subseteq M_{\mathbb{R}}$ $\rightsquigarrow$ Tropical curves inside $N_{\mathbb{R}}$



$\Sigma_P \overset{\text{Moduli}}{\rightsquigarrow} \Sigma_Q \rightsquigarrow$?

Toric surface fan

"Primary"

tropical curves in Σ_P

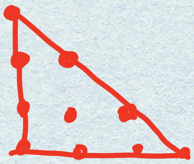
"secondary"

First connection: The secondary polytope is already analogous to the Hilbert scheme.

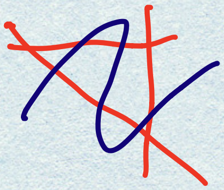
$\Big\{$ Both are about embedded objects.

Toric geometry:

The toric variety of the secondary polytope:

$\mathcal{P}$  $\rightsquigarrow \quad \mathbb{P}^9 \supseteq (\mathbb{C}^*)^9$

All monomials have nonzero coefficient.

 $\mathbb{C}^*$

Coefficient up to scaling (≃ Hilbert scheme.)

$(\mathbb{C}^*)^2 \circlearrowright X_{\mathcal{P}} \cong \mathbb{P}^2$; also act on $\mathbb{P}^9$

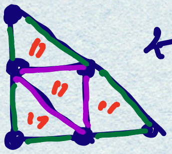
Thm (GKZ) The toric variety X_Q is the ("best") Chow quotient $\mathbb{P}^9 /\!\!/_{Ch} (\mathbb{C}^*)^2$

also: Billera-Sturmfels.

$$\begin{tikzcd} \mathbb{P}^9 \arrow[r, dashed, leftrightarrow] & \mathcal{Y} \arrow[d, "\text{family of surfaces.}"] \\ & X_Q \end{tikzcd}$$

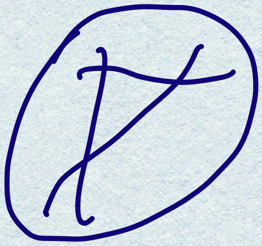
fibers over boundary are "broken" surfaces.

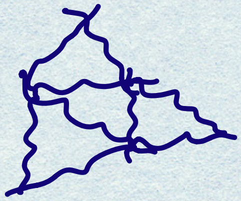
Who sees what?

Given  $\longleftrightarrow$ 

• Toric person: A toric degeneration associated to

via the "toric dictionary"





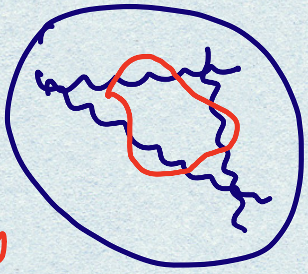
4 surfaces

These are the orbits in the Chow quotient

• Tropical person: a curve in $(\mathbb{C}\{\{t\}\}^{\times})^2$ of degree 2 with this tropicalization.

Lesson: tropicalizations know how curves "want" to degenerate

Setup: $(\mathbb{P}^2 | \partial \mathbb{P}^2)$



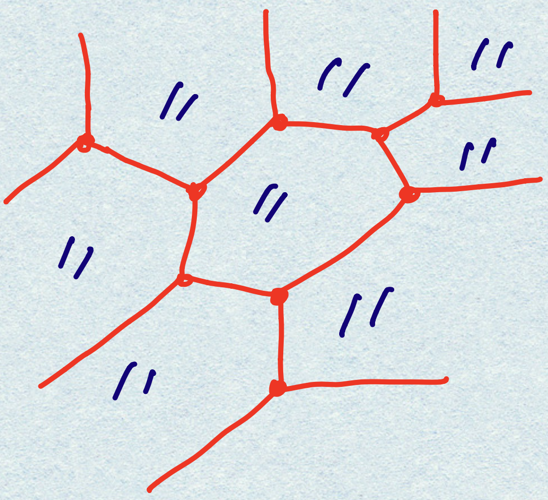
$\mathrm{Hilb}^{\circ}(\mathbb{P}^2 | \partial \mathbb{P}^2)$ nice locus

Take a family over $\mathbb{C}((t))$ of such objects.

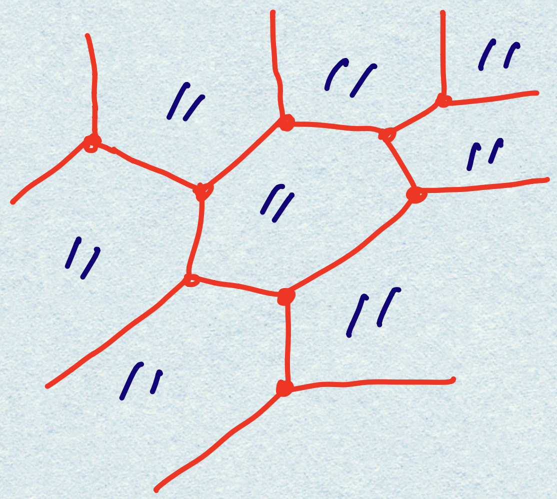
- How to compactify?

- Tevelev says: ① Compute the tropicalization:

$$\begin{array}{ccccc} Z_\eta & \hookrightarrow & \mathbb{P}^2_{\mathbb{C}((t))} & & \\ \cup\!\shortmid & & \cup\!\shortmid & & \\ Z^{\circ}_\eta & \hookrightarrow & (\mathbb{C}((t))^{\times})^2 & \xrightarrow[\text{trop}]{} & \mathbb{R}^2 \end{array}$$



② Turn this picture



9 toric surfaces glued together.

Into a degeneration of $\mathbb{P}^2$

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ \operatorname{Spec} \mathbb{C}[[t]] \end{array} \left\{ \begin{array}{l} \text{and take the limit.} \\ \text{limit in } \mathcal{Y} \text{ of} \\ \text{the curve } Z_\eta. \end{array} \right.$$

Output: Tropical geometry tells you to study:

- A tropical curve $\Gamma \hookrightarrow (X|D)^{trop}$ ⇜ $\mathbb{R}^2$
- Induced degeneration $\mathcal{Y}_\Gamma$ of $(X|D)$ ⇜ $\mathbb{P}^2$
- A subscheme inside $\mathcal{Y}_\Gamma$. w/ "nice" condition.

Main Result: ("Logarithmic DT theory") we construct

these Logarithmic Hilbert schemes.

secondary toric variety

Algebraic world

Hilbert scheme

... and moduli of 1-dim subschemes of degenerations

Tropical world

Secondary polytopes

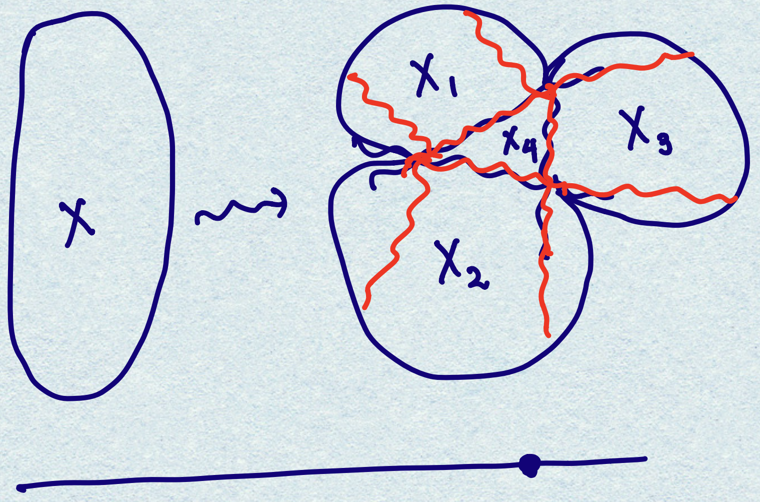
... moduli of 1-dim embedded polyhedral complexes in a fan

• In general the moduli of embedded tropical curves that we construct are much more general but they have fewer good properties. The secondary polytope remains the GOLD standard!

Some basic theorems: • New versions of old formulas

• Degeneration formalism

If we degenerate some $X \rightsquigarrow \bigcup (X_i | D_i)$



This determines a fan Σ_X and

$$DT(X) = \sum_{\substack{\gamma : \text{ tropical} \\ \text{curves in } \Sigma_X}} \text{“}\mathop{\ast}_i\text{''}\, DT^{\log}(X_i | D_i)$$

In favourable situations the inside term collapses to something combinatorial.

→ "Correspondence theorems" (Mikhalkin, Nishinou-Siebert, et al)

Generalized secondary fans

Recall 2 invariants χ and β from before.

- A degree d plane curve C has arithmetic genus given by $\binom{d-1}{2} \longleftrightarrow \chi(\mathcal{O}_C)$.

- To change $\chi(\mathcal{O}_C)$ we study relative Hilbert schemes of points:

$$\mathcal{C}/\mathcal{H} \rightsquigarrow \mathrm{Hilb}^n_{\log}(\mathcal{C}/\mathcal{H}).$$

Combinatorially: higher Euler characteristic versions of the secondary fan

$$\begin{tikzcd} & & \Gamma \arrow[r, hook] \arrow[d] & \mathbb{R}^2 \times \Sigma_Q \arrow[dl] \\ P \arrow[r, squiggly] & Q \arrow[r, squiggly] & \Sigma_Q & \end{tikzcd}$$

Primary   Secondary

moduli of n points on Γ

What is the object of study?

If Σ is a fan $PP^*(\Sigma)$ is the ring of piecewise polynomial functions on Σ.

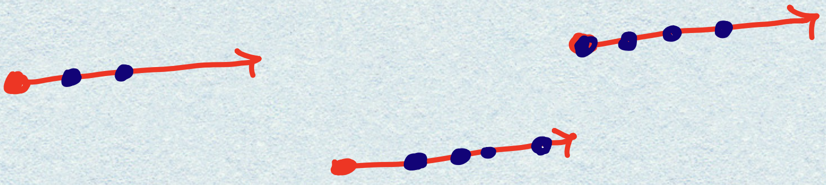
So what is $PP^*(\Sigma_Q)$

$PP^*(\Sigma_Q^{[n]})$

$PP^*(\Sigma)$ gives rise to universal relations in log DT theory.

No idea. But surely this is beautiful.

Example: Take $\Sigma = \mathbb{R}_{\geq 0}$ & study the "fan" of $\leq n$ unlabelled tropical points on Σ:



$PP^*(\Sigma^{[n]}) \simeq$ Quasi symmetric polynomials in n variables.

Hopf algebra structure