The cycle of curves in a toric variety

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Premise

Recent direction in the study of $\overline{\mathcal{M}}_{g,n}$: **logarithmic intersection theory**.

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Motivation from **logarithmic Gromov–Witten theory** and the study of curve counting using tropical and degeneration methods.

The case study for the day: **higher double ramification cycles**, i.e. cycles in $\overline{\mathcal{M}}_{g,n}$ of curves admitting a map to a given toric variety.

People

Main results today are work with Samouil Molcho (ETH Zürich) in our paper "A case study of intersections on blowups of the moduli of curves".

Connections to enumerative geometry are work with Renzo Cavalieri (Colorado State) and Hannah Markwig (Tübingen) and Ajith Urundolil Kumaran (Cambridge).

Parallel and related work by Bae, Barrott, Holmes, Herr, Molcho, Nabijou, Pandharipande, Pixton, Schwarz, Schmitt.

Core goals

Logarithmic intersection theory is a refinement and enlargement of intersection theory. The input is a pair (X, D) a normal crossings pair:

$$\mathrm{CH}^\star(X) \hookrightarrow \mathrm{logCH}^\star(X, D) \twoheadrightarrow \mathrm{CH}^\star(X).$$

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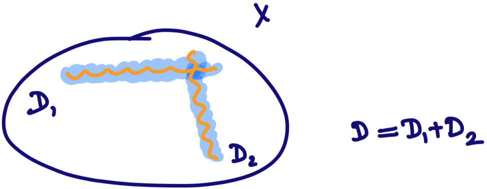
I will explain:

- What is the logarithmic Chow ring of (X, D)?
- How it is motivated by the higher double ramification cycle problem.
- The geometry and combinatorics of working in the ring $\mathsf{logCH}^\star(X, D)$.

Logarithmic intersection theory

Setting: varieties with boundary

Let (X, D) be a **normal crossings pair** with X projective.



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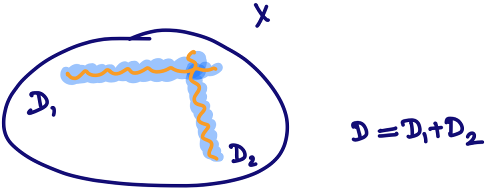


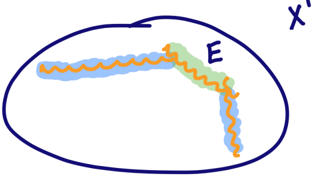
It is **stratified** by connected components of (potentially self) intersections of the components of the divisor D.

Logarithmic intersection theory

A **logarithmic blowup of** (X, D) is any blowup

$$X' \to X,$$

with centre that is **monomial** with respect to D.



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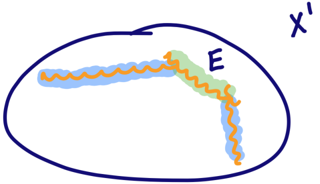

The blowup $\pi : X' \to X$ induces $\pi^\star : \mathrm{CH}^\star(X) \hookrightarrow \mathrm{CH}^\star(X')$. Define

$$\mathrm{logCH}^\star(X, D) := \varinjlim_{X' \to X} \mathrm{CH}^\star(X').$$

Proper pushforward gives:

$$\mathrm{logCH}^\star(X, D) \to \mathrm{CH}^\star(X).$$

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If (X, D) is $(\overline{M}_{0,n}, \partial\overline{M}_{0,n})$ then $\log\mathrm{CH}^\star(\overline{M}_{0,n}, \partial\overline{M}_{0,n})$ is completely controllable – find your local tropical geometer.

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Interest today is $\log\mathrm{CH}^\star(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$.

... why would you study it? What extra information does it contain? How do you work with it?

Double ramification cycles

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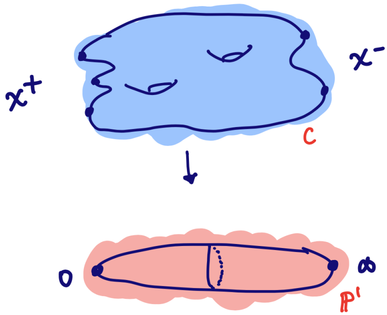

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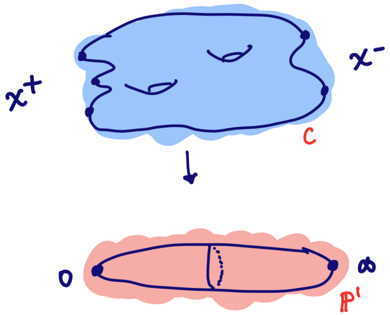

Basic Facts:

Locus where the line bundle $\mathcal{O}(\sum x_i p_i)$ is trivial.

Expected codimension g in $\mathcal{M}_{g,n}$.

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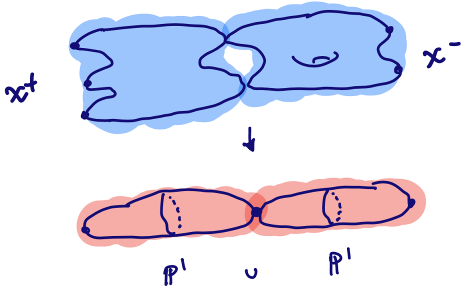

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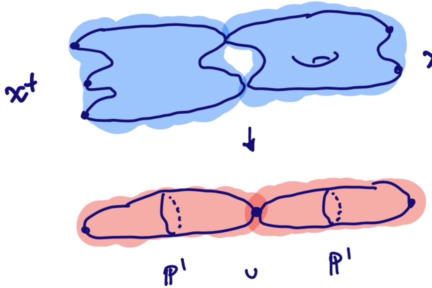


Furnishes an algebraic cycle class $\mathrm{DR}(\mathbf{x})$ in $\overline{\mathcal{M}}_{g,n}$ of codimension g.

Why study it?

- Basic object in higher genus enumerative geometry.
- Intersection theory on the universal Picard variety over $\overline{\mathcal{M}}_{g,n}$.
- Gives relations in the Chow ring of $\overline{\mathcal{M}}_{g,n}$.

Connections to integrable systems, meromorphic differentials, tropical geometry, enumerative geometry of surfaces and threefolds, GW/DT correspondence, etc.

History

The study of $\mathrm{DR}_g(\mathbf{x})$ was proposed by Eliashberg in 2001.

- (Faber–Pandharipande '03) The cycle lies in the **tautological** part of the cohomology of $\overline{\mathcal{M}}_{g,n}$.
- (Hain '11, Grushevsky–Zakharov '12) Explicit formula on the locus of **compact type** curves.
- (Janda–Pandharipande–Pixton–Zvonkine '16) Complete formula[1] on $\overline{\mathcal{M}}_{g,n}$.

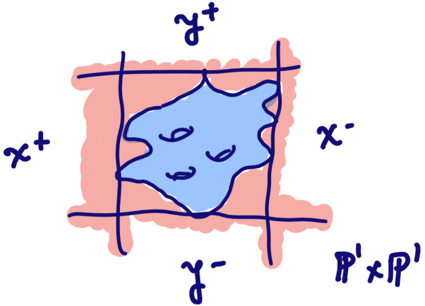
[1]For glorious details see Pixton's ICM lecture `https://youtu.be/p3r1fsHmcG8`

Towards the logarithms

Symptoms of an incomplete story

One can formulate a similar problem for $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathsf{DDR}_g^\circ(\mathbf{x}, \mathbf{y}) = \left\{(C, \underline{p}) : C \text{ admits a map to } \mathbb{P}^1 \times \mathbb{P}^1 \text{ with tangency } \mathbf{x} \text{ and } \mathbf{y}\right\}.$$



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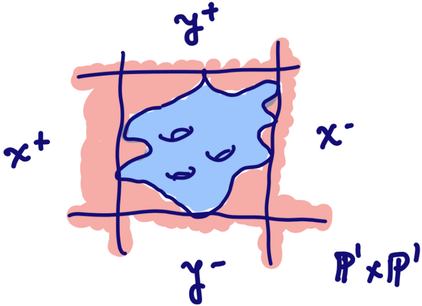

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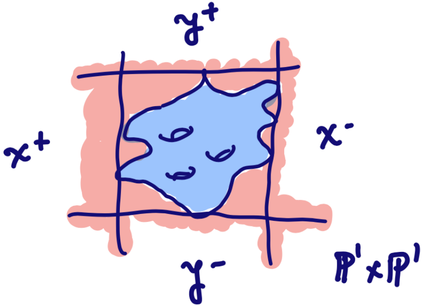

Compactification from **logarithmic Gromov–Witten theory**.

More generally, **maps to any toric variety** with tangency with the boundary.

Symptoms of an incomplete story

We have a simple equality on the **interior** $\mathcal{M}_{g,n}$:

$$\mathsf{DDR}^{\circ}_{g}(\mathbf{x},\mathbf{y}) = \mathsf{DR}^{\circ}_{g}(\mathbf{x}) \cap \mathsf{DR}^{\circ}_{g}(\mathbf{y})$$

The equality fails on the **compact** moduli space $\overline{\mathcal{M}}_{g,n}$

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Today's Story

Lifts of DR and DDR to $\mathsf{logCH}^\star(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ that are better behaved.

Spoiler: information in the logarithmic DR

The logarithmic higher DR cycles are basic objects in enumerative geometry. As a preview:

- Intersections with logarithmic DR encode complete information about Hurwitz numbers of $\mathbb{P}^1$ and Severi degrees of $\mathbb{P}^2$.
- Higher DR intersection calculations lead immediately to correspondence theorems in (refined) tropical curve counting.
- Fundamental part of studying Gromov–Witten theory under normal crossings degenerations.

Building classes in logarithmic Chow

Constructing classes I: cycles

Suppose we have a map

$$\iota : (W, E) \to (X, D)$$

with preimage of D contained in E. Let $X' \to X$ be a blowup at $Z \subset X$.

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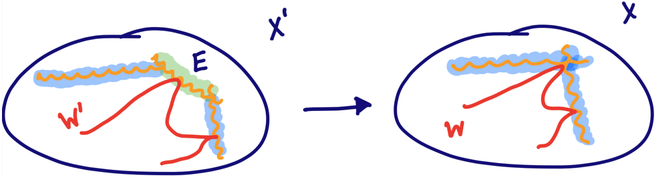

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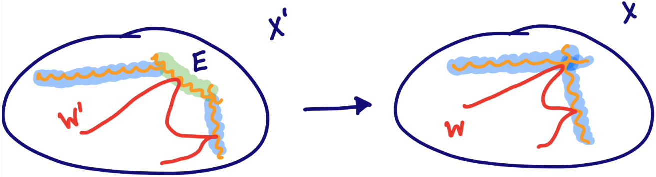

Key Fact!

It is the strict transform and **not** cohomological pullback!

The toric incarnation

Suppose $X' \to X$ is a toric blowup and $W \to X$ is torus equivariant.

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However, the category of toric varieties admits fibre products. The two fibre products play the roles of the two lifts[2].

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Constructing classes II: cocycles

Another source of classes is the **tropicalization** of (X, D).

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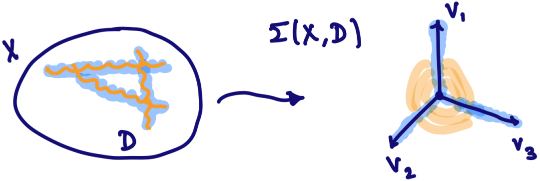
The tropicalization $\Sigma(X, D)$ is a cell complex with conical cells.



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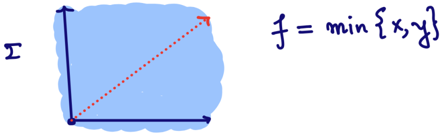

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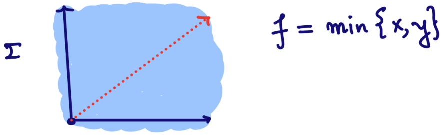

Theorem (Holmes–Schwarz, Molcho–R)

There is a ring homomorphism

$$\mathrm{PP}(\Sigma(X, D))) \to \mathrm{logCH}^\star(X, D).$$

The construction has expected functoriality properties in (X, D).

Logarithmic and higher DR

A reminder of the problem...

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The results will control the higher DR classes via the logarithmic promotions of usual DR classes.

First results

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The logarithmic lifts restore the product rule so:

Corollary

All higher double ramification cycles lie in the tautological ring of the moduli space of curves.

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Further results

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Theorem (R–Urundolil Kumaran '22)

Logarithmic GW cycles of toric varieties lie in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

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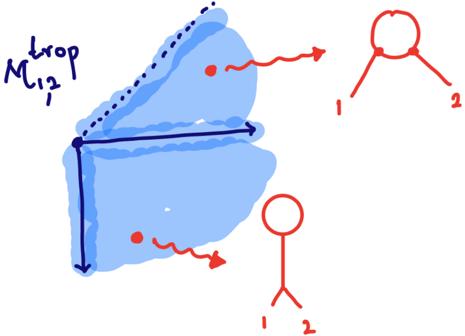

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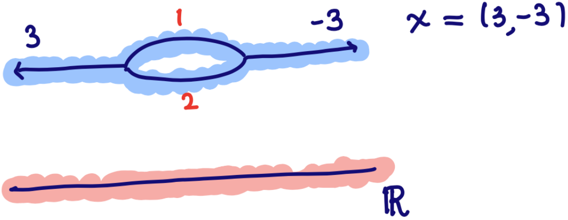

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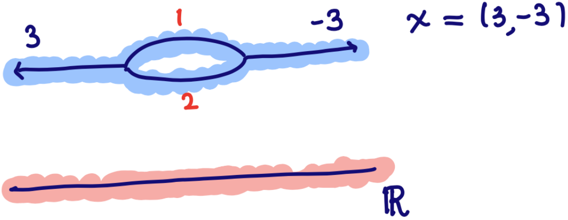

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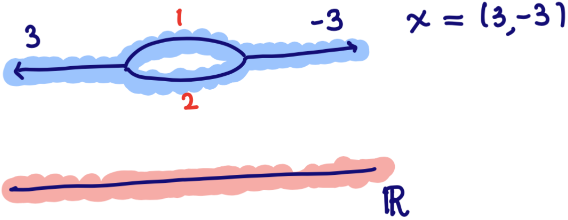

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Theorem (R '19)

Subdivide $\mathcal{M}_{g,n}^{\mathrm{trop}}$ *so this subset is included. The lift of* $\mathrm{DR}_g(\mathbf{x})$ *to this subset is stable under further blowups.*

A few concluding words

Phenomena of this flavour seem to be common – degeneration formulas in GW/DT theory, Fourier–Mukai transforms for degenerations of abelian varieties, and results of an arithmetic nature.

Thanks!