Quantum Geometry of Matroids

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work with J. Usatine
@
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Enumerative

Geometry

THE PARADIGM:

$$X \rightsquigarrow M(X) \rightsquigarrow \text{"Quantum" Invariants of } X$$

space $\qquad$ moduli space

The invariants are called CURVE COUNTING OR GROMOV-WITTEN INVARIANTS of X.

The Paradigm:

X $\rightsquigarrow$ $M(X)$ $\rightsquigarrow$ Gromov–Witten theory of X

space — moduli space

First $\rightsquigarrow$ Contemplate curves in X with fixed topology

Second $\rightsquigarrow$ Use topology of $M(X)$ e.g. χ_{top}, natural integrals.

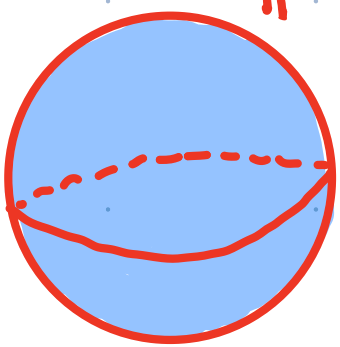
An Example:

$\mathbb{P}^2 \rightsquigarrow \{$Curves in $\mathbb{P}^2$ of degree d & genus $0\} \longrightarrow$ $\mathbb{C}$-manifold of dimension 8

$\Big\downarrow$ Passage through 8 general points

12

$\mathbb{P}^1$

 $\longrightarrow \mathbb{P}^2$

An Example:

$\mathbb{P}^2 \rightsquigarrow \{$ Curves in $\mathbb{P}^2$ of degree d & genus $0\}$

Passage through 8 general points

12

More generally: Curves of degree d & genus 0 through $3d-1$ points — $N_{\mathbb{P}^2}(d)$

HIGHLIGHTS:

The numbers $N_{\mathbb{P}^2}(d)$ have remarkable structure:

$$N(d) = \sum_{\substack{d_1+d_2=d \\ \text{nonzero}}} N(d_1)\cdot N(d_2)\cdot d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_2-2} \right)$$

Proved by Kontsevich in the 1990's

via stable maps

NOTES ON GW THEORY

- Direct connections to integrable systems
- If $X = \mathbb{P}^1$ closely linked with Hurwitz theory
- Using higher genus theory, gives access to geometry of $\overline{\mathcal{M}}_{g,n}$.

[a word about the virtual class]

... and other fun stuff...

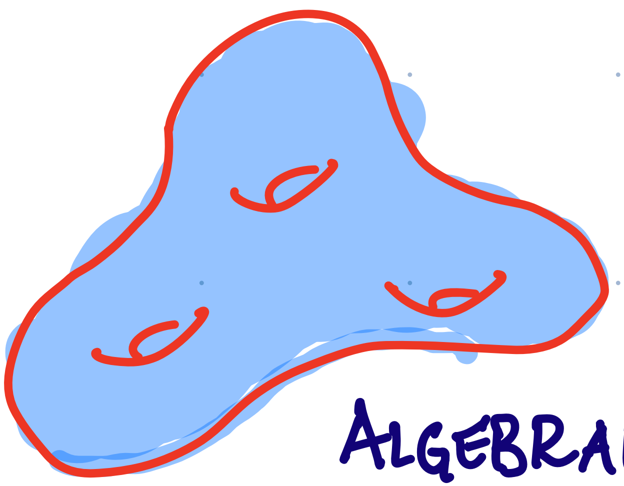
Tropical

Geometry

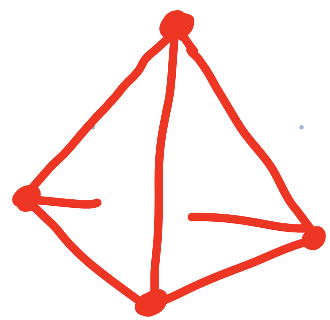
TROPICAL GEOMETRY...

Theory of SOME VERY SPECIAL objects in polyhedral geometry

Strong analogy with algebraic geometry.



ALGEBRAIC



TROPICAL

TROPICAL GEOMETRY... by analogy

Curve C $\longleftrightarrow$ (Metric) graph G

Algebraic variety X $\longleftrightarrow$ Polyhedral/simplicial complex X^{trop}

Curve in X $\longleftrightarrow$ Graph in X^{trop}

Both sides: Riemann–Roch, Bezout's theorem, various other basic results.

TROPICALIZATION... simplest instance

$X \subseteq (\mathbb{C}^*)^n$ then take image of X under

$$\mathrm{trop}: (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$$
$$z \longmapsto (\log_\infty |z|)$$

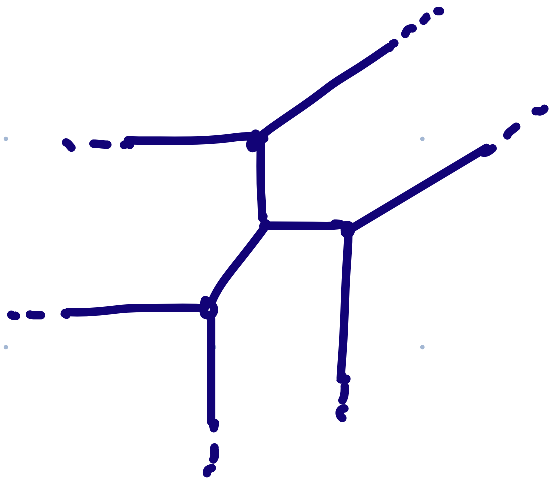
The tropicalization of X is $\mathrm{trop}(X)$

- There are fancier versions...

TROPICALIZATION ... a couple of examples

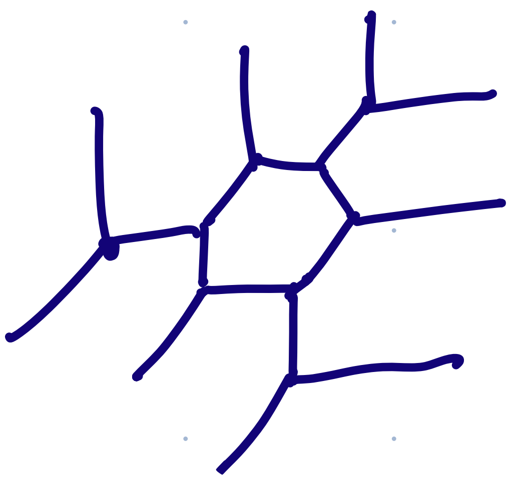
$X = \mathbb{V}($ degree d polynomial in $x, y)$ in $(\mathbb{C}^*)^2$

for $d = 2$



tropical conic...

for $d = 3$



tropical cubic...

... in $\mathbb{R}^2$

STRUCTURE THEOREM (Bieri – Groves)

Let $X \subseteq (\mathbb{C}^*)^n$ be an algebraic variety.

Then $\operatorname{trop}(X)$ is a polyhedral complex

with dimension equal to $\dim_{\mathbb{C}} X$

CENTRAL QUESTION: What does $\operatorname{trop}(X)$ know

about X?

A TROPICAL PLANE CURVES

Take a piecewise integer-affine function eg:

$$f = \min\{a + bx + cy\}$$

$a \in \mathbb{R}$

$b, c \in \mathbb{Z}$.

DEF: A tropical curve $\Gamma \subseteq \mathbb{R}^2$ is the set where $f : \mathbb{R}^2 \to \mathbb{R}$ is NON-DIFFERENTIABLE

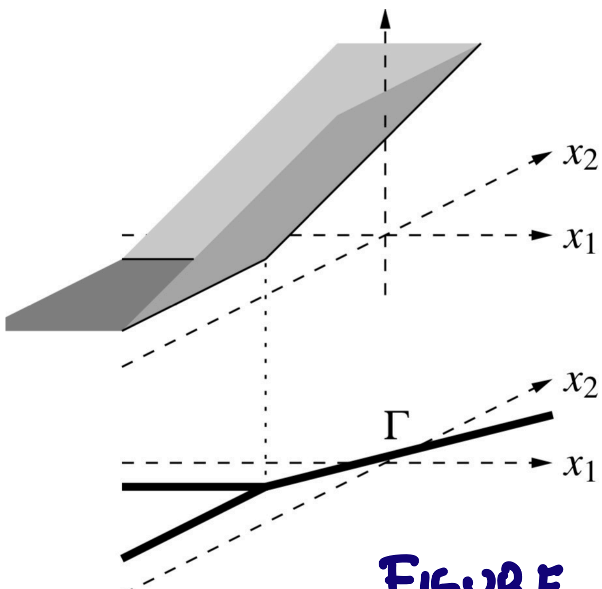


FIGURE by GATHMANN

A TROPICAL PLANE CURVES

Every tropical plane curve is actually the tropicalization of a curve in $(\mathbb{C}^*)^2$.

In other situations this INVERSE PROBLEM is a central question in the subject.

[More on this later]

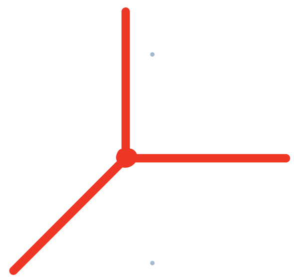
Tropical

Enumerative

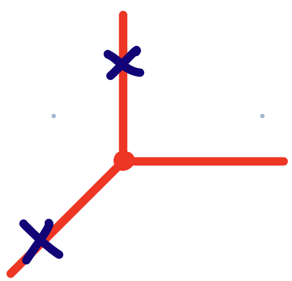
Geometry

A VERY SILLY EXAMPLE... lines in the plane

A tropical line in $\mathbb{R}^2$ up to translation is

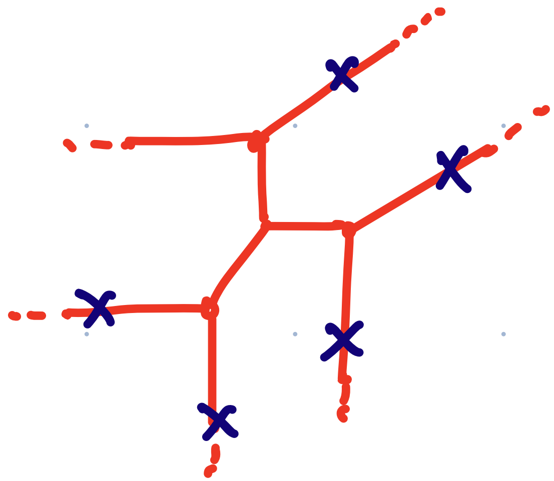


Exercise: Fix p_1 & p_2 in $\mathbb{R}^2$ GENERAL; there exists a unique line through:



$N(1) = 1$

Less Silly Example ... conics in the plane



$N(2) = 1$

Easy: Convince yourself this is rigid

Note: Not allowed to change slopes of any edges or rays.

THE CORRESPONDENCE THEOREM... of Mikhalkin

i. Finite number of genus 0 tropical curves in $\mathbb{R}^2$ of degree d through $3d-1$ points.

ii. Each comes with a combinatorial multiplicity denoted m_Γ such that:

$$N^{trop}_{\mathbb{P}^2}(d) = \sum_{\Gamma} m_\Gamma$$

is well-defined.

THE CORRESPONDENCE THEOREM... of Mikhalkin

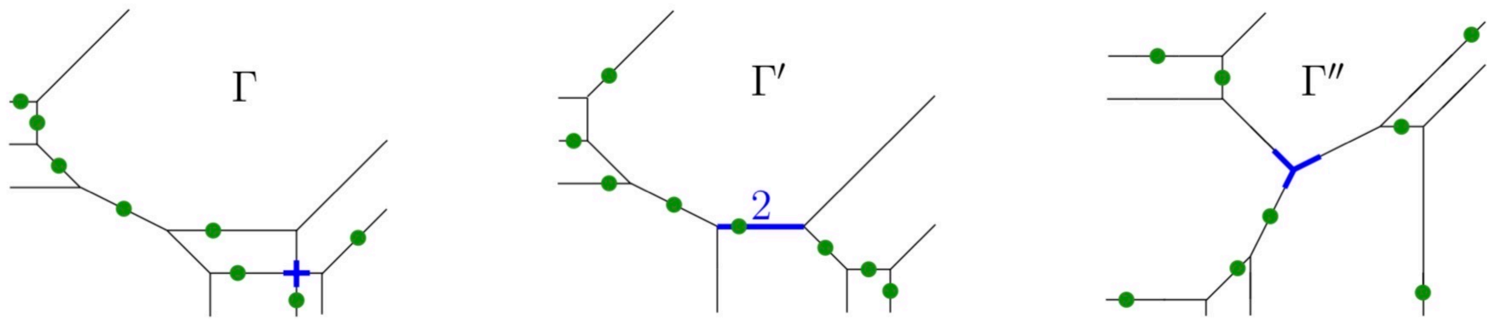
There is an equality of counts:

$$\boxed{N_{\mathbb{P}^2}(d) = N^{trop}_{\mathbb{P}^2}(d)}$$

Gathmann & Markwig then reproved the Kontsevich recursion TROPICALLY

WHAT DOES IT LOOK LIKE?

When calculating $N_{\mathbb{P}^2}(3)$:



Types of curves $\longleftrightarrow$ Lattice subdivisions of polygons

[POLYMAKE is very good at this...]

HOW TO PROVE SUCH A THING?

Key: Directly compare MODULI SPACES

$$\mathrm{trop}(M(X)) = M(X^{\mathrm{trop}})$$

tropicalization of moduli of curves in X

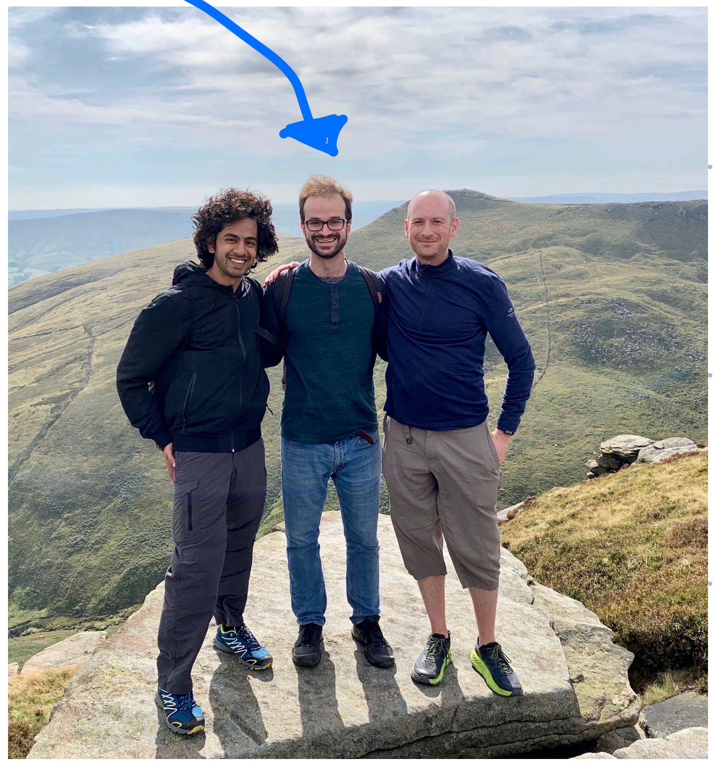
Moduli of tropical curves in X^{trop}.

Now connect tropical space $M(X^{\mathrm{trop}})$

to $H^*(M(X))$.

ACTUALLY COMPUTING...

Dan Corey has recently implemented this, building on POLYMAKE (ongoing)

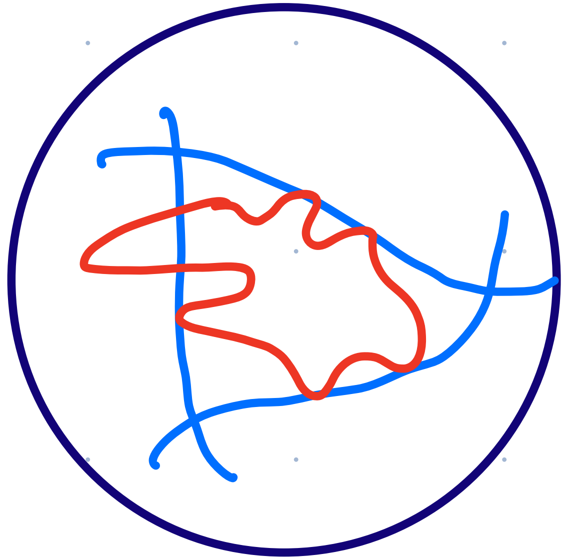


w/ Hannah & Dan: exploring structure of counts (& generalizations)

SPECIFICALLY...

w/ Dan & Hannah

Studying curves $C \subseteq \mathbb{P}^2$ with prescribed tangency along 3 coordinate lines



EXPLORING: How the count behaves as we vary the tangency profile.

- Under some circumstances, governed by a polynomial.

Enumerative

Geometry

& Matroids

SLEIGHT OF HAND...

We traded curves in $\mathbb{P}^2$ for curves in

$$(\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$$

$$N_{\mathbb{P}^2}(d) = N_{(\mathbb{C}^*)^2}(d) = N_{\mathbb{R}^2}(d) = N^{\mathrm{trop}}_{\mathbb{P}^2}(d)$$

by secretly passing through logarithmic enumerative geometry

SLEIGHT OF HAND...

General form of enumerative geometry is about

PAIRS : (X, D) with X a variety

$D \subseteq$ a nice divisor.

For our purposes:

$X = \mathbb{P}^n$; $D =$ any union of hyperplanes.

Mikhalkin: $X = \mathbb{P}^2$; $D = 3$ general lines.

HYPERPLANE ARRANGEMENTS...

$\mathcal{A}$ $\rightsquigarrow$ Gromov-Witten theory of the pair

arrangement in $\mathbb{P}^n$

$(\mathbb{P}^n, \mathcal{A})$

[via deConcini & Procesi's WONDERFUL MODEL]

Basic Question Does tropical correspondence hold?

SENSIBLE..?

Given $(\mathbb{P}^n, \mathcal{A})$ $\rightsquigarrow$ Tropicalization of $\mathbb{P}^n \setminus \mathcal{A}$. by earlier procedure.

<u>(Abuse of) Terminology:</u> The matroid of $\mathcal{A}$ is the tropicalization of $\mathbb{P}^n \setminus \mathcal{A}$

Asking for GW to depend only on matroid

TWO THINGS ABOUT MATROIDS...

- There exist matroids that do not come from ANY hyperplane arrangement. (NON-REALIZABLE)

- Two arrangements with the same matroid may be geometrically very different

(REALIZATION SPACE)

INVARIANTS OF ARRANGEMENTS

Given $\mathcal{A}$ and $\mathbb{P}^n$

- Betti numbers (even cohomology ring!) of $\mathbb{P}^n \setminus \mathcal{A}$ are combinatorial

- But $\pi_1(\mathbb{P}^n \setminus \mathcal{A})$ is NOT even for $n = 2$

[see Rybinkov]

A RESULT:

THEOREM (w/ Usatine) Tropical correspondence holds for all pairs $(\mathbb{P}^n, \Delta)$ & in particular GW theory is COMBINATORIAL.

$\rightsquigarrow$ Quantum deformations of matroid Chow rings studied by Adiprasito–Huh–Katz

Combinatorial complexity is enormous!

A CONJECTURE: (w/ Usatine)

Gromov–Witten theory is well-defined for ARBITRARY, potentially non-realizable matroids

→ Predicts several remarkable vanishings!

→ No viable computation methods

In theory, it is calculation in the Stanley–Reisner ring of a very specific polytope.

A CONJECTURE: (w/ Usatine)

Gromov–Witten theory is well-defined for ARBITRARY, potentially non-realizable matroids

The conjecture would take GW theory outside the world of algebraic geometry!

QUANTUM COHOMOLOGY

- Given $(\mathbb{P}^n, \mathcal{A})$ one can construct a ring: the cohomology of the wonderful model.
- The CHROMATIC POLYNOMIAL comes from this ring.
-

The theorem gives a canonical deformation to quantum cohomology.

What does it say about $\mathcal{A}$?

THANKS !