# ALGEBRAIC GEOMETRY : PART II

A brief introduction to algebraic geometry

"This subject is about the ring of polynomials

$k[X_1, .., X_n]$ and the geometry

that it and its quotients by ideals.

$\boxed{\text{STARTING POINT}}$ Let $k$ be a field

$$J \subseteq k[X_1, .., X_n] \twoheadrightarrow \frac{k[X_1, .., X_n]}{J} = A$$

an ideal

This is how we organize the information in $k[X_1, .., X_n]$.

Guiding Question: On what space is the ring $A$

naturally the ring of functions?

$k[x_1, \ldots, x_n]$ is naturally defined on $k^n$. Given $f \in k[x_1, \ldots, x_n]$ we get

$$ev_f : k^n \longrightarrow k \quad \text{by evaluation.}$$

Given $f_1, f_2 \in \mathfrak{I}$, $f_1$ is meant to be equal to $f_2$ (and equal to the zero function). Thus, whatever subset of $k^n$ we want, $ev_{f_1}$ will only be well defined on $\underbrace{\mathbb{V}(f_1) \cap \mathbb{V}(f_2)}_{\text{VANISHING LOCUS.}}$

Given $\mathfrak{I}$, we have $\mathbb{V}(\mathfrak{I}) = \{ p \in k^n \mid f(p) = 0 \ \forall f \in \mathfrak{I} \}$

and $k[\underline{X}] / \mathfrak{I}$ is the RING OF FUNCTIONS.

EVENTUALLY: we will want "spaces" $X$ st "locally" $X$ has the form $\mathbb{V}(I)$ (or something close). This is a bit like:

Calculus in $\mathbb{R}$ $\rightsquigarrow$ Calculus in $\mathbb{R}^n$ $\rightsquigarrow$ Calculus on manifolds

$\rightsquigarrow$ Differential geometry

AFFINE SPACE Affine $n$-space or "affine space of dimension $n$" is denoted as $\mathbb{A}^n_k$ and as a set is $k^n$. We think of $\mathbb{A}^n_k$ as coming with a natural ring of functions

$$k[\underline{x}] := k[x_1, \dots, x_n].$$ Each $f \in k[\underline{x}]$

gives $f : \mathbb{A}^n_k \longrightarrow k$

⚠ If $k$ is finite, then two polynomials can represent the same <u>function</u>.

FUNDAMENTAL FACTS:

(1) The ring $k[\underline{X}]$ is a UFD &

(2) Given an ideal $J \subseteq k[\underline{X}]$ there exists a set $f_1, .., f_r$ st $\langle f_1, .., f_r \rangle = J$

$\boxed{\text{HILBERT BASIS THEOREM}}$

DEFINITION: An affine variety (closed algebraic subset of $\mathbb{A}^n_k$) is given by;

$$\mathbb{V}(S) = \{ p \in \mathbb{A}^n_k \mid f(p) = 0 \} \text{ for some}$$

$S \subseteq k[\underline{X}]$.

⚠ Some people will ask also for "irreducible", more on this later.

## EXAMPLES & WORDS

(1) The affine varieties in $\mathbb{A}^1_k$ are precisely the finite sets.

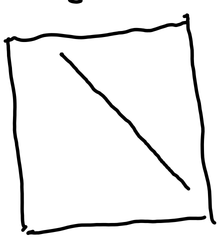
(2) If $Z = V(\{f\})$ is the vanishing of a <u>single</u> polynomial, it is called a <u>hypersurface</u>

<u>Note</u>: In general, this is useful when $k = \bar{k}$ is algebraically closed. But, $k = \mathbb{R}$ is good for pictures.

**PICTURES** over $\mathbb{R}$

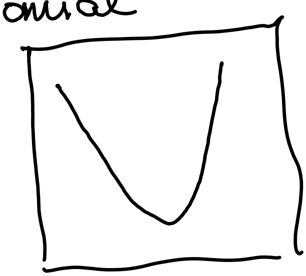
$f \in \mathbb{R}[x,y]$ a degree $d$ polynomial

$d=1$

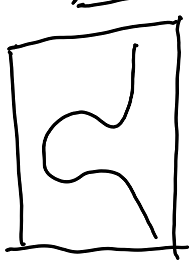


$\rightarrow V(f)$

$d=2$

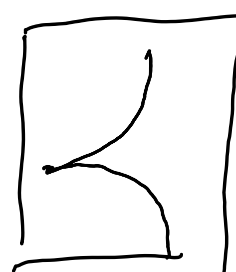


$d=3$



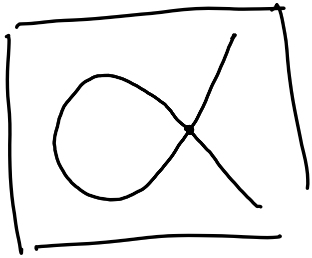
for example

$d=3$, $y^2 = x^3$

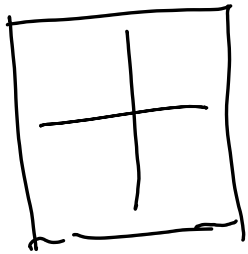


$\boxed{\text{CUSP}}$

$\boxed{d=3}$ $f = y^2 - x^3 - x^2$

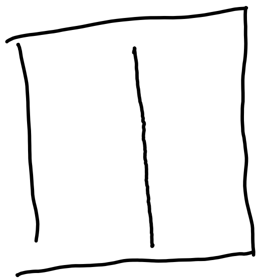


$\boxed{d=2}$ $f = xy$



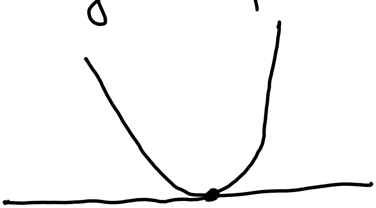
SOMETHING DIFFERENT HERE

$\boxed{d=2}$ $f = x^2$



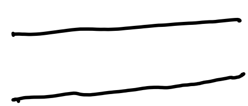
SOMETHING VERY DIFFERENT HERE!

**EXPECTATIONS** Given $f \in k[x,y]$ of degree $d$ and $\ell \subseteq \mathbb{A}^2_k$ a line, $V(f) \cap \ell$ is "probably $d$ points", but:



I. $d=2$ ;

II $d=2$ ;



$d=1$

<u>THEOREM</u> • Given $S \subseteq k[\underline{X}]$ and $\mathcal{I}_S$ the ideal generated by $S$, there is an equality

$$\mathbb{V}(S) = \mathbb{V}(\mathcal{I}_S)$$

• Given and $S$, there exists a finite set $f_1, \dots, f_r$ st

$$\mathbb{V}(S) = \bigcap_{i=1}^{r} \mathbb{V}(f_i)$$

<u>PROOF</u>: • Clear from the definition of an ideal

• Apply the Hilbert basis theorem.

<u>PROPOSITION</u>: • If $S, T \subseteq k[\underline{X}]$ and $T \subseteq S$ then $\mathbb{V}(S) \subseteq \mathbb{V}(T)$

• $\bigcap_i \mathbb{V}(\mathcal{I}_i) = \mathbb{V}(\sum_i \mathcal{I}_i)$

• $\mathbb{V}(\mathcal{I}) \cup \mathbb{V}(\mathcal{I}') = \mathbb{V}(\mathcal{I} \cap \mathcal{I}')$.

Pf: All but the last statement are clear in light of the preceding theorem.

For the last: "$\subseteq$" is clear. Conversely, take $p \in \mathbb{V}(\mathfrak{I} \cap \mathfrak{I}')$ and $p \notin \mathbb{V}(\mathfrak{I})$. Then there exists $g \in \mathfrak{I}$ st $g(p) \neq 0$. For every

$f \in \mathfrak{I}'$, $fg \in \mathfrak{I} \cap \mathfrak{I}'$ and thus,

$fg(p) = 0$. Thus $f(p) = 0 \;\forall f \in \mathfrak{I}'$

so $p \in \mathbb{V}(\mathfrak{I}')$.

$\square$

IRREDUCIBILITY

A variety $Z$ is reducible if

$Z = Z_1 \cup Z_2$ $Z_i$ varieties

means that $Z = Z_1$ or $Z_2$.

Otherwise it is IRREDUCIBLE.

**When are two varieties the same?**

If $\mathbb{V}(\mathcal{J}) = \mathbb{V}(\mathcal{J}')$ then what can we say about $\mathcal{J}$ and $\mathcal{J}'$?

**NOT ALWAYS!** $\mathbb{V}(f) = \mathbb{V}(f^k)$ $k \in \mathbb{N}_{>0}$

But, given a variety $Z$, we can take

$$I(Z) = \{ f \mid f(p) = 0 \ \forall p \in Z \}.$$

PROPOSITION:
- $Z = \mathbb{V}(S)$, then $S \subseteq I(Z)$
- $Z = \mathbb{V}(I(Z))$
- $Z = Z' \iff I(Z) = I(Z')$

So $I(Z)$ is the largest ideal that could give $Z$ as a vanishing locus.

We have built an association

$$\left\{ \begin{array}{c} \text{affine varieties} \\ Z \subseteq \mathbb{A}^n_k \end{array} \right\} \longrightarrow \left\{ \text{Ideals in } k[\underline{x}] \right\}$$

which is an injective association

$$Z \longmapsto I(Z) = \{ f \in k[\underline{x}] \mid f(p) = 0 \ \forall p \in Z \}$$

with left side inverse

$$I(Z) \longmapsto \mathbb{V}(I(Z)).$$

Ideals obtained as $I(Z)$ have the following property:

(*) If $f^d \in I(Z)$ then $f \in I(Z)$.

DEF: Given an ideal $\mathfrak{I} \subseteq k[\underline{x}]$, its radical

$$\sqrt{\mathfrak{I}} = \{ f \in k[\underline{x}] \mid f^d \in \mathfrak{I} \}.$$ In addition

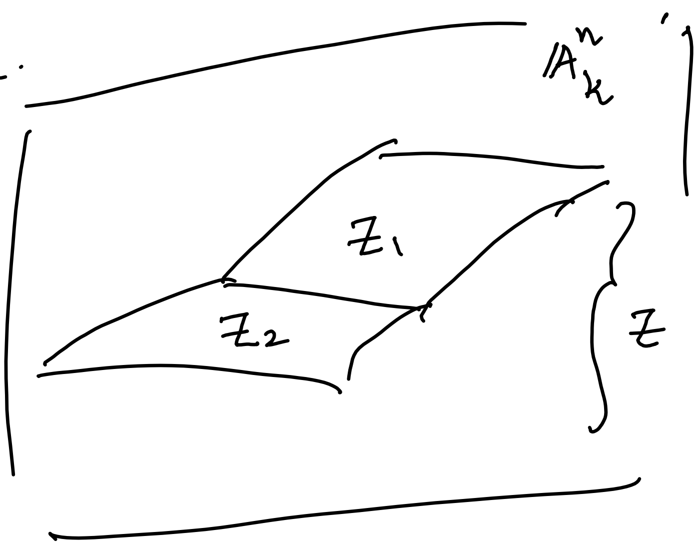
$\mathfrak{I}$ is called radical if $\mathfrak{I} = \sqrt{\mathfrak{I}}$.

OBSERVE: $\sqrt{\sqrt{\mathfrak{I}}} = \sqrt{\mathfrak{I}}$, so $\sqrt{\mathfrak{I}}$ is radical.

Let $Z = \mathbb{V}(\mathfrak{J})$.

Denote $\mathcal{O}(Z)$ the <u>coordinate ring</u> $k[\underline{x}]/\mathfrak{J}$.

<u>PROPOSITION</u>: $I(Z)$ is prime if and only if $Z$ is irreducible.




**PROOF BY PICTURE**

Pick $f_1$ vanishing on $Z_1$ not $Z_2$, and $f_2$ vanishing on $Z_2$ not $Z_1$. Then $f_1 f_2$ vanishes on $Z_1 \cup Z_2 = Z$. Thus, $f_1 \notin I(Z)$ & $f_2 \notin I(Z)$ but $f_1 f_2 \in I(Z)$.

**ACTUAL PROOF** Let $I_1 = I(Z_1)$ and $I_2 = I(Z_2)$ with $Z = Z_1 \cup Z_2$, $Z_i \neq Z$.

such that $I(Z) = I_1 \cap I_2$.. Choose

$$f_1 \in I_1 \setminus I_2 \quad \text{and} \quad f_2 \in I_2 \setminus I_1.$$

(since we have $I_1 \not\subseteq I_2 \not\subseteq I_1$, this is possible)

and observe $f_1 f_2 \in I(Z)$.

$\boxed{\text{Conversely}}$ if $Z$ is irreducible, we claim

for $f_1 f_2 \in I(Z)$, $f_1$ or $f_2 \in I(Z)$.

Indeed, if not, $\boxed{Z_i = V(f_i) \cap Z}$ and

observe that $Z = Z_1 \cup Z_2$ $\boxed{\text{why?}}$

$\square$.

$\boxed{\text{HILBERT'S NULLSTELLENSATZ}}$ Let $k = \bar{k}$. Then

$$I(V(\mathfrak{J})) = \sqrt{\mathfrak{J}}$$

(say explicitly what this means).

So, affine varieties don't see the difference between $\mathbb{V}(f)$ & $\mathbb{V}(f^2)$. Of course, one could remember it "by design", and think about the data: $(Z, k[X]/\mathfrak{J})$ ← this is a scheme

Often convenient, but we won't get there till Part III. The real geometry is all in what we'll do though.

$k = \bar{k}$ now unless otherwise stated

MORPHISMS Let $V \subseteq \mathbb{A}^n_k$ & $W \subseteq \mathbb{A}^m_k$. A morphism is a map

$\phi : V \longrightarrow W$ st there exist

$f_1, \ldots, f_m \in \mathcal{O}(V)$ with $\phi(p) = (f_1(p), \ldots, f_m(p))$

$\boxed{\text{What does this say?}}$

The elements of $\mathcal{O}(V)$ are functions with values in $k$. With $m$ of them we map to $\mathbb{A}^m_k$ so

$\phi = (f_1, \dots, f_m) : V \longrightarrow \mathbb{A}^m_k$. We want these points to $\boxed{\text{satisfy the equations of } W}$.

$\boxed{\text{RING THEORETIC PERSPECTIVES}}$

I. Given any ring $A \cong k[x_1, \dots, x_n] / \mathfrak{g}$ (k-algebra)

a homomorphism

$k[x_1, \dots, x_m] \longrightarrow A$ is equivalent to

the data of the images of $x_1, \dots, x_m$, so just $m$ elements of $A$.

the condition that $f_1(p), \ldots, f_m(p)$ satisfy the equations of $W$ is equivalent to saying that

$$\begin{tikzcd} k[x_1, \ldots, x_m] \arrow[rr] \arrow[dr] & & A = \mathcal{O}(V) \\ & \mathcal{O}(W) \arrow[ur, dashed] & \end{tikzcd}$$

extends.

THEOREM: The morphisms $\varphi : V \to W$ are in bijection with ring homomorphisms

$$\varphi^* : \mathcal{O}(W) \to \mathcal{O}(V)$$

Given by $\varphi \longleftrightarrow \varphi^*$.

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II. The functions (regular/polynomial/algebraic) on $W$ are $W \xrightarrow{f} \mathbb{A}^1_k$

If we had $V \xrightarrow{\varphi} W$, we'd be able to produce a $\boxed{\text{pullback}}$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[rr, bend right] & W \arrow[r, "f"] & \mathbb{A}^1_k \end{tikzcd}$$

$$\varphi^* f \in \mathcal{O}(V)$$

$\rightsquigarrow$ $\boxed{\varphi \text{ must produce } \varphi^* : \mathcal{O}(W) \to \mathcal{O}(V)}$

But $\varphi^*$ determines $\varphi$ by our previous discussion

CHECK!

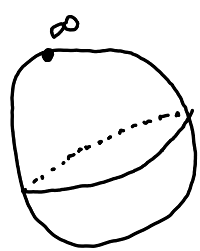
So ultimately, affine varieties are equivalent objects to rings of the form $k[x_1, \ldots, x_n]/_{J}$ and maps between them $\boxed{\text{FINITE TYPE } k\text{- ALGEBRAS}}$

We nonetheless keep the geometric perspective to correctly $\boxed{\text{GLOBALIZE}}$ to general varieties. Local-to-global transition is a little subtle compared to manifolds, so we start with a broad class of examples.

## $\boxed{\text{PROJECTIVE VARIETIES}}$

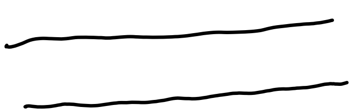
$\mathbb{P}^1_k$ : "COMPACTIFYING" $\mathbb{A}^1_k$, perhaps $k = \mathbb{C}$ for pictures.

$\mathbb{P}^1_{\mathbb{C}}$

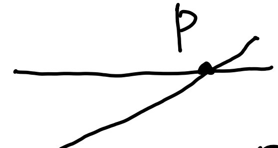



$\mathbb{A}^2_k$ : • Given lines $l_1, l_2$, what is $\#|l_1 \cap l_2|$?

0 : 1 or $\infty$ $\longrightarrow$ But the answer <u>should</u> be 1.



is a limit of




as $p \to \infty$

$l_1 = \mathbb{V}(x)$, $l_2(t) = \mathbb{V}(x + 1 + \frac{1}{t} \cdot x)$

as $t \longrightarrow \infty$ $l_1 \cap l_2(t)$ becomes empty.

$\boxed{\text{"LOST" intersection point at } \infty.}$

$\mathbb{P}^n_k$ will fix these "issues"

$\boxed{\text{DEFINITION}}$ Given a $k$-vector space $V$ (f-d)

define $\mathbb{P}(V) = V \setminus \{\vec{0}\} \big/ k^*$ acting by scaling.

In particular $\mathbb{P}^n_k = \mathbb{P}(k^{n+1})$

$\boxed{\text{STANDARD CANON}}$

- Homogeneous coordinates
- Homogeneous polynomials
- Affine patches
- Projective varieties.
- Homogenization of polynomials and $\mathbb{A}^n_k \subseteq \mathbb{P}^n_k$

Let's explore $\mathbb{P}^n_k$

- Points are given by $(z_0, .., z_n)$ not all zero with implicit ambiguity that $(z_0, .., z_n) \sim (\lambda z_0, .., \lambda z_n)$ for all $\lambda \in k^*$ (Notation: $[z_0, .., z_n] = (z_0 : \cdots : z_n)$
$= [z_0 : \cdots : z_n]$.

- Polynomials don't take on well-defined values i.e. $f(z_0, .., z_n)$ changes based on what representative of a point you plug in

Covering $\mathbb{P}^n$: Homogeneous coordinates are $(z_0 : \cdots : z_n)$ but if we consider

$$\mathbb{P}^n \setminus \{z_0 = 0\} = \left\{ \left(1 : \frac{z_1}{z_0} : \cdots : \frac{z_n}{z_0}\right) \right\}$$

$z_i / z_0 \in k^*$

$\mathbb{P}^n$ inherits a Zariski topology

# Monday February 3

① $V(I) = \emptyset \Leftrightarrow I \supseteq (Z_0^m, \dots, Z_n^m)$

② $V(I) \ne \emptyset$ then $I^h(V) = \sqrt{I}$

- Projective varieties $I^h$
- Projective nullstellensatz
- Affine patches
  - Zariski topology
- Parallel lines in $\mathbb{P}^2$.
- Irreducible subvarieties, irreducible decomposition
  ↳ Key: Non-primality detected by hom. forms.
- Projective closure

Zariski topology

I LECTURED OFF THE CUFF — DON'T TAKE THESE NOTES TOO SERIOUSLY FOR THIS DAY

## Function field

$V$ irreducible projective,

$$k(V) = \{ F/G \mid F, G \in k[\underline{Z}] \text{ hom. same degree}, G \notin I^h(V) \} / \sim$$

$$\frac{F_1}{G_1} \sim \frac{F_2}{G_2} \Leftrightarrow F_1 G_2 = F_2 G_1$$

Transitive?

If
$$F_1 G_2 = G_1 F_2$$
$$F_2 G_3 = G_2 F_3$$

$$F_1 \boxed{G_2} G_3 = G_1 \underline{F_2 G_3} = G_1 \boxed{G_2} F_3$$

$$\boxed{F_1 G_3 = G_1 F_3}$$

closed subvariety, open subvariety.

**Prop.** $V \subseteq \mathbb{P}^n$ irred., and $W \subseteq V$ proper closed, then $V \setminus W$ is dense

Pf: want to show, if $f$ vanishes on $V \setminus W$ it vanishes on $V$.

Take such $f$. Since $V \neq W$, $\exists$

$$g \in I^h(W) \setminus I^h(V) \text{ by Null.}$$

Now, $fg$ vanishes on $V$, but $I^h(V)$ prime and $g \notin I^h(V)$ so

$$f \in I^h(V).$$

Sketch

**MORAL**: Smaller

**LEMMA**: If $\overline{V} \subseteq \mathbb{P}^n$ is closure of $V \subseteq \mathbb{A}^n$ then $FF(k[V]) = k(\overline{V})$

Pf $f/g$ on $V$ multiply by $z_0^m$ and homog; $p/q$ $\to p(1, \ldots)/q(1, \ldots)$

Affine varieties $X \subseteq \mathbb{A}^n$ can be closed up in $\mathbb{P}^n$ as follows.

How this works for $X = \mathbb{V}(f)$, $f \in k[X_1, .., X_n]$

Let the total degree of $f$ be $d$, then define a new (homogeneous!) polynomial

$$F(Z_0, .., Z_n) = Z_0^d \cdot f\left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0}\right)$$

<u>In practice</u>: Given $X_1^2 + 3X_2 + X_3^3$

want to make it homogeneous by adding a variable:

$$Z_0 Z_1^2 + 3Z_0^2 Z_2 + Z_3^3$$

If we set $Z_0 = 1$ we recover the older polynomial

Ideals : we need a definition

DEF : An ideal is homogeneous if it is gen. by hom. elements

LEMMA. TFAE

(i) $I$ is homogeneous

(ii) If $f \in I$ then the homogeneous parts $f_{[r]} \in I$.

Pf : $\boxed{(i) \Rightarrow (ii)}$ Say $I = \langle g_j \rangle$ of hom. degree $d_j$ and let $f \in I$.

Write $f = \sum \overset{\text{ARB.}}{h_j} \overset{\text{HOM.}}{g_j}$. Now split

$h_j = \sum_r h_{j[r]}$ hom. degree $r$ pieces.

$f_{[r]} = \sum h_{j[d_j - r]} g_j$ but $g_j \in I$, so $f_{[r]} \in I$.

$\boxed{ii \Rightarrow i}$ This is a triviality.

DEF: A projective variety is

$$\mathbb{V}(I) \subseteq \mathbb{P}^n$$

$$= \{ P \in \mathbb{P}^n \mid f(P) = 0 \ \forall\, P \in \mathbb{P}^n \}$$

$$\forall f \in I.$$

where $I$ is a homogeneous ideal.

$\boxed{\text{TALK ABOUT } \mathbb{V}(I) \cap \mathbb{A}^n}$ dehomogenization

If $I \supseteq (Z_0^m, \ldots, Z_n^m)$ then

$$\mathbb{V}(I) = \emptyset \ !$$

→ irreducibility

PROPOSITION: • Every proj. variety is a finite union of irreducibles

• irreducible iff $I^h(V)$ is prime.

PROOFS SAME AS AFFINE CASE

On $\mathbb{P}^n$ we can change coordinates by elements of $\mathbb{P}GL(n+1, \mathbb{C})$, i.e. $(n+1)\times(n+1)$ matrices ~~of~~ invertible, but up to scalar.

Let $\mathcal{O}_{\mathbb{P}^n}(d) = \{ f \in k[z_0, \dots, z_n] \mid f \text{ is homogeneous of degree } d \}$,

- Each $\mathcal{O}_{\mathbb{P}^n}(d)$ has the structure of a vector space of dimension $\binom{n+d}{d}$ ← Number of monomials

Let $f_d \in \mathcal{O}_{\mathbb{P}^n}(d)$.. $\boxed{\text{Terminology}}$

$V(f_1)$ is a hyperplane, and $V(f_1) \cong \mathbb{P}^{n-1}$

$V(f_2)$ is a quadric hypersurface

$V(f_3)$ is a cubic

quartic

quintic

$\vdots$

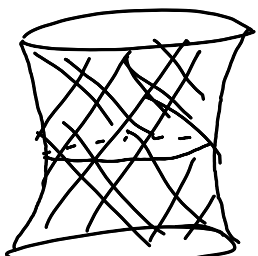
All hyperplanes are the same:

Pf: change coordinates such that $f_1 = Z_0$,

$\mathbb{V}(f_1) \cong \mathbb{V}(Z_0) \cong \mathbb{P}^{n-1}$

Quadrics are not always the same!

One example: $\mathbb{V}(Z_0 Z_3 - Z_1 Z_2)$ in $\mathbb{P}^3$.



observe: This is the image of the map

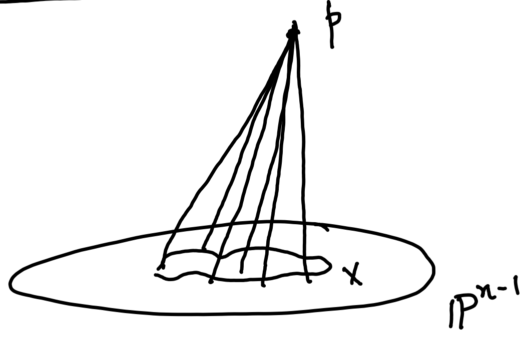
$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$((X_0, X_1), (Y_0, Y_1)) \longmapsto (X_0 Y_0, X_0 Y_1, X_1 Y_0, X_1 Y_1)$$

There are other types of quadrics

$\mathbb{P}^n$

$\boxed{\text{Cones}}$



we define the cone $\mathrm{Cone}(X,p)$ for $X \subseteq \mathbb{P}^{n-1}$

$$\mathrm{Cone}(X,p) = \bigcup_{q \in X} \mathrm{line}(p,q)$$

why is <u>this a</u> projective variety?

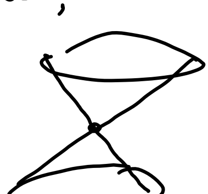
Change coordinates such that $\mathbb{P}^{n-1} = \mathbb{V}(Z_n)$
& $p = (0:\dots:1)$

Then, if $X = \mathbb{V}(\{F_\alpha(Z_0,\dots,Z_{n-1})\})$

then $\mathrm{Cone}(X,p) = \mathbb{V}(\{F_\alpha(Z_0,\dots,Z_{n-1})\}$

thought of as polynomials in $1+n$ variables

<u>FACT</u>: The quadric first introduced is not a cone, there is a quadric cone



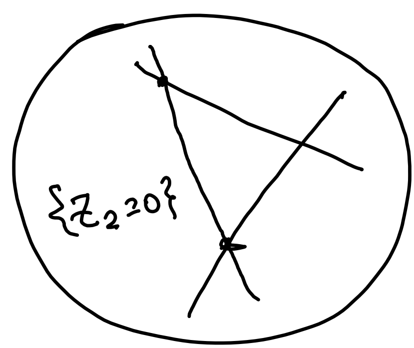
EXAMPLES $k = \bar{k}$

PROPOSITION: Let $C = \mathbb{V}(F)$, with $F$ irreducible in $k[Z_0, Z_1, Z_2]$, hom. degree $d$. Then for $\ell$ a line, $\# C \cap \ell \leq d$, and in fact

$\exists$ $m_p(C, \ell)$ for $p \in \ell \cap C$ st

$$\sum_{p \in C \cap \ell} m_p(C, \ell) = d.$$

Pf.: Choose coordinates such that




- $\ell = \{Z_2 = 0\}$, $C = \mathbb{V}(F)$

and since $C$ doesn't contain a line, assume $(0:1:0) \notin C$

- Thus, $C \cap \ell \subseteq \mathbb{P}^2 \setminus \{Z_0 = 0\} \cong \mathbb{A}^2$

Pick coordinates $x = \frac{Z_1}{Z_0}$, $y = \frac{Z_2}{Z_0}$ in

$\mathbb{A}^2_k \subseteq \mathbb{P}^2_k$. Then notice, $f(x,y) = F(1,x,y)$

has degree $d$ in $x$ since $(0:1:0) \notin C$.

Now, $\ell \cap C = \{f(x,0) = 0\}$ which is as expected.

$\square$

Further comment: If $C_1$ & $C_2$ have degrees $d_1$ & $d_2$, the $\exists$ $m_p(C_1, C_2)$ $\forall p \in C_1 \cap C_2$ st

$$\sum m_p(C_1, C_2) = d_1 d_2 \quad \boxed{\text{Intersection multiplicities}}$$

$\boxed{\text{Change of coordinates}}$* Any collection of $n+2$ points in general position are projectively equiv.

↪ no $n+1$ contained in a hyperplane.

(For $N$ points in $\mathbb{P}^n$, no $n+1$ or fewer are dependent as vectors in ~~$\mathbb{C}$~~ $k^{n+1}$)

<u>Proof of</u> $*$: Send $p_1, .., p_{n+1}$ to $e_1, ... e_{n+1}$ by $\varphi$. Then by linear position $p_{n+2}$ sent to a ~~coordinate~~ to point w/ ALL nonzero coordinates

$\longrightarrow$ After dilating by diagonal matrices, we can take $\varphi(p_{n+2}) = (1, .., 1)$.

<u>THM</u>: In $\mathbb{P}^2$, there is a unique conic passing through 5 points $p_1, .., p_5$ in general position. It is irreducible.

<u>LEMMA</u>: Two conics intersect in at most 4 points in $\mathbb{P}^2_k$. Precisely, $F$ & $G$ coprime, $\deg \leq 2$,

$\# V(G) \cap V(F) \leq \deg(F) \cdot \deg(G)$.

Pf: Exercise involving conic sections & changing coordinates. Similar to $\ell \cap C$ lemma

<u>Pf of thm</u>: If $C = V(F) \subseteq \mathbb{P}^2$ is a conic through $p_1, \ldots, p_5$, $C$ is irreducible (why?)

If $C' = V(F')$ is another, then

$\# \; C \cap C' \geq 5$ which is impossible

unles $C = C'$.

To see why there is one, there are 6 coeff's

$$F = a_0 Z_0^2 + a_1 Z_1^2 + a_2 Z_2^2 + a_3 Z_0 Z_1 + a_4 Z_0 Z_2 + a_5 Z_1 Z_2$$

with $a_i \in k$. $F(p_i) = 0$ is a linear condition in the $a_i$'s, so with 5 equations, there is a solution. $\square$

Rational Normal Curve

$\nu_n : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$

$[x_0 : x_1] \longmapsto [x_0^n : x_0^{n-1} x_1 : \dots : x_1^n]$

$= [z_0 : \dots : z_n]$

Image is called the rational normal curve

Let Id be the ideal of $2 \times 2$ minors

of $\begin{bmatrix} z_0 & z_1 & \cdots & z_{n-1} \\ z_1 & z_2 & \cdots & z_n \end{bmatrix}$

$\boxed{\mathrm{im}(\nu_n) = \mathbb{V}(I_d)} \longrightarrow$ Example sheet II.

I'll do the $d=3$ case explicitly

$\nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$[x_0 : x_1] \longmapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$

Consider the 3 quadrics given by

$$Q_0 = \mathbb{V}(Z_0 Z_2 - Z_1^2)$$

$$Q_1 = \mathbb{V}(Z_0 Z_3 - Z_1 Z_2) \quad \&$$

$$Q_2 = \mathbb{V}(Z_1 Z_3 - Z_2^2)$$

Easy check: $\operatorname{im}(\nu_3) \subseteq Q_i \quad \forall i$

Converse: Given $p = [Z_0 : \dots : Z_3]$ on $\cap Q_i$, either $Z_0$ or $Z_3$ is nonzero. If $Z_0 \neq 0$, then $p = \nu_3([Z_0 : Z_1])$

and if $Z_3 \neq 0$ then $p = \nu_3([Z_2 : Z_3])$

Note: $Q_i \cap Q_j \neq C$, in fact

$Q_i \cap Q_j = C \cup \ell$, where $\ell \subseteq \mathbb{P}^3$ is a line.

This is called the $\boxed{\text{twisted cubic}}$

$\boxed{\text{Segre surface}}$ $\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ as

$$([X_0 : X_1], [Y_0 : Y_1]) \longmapsto [X_0 Y_0 : X_0 Y_1 : Y_1 X_0 : Y_1 X_1]$$

Given by $\mathbb{V}(Z_0 Z_3 - Z_1 Z_2)$

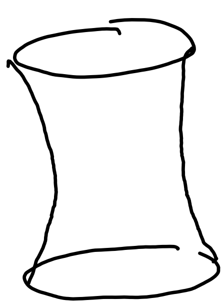
$\mathrm{im}(\sigma_{1,1})$ is $\Sigma_{1,1}$ the Segre surface.

Observe : • There exist two lines on $\Sigma_{1,1} \subseteq \mathbb{P}^3$ that are disjoint

• There exist two lines on $\Sigma_{1,1}$ that intersect

• In fact there are $\infty$-many line.

Parta :



Quadric surface.

THINGS TO KNOW THAT WE CAN'T PROVE YET

- Given $f_3(z_0, \dots, z_3) \in \mathcal{O}_{\mathbb{P}^3}(3)$, the choices of coefficients lie in $\mathbb{C}^{20}$. For a dense open set $U \subseteq \mathbb{C}^{20}$, if we take

$$V(f_3(z_0, \dots, z_3)) = S$$ this surface has 27 lines.

- If we take $f_d(z_0, \dots, z_3) \in \mathcal{O}_{\mathbb{P}^3}(d)$ for a denset of coefficient choices in $\mathbb{C}^{\binom{d+3}{3}}$,

$V(f_d)$ contains no lines.

why this "dense set": For the first one, it's because these are manifolds $\boxed{\text{SMOOTHNESS}}$

The Grassmannian $G(k,n) = \{k\text{-dim subspaces of } k^n\}$

is also a projective variety.

How? : Give $W \subseteq V \cong k^n$ of dimension $r$

associate any basis $\langle v_1, \dots, v_r \rangle$. This association

is not unique. Given $M = [v_1 \dots v_r]$ & $M' = [u_1 \dots u_r]$

which ~~st~~ tells us: $GL(r)$ acts on

$\{r \times n \text{ matrices}\}$ by left multiplication

without changing row span

Now: there are $\binom{n}{r}$ $r \times r$ minors of this matrix,

which are well-defined up to scaling.

so $\{k\text{-dim'l subspaces}\} \longrightarrow \mathbb{P}\left(k^{\binom{n}{r}}\right)$.

LEMMA: • This is injective • Image is cut out by polynomials.

Back to theory:

Morphisms: A rational map

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^m$$

given $F_0, \dots, F_m$ homogeneous degree $d$ in variables $Z_0, \dots, Z_n$ we obtain a map

$$\mathbb{P}^n \setminus \bigcap_j \mathbb{V}(F_j) \longrightarrow \mathbb{P}^m$$

$$P = (Z_0 : \dots : Z_n) \longmapsto (F_0(P), \dots, F_m(P))$$

This intersection is called the base locus or locus of indeterminacy. $\boxed{\text{CREMONA}}$

- If $X \subseteq \mathbb{P}^n$ is $\mathbb{V}(I)$, if $F_0, \dots, F_m \in k[\underline{Z}]$ not all in $I$, then they determine a rational map

$$X \setminus \bigcap_j (X \cap \mathbb{V}(F_j)) \longrightarrow \mathbb{P}^m$$

Regularity, morphisms.

Dominant rational map $\leftrightarrow$ $\varphi : X \dashrightarrow Y$

st $\mathrm{im}(\varphi)$ is dense

<u>THEOREM</u>: • Function field of projective closure

• If $\varphi : X \dashrightarrow Y$ is birational

the $k(X) \cong k(Y)$ as fields.

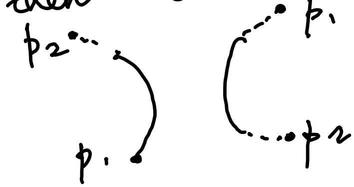
: Given $f \in k[z_1, \ldots, z_n]$ we get a homogenization $F(Z_0, \ldots, Z_n)$ given by

$$F := Z_0^d \cdot f\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right)$$

$\mathbb{V}(F)$ is the projective closure in the Zariski topology of $\mathbb{V}(f) \subseteq \mathbb{A}^n_k = \mathbb{P}^n \setminus \{Z_0 = 0\}$.

<u>Ex</u>: Affine hyperbola $X = \mathbb{V}(z_1^2 - z_2^2 - 1) \subseteq \mathbb{A}^2_k$

then closure $\overline{X} = \mathbb{V}(Z_1^2 - Z_2^2 - Z_0^2)$



Let $I$ be an ideal in $k[z_1, \dots, z_n]$ then

$$I^h = \langle f^h \mid f \in I \rangle \subseteq k[z_0, \dots, z_n]$$

where $f^h$ is the homogenization of $f$.

PROPOSITION: If $X = \mathbb{V}(I)$, inside $\mathbb{A}^n_k$ then $\overline{X}$ in $\mathbb{P}^n_k$ is $\mathbb{V}(I^h)$

Pf: $\mathbb{V}(I^h)$ contains $X$ and is closed.

Want to check it is the smallest such:

Say $Y$ is closed & contains $X$. Then $Y = \mathbb{V}(J)$. Any $F \in J$ is given by $x_0^d \cdot f^h$ for $f \in k[z_1, \dots, z_n]$

If $x_0^d f^h$ is zero on $X$ $\Rightarrow f = 0$ on $X$

$\Rightarrow f \in I(X) = \sqrt{I}$, so $f^m \in I$, so $(f^m)^h = (f^h)^m \in I^h$ so $x_0^d f^h \in \sqrt{I^h}$.

Therefore $J \subseteq \sqrt{I^h}$, which means

$$Y \supseteq \mathbb{V}(\sqrt{I^h}) = \mathbb{V}(I^h) \text{ as required.}$$

$\boxed{\text{START OF 10/2 LECTURE}}$

LEMMA:
Given $X \subseteq \mathbb{A}^n$ and $\overline{X}$ the projective closure,

$$k(\overline{X}) \cong k(X).$$

Pf sketch: given

$$\frac{f(z_1, \dots, z_n)}{g(z_1, \dots, z_n)} \in k(X) \quad \text{let } m = \text{max degree}$$

so

$$F = Z_0^m f\left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0}\right)$$

$$G = Z_0^m g\left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0}\right)$$

$$F/G \in k(\overline{X}).$$

Conversely given $\frac{F}{G}$, $\dfrac{F(1, z_1, \dots, z_n)}{G(1, z_1, \dots, z_n)} \in k(X)$

Checking inverses is a pain but straightforward

Ex: If $X = \mathbb{P}^n$ $k(X) \cong k(z_1, \dots, z_n)$

---

MORPHISMS & RATIONAL MAPS: Let $X$ be projective irred.

- Rational functions $\phi \in k(X) = \mathrm{Frac}(k[X])$
  $\phi = F/G$ $\quad k[X] = k(X)$

give $X \setminus \mathbb{V}(G) \longrightarrow k$

- Rational maps: Given $F_0, \dots, F_m \in \mathcal{O}_{\mathbb{P}^n}(d)$,

we get $\varphi : \mathbb{P}^n \setminus \bigcap \mathbb{V}(F_i) \dashrightarrow \mathbb{P}^m$

$$[a_0 : \dots : a_n] \longmapsto [\dots : F_j(\underline{a}_i) : \dots]$$

is well defined away from the base locus.

BROKEN ARROW: partially defined.

Given $X \subseteq \mathbb{P}^n$ and $\mathbb{P}^n \overset{\varphi}{\dashrightarrow} \mathbb{P}^m$
$\quad \cup\!\!\mid$
$\quad X$

such that $\varphi(p)$ is well-defined $\forall\, p \in X$

this is called a morphism:

$$X \xrightarrow{\varphi} \mathbb{P}^m. \quad \text{If } \varphi(X) \subseteq Y$$

$Y \subseteq \mathbb{P}^m$ proj. then $\varphi$ is a morphism

$$\boxed{X \longrightarrow Y}.$$

<u>Examples</u>: • Veronese from $\mathbb{P}^d \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$

• Projection from a point: $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$

$x_1^2 = x_0 x_2$

proj from $[0:1:0]$

check regularity

In <u>coordinates</u>: $\{Z_n = 0\} = \mathbb{P}^{n-1}$; $p = [0:\dots:1]$

$\boxed{\text{Isomorphism definition}}$

Check that $V(x_1^2 - x_0 x_2) \cong \mathbb{P}^1$.

$\boxed{\text{Domain of } \varphi}$ $(F_i)$ & $(G_j)$ : $V \longrightarrow \mathbb{P}^m$

are equivalent if $F_i G_j - F_j G_i \in I^h(V)$.

<u>Domain</u>: there exists a point w/ a regular rep.

<u>Dominant</u>: $\varphi(\operatorname{dom} \varphi)$ is dense.

Birational: $X \overset{\varphi}{-\to} Y$

& $Y \underset{\psi}{-\to} X$ st

$\varphi \circ \psi$ & $\psi \circ \varphi$ are identity on an open dense.

Let $X$ & $Y$ be proj. irred.

THEOREM: $X$ & $Y$ birational $\iff$

$$k(X) = k(Y)$$

Sketch: One direction is obvious. Conversely,

given $k(X) \xrightarrow{\sim} k(Y)$ write

$k(X)$ & $k(Y)$ as function fields of complements of hypersurfaces

to reduce to the affine case.

$$k(X) = k(x_1, \ldots, x_n) \quad x_i = X_i / X_0$$

$$k(Y) = k(y_1, \ldots, y_n) \quad \& \quad y_j = Y_i / Y_0$$

so clears denominators and write a

rational map. Invert. $\square$.

FINALLY: A (quasi-projective) variety is an open in a projective variety.

THEOREM: A product of (quasi)-projective varieties remains quasi-projective

Proof is based on the fact that

$\mathbb{P}^n \times \mathbb{P}^m$ is projective:

$$\sigma_{m,n}: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

$$((\underline{x}_i), (\underline{y}_j)) \longmapsto (\cdots : x_i y_j : \cdots)$$

Equations: $\{Z_{ij} Z_{kl} - Z_{il} Z_{kj} = 0\}$

$Z_{ij}$ are the coordinates on $\mathbb{P}^{(n+1)(m+1)}$

Under this structure, $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^n$ is a morphism.

Equations / Map

- Map is clearly injective, as given

$c = [\dots : C_{ij} : \dots]$ and $a, b$ st $\sigma(a,b) = c$

w/ $a \in \mathbb{P}^n$ $b \in \mathbb{P}^m$

WLOG: take $C_{0,0} = A_0 = B_0 = 1$. This forces

$B_i = C_{0,i}$ and $A_j = C_{j,0}$

which uniquely determines $a$ & $b$.

- Image is closed. If $C = [\dots : C_{ij} : \dots]$

solves the equations, take $C_{0,0} = 1$ and

for any $k, l \neq 0$, $Z_{0,0} Z_{kl} - Z_{k,0} Z_{0,l}$

gives $\boxed{C_{k,l} = C_{k,0} C_{0,l}}$

Take $A_0 = B_0 = 1$, $A_k = C_{k,0}$ $B_l = C_{0,l}$

all done. $\square$.

# SMOOTHNESS & TANGENT SPACES

$X = \mathbb{V}(f) \subseteq \mathbb{A}^n_k$, $f$ irreducible $P \in X$.

Line is $L = \{(a_1 + b_1 t, \dots, a_n + b_n t) \mid t \in k\}$

$\underline{b} \in k^n \setminus \{0\}$

$X \cap \underline{L}$: $f(a_1 + b_1 t, \dots, a_n + b_n t)$

$= g(t) = \sum_{r \geq 1} c_r t^r$

$c_0 = f(P) = 0$, $c_1 = \sum b_i \cdot \frac{\partial f}{\partial z_i}(P)$

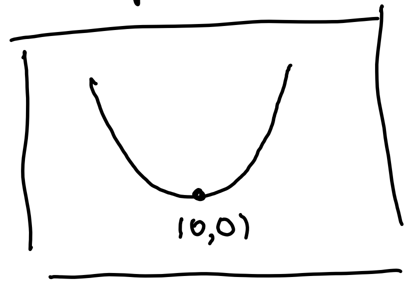
$L$ is tangent iff $L$ lies in, $(P = (a_1, .., a_n))$

$$T^{aff}_{X,p} = \mathbb{V}\left(\sum \frac{\partial f}{\partial z_i}(P)(z_i - a_i)\right)$$

we'll mostly use affine case

(Similar in projective case) either way we get a linear subvariety of dim $n-1$ or $n$.

**Ex**: $\mathbb{V}(y - x^2)$
$\quad\quad\quad\quad \overset{\shortparallel}{f}$

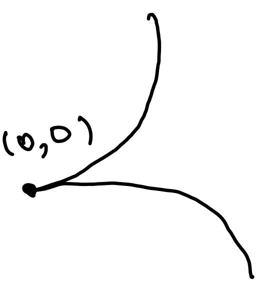



$\dfrac{\partial f}{\partial x} = -2x$

$\dfrac{\partial f}{\partial y} = 1,$

Tangent line: $(y = 0)$ TY

**Ex**: $\mathbb{V}(y^2 - x^3)$
$\quad\quad\quad\quad f$




$f^{lin} = 0 \qquad \mathbb{V}(0) = \mathbb{A}^2$

---

smooth points are where tangent space has dim $n$.

PROPOSITION: Smooth points are dense

Pf: singular points solve $f$ & $f^{lin}$

Pf: If $\frac{\partial f}{\partial z_i} = 0 \ \forall i$ then either

① $f$ is constant in char $k = 0$

② $f = g^p$ for $f \in k[z_1^p, \dots, z_n^p]$.

For $X = \mathbb{V}(I) \subseteq \mathbb{A}^n$ general, then for $P \in X$,

$$T_{X,P} = \left\{ (\underline{v}) \in k^n \;\middle|\; \sum v_i \frac{\partial f}{\partial z_i}(P) = 0 \text{ for all } f \in I(X) \right\}$$

$\subseteq k^n$

$$T_{X,p}^{aff} = p + T_{X,p}.$$

DEFINITIONS Smoothness, dimension

PROPOSITION: Generic smoothness.

DEFINITION: Let $X = V(I) \subseteq \mathbb{A}^n_k$ be a variety and let $P \in X$. Then

$$T_{X,P} = \left\{ \underline{v} \in k^n \;\middle|\; \sum v_i \frac{\partial f}{\partial z_i}(P) = 0 \;\; \forall f \in I(V) \right\} \subseteq k^n.$$

If $X \subseteq \mathbb{P}^n$ is quasi-projective, define $T_{X,P}$ by choosing $U \subseteq X$ affine and containing $P$.

Note: If $X$ affine then $T^{aff}_{X,P} = T_{X,P} + P$

DEFINITION: Let $X$ be an irreducible variety.

(i) $\dim X = \min \{ \dim T_{X,P} \mid P \in X \}$

(ii) Let $P \in X$. Then this point is smooth or nonsingular if $\dim T_{X,P} = \dim X$

Dimension of $X$ reducible is the maximum over the components.

THEOREM : The set of smooth points in $X$ is nonempty and open.

Proof : Say $I(X) = \langle f_j \rangle$. Then if $P \in X$,

$$T_{X,P} = \left\{ v \in k^n \;\middle|\; \langle v, f^{lin}(P) \rangle = \sum v_i \frac{\partial f_j}{\partial z_i}(P) = 0 \right\}$$

and so $$\dim T_{X,P} = n - \operatorname{rank}\left( \frac{\partial f_j}{\partial z_i}(P) \right).$$

In particular, the locus where the rank of a matrix is $\leq n-r$ for any $r$ is a closed subvariety, given by the vanishing of $(n-r)\times(n-r)$ minors

PROPOSITION: If $X$ and $Y$ are birational, then they have the same dimension.

Follows from the following lemma:

Given $X \overset{\varphi}{\dashrightarrow} Y$ and $p \in \mathrm{dom}(\varphi)$

we have a linearized

$$d\varphi : T_{X,p} \longrightarrow T_{Y,\varphi(p)}.$$

$$d\varphi_p(\underline{v}) = \left( \sum_{i=1}^{n} v_i \frac{\partial f_j}{\partial z_i}(p) \right)_j$$

where $(f_1, \dots, f_m)$ are local expressions for $\varphi$ regular at $p$.

PROPOSITION:

- $d\varphi_p(T_{X,p}) \subseteq T_{Y,\varphi(p)}$
- $d\varphi_p$ is independent of representation
- Composition: $X \dashrightarrow Y \dashrightarrow Z$

Corollary: Dimension is a birational invariant

Pf of Prop:

(1) Affine pieces are sufficient

Let $g \in I(Y)$ and $(f_1, .., f_m)$ the map.

Then $h(f_1, .., f_m)$ is regular at $p$

vanishing on $X$ when regular

$q = \varphi(p)$

Then Chain rule:

$$\boxed{\frac{\partial h}{\partial z_i}(p) = \sum_j \frac{\partial g}{\partial w_i}(q) \frac{\partial f_j}{\partial z_i}(p)}$$

so for $v \in T_{X,p}$ $\cancel{d\varphi(p)}$ $d\varphi_p(v) \in T_{Y,q}$.

- Given an irreducible variety $X$, its birational class is captured by $k(X)$

For example, $X = \mathbb{A}^2, \mathbb{P}^2, Bl_0 \mathbb{A}^2, \mathbb{P}^1 \times \mathbb{P}^1, \ldots$

all have $\boxed{k(X) \simeq k(z_1, z_2)}$

I've regularly mentioned, if $E = \mathbb{V}(y^2 - x(x-1)(x+1))$

then $E$ is an (affine elliptic curve) so

$$\boxed{k(E) \not\simeq k(\mathbb{P}^1)}$$

In general, what does the function field look like?

As a consequence of what will now follow:

THEOREM: Every variety $X$ is birational to a hypersurface.

## TRANSCENDENTAL EXTENSIONS

Let $K/k$ be a finitely generated field extension of $k$.

$K/k$ is a $\boxed{\text{pure transcendental extension}}$ if

$$K = k(x_1, \dots, x_n) \text{ for } x_1, \dots, x_n \in K$$

algebraically independent over $k$.

PROPOSITION: Let $K/k$ be a finitely generated field extension. There exists a pure ~~&~~ transcendental extension $K_0 = k(x_1, \dots, x_n)$ st

$K/K_0$ is finite & separable. Moreover,

$$K = K_0(y) \text{ for some } y \in K.$$

Remark: This integer $n$ is unique, called the transcendence degree. ⟵ Dimension!

$\boxed{\text{Proof in char}(k) = 0}$ By finite generation, $K = k(x_1, \dots, x_m)$

There is a maximal subset $\{x_i\}$ that is algebraically independent. Reorder it so $\{x_1, \dots, x_n\}$ is independent

Now $x_{n+1}, .., x_m$ are algebraic over $k(x_1, .., x_m)$ so $K/k(x_1, .., x_n)$ is finite. When $\mathrm{char}(k) = 0$ separability is automatic. Finally, primitive element. $\square$

$\mathrm{Char}(k) = p > 0$ this is still true but it requires a little bit more work. $\boxed{\text{Not examinable}}$

PROPOSITION: Let $K = k(x_1, .., x_n)$ pure transc. and let $x_{n+1}$ be algebraic over $K$. Then

$$I := \{ g \in k[z_1, .., z_m] \mid g(\underline{x}) = 0 \}$$

is principal, ie $(f)$. If $f$ contains $z_i$ then

$$\{x_1, .., x_{i-1}, x_{i+1}, ..., x_{n+1}\} \text{ is indpdt.}$$

$$\boxed{k[x_1, .., x_n] = k[z_1, .., z_{n+1}] / I = (f).}$$

Pf: $\{x_1, .., x_n\}$ are indpdt so $R = k[x_1, .., x_n]$ is isomorphic to $k[z_1, .., z_n]$, so is a UFD.

(I) Let $h \in K[T]$ be $x_{n+1}$'s minimal poly. and $b$ be the LCM of the coeff's. clear denominators, so

(2) $b h = f(x_1, \dots, x_n, T)$ for $f$ irreducible (by the Gauss' Lemma) UFD

$\in k[x_1, \dots, x_n]$

(II) Let $g \in k[\underline{z}_i]$. In $K[T]$, we know $g$ is a multiple of $h$ so by Gauss' Lemma $g$ is a multiple of $f$, so $I$ is principal

(III) • For last part, assume $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ was dependent. There exists $g \in I$ not involving $z_i$ but $g$ is a multiple of $f$ ↯

□

$\boxed{\text{Straightforward Corollary}}$: If $X$ is irreducible,

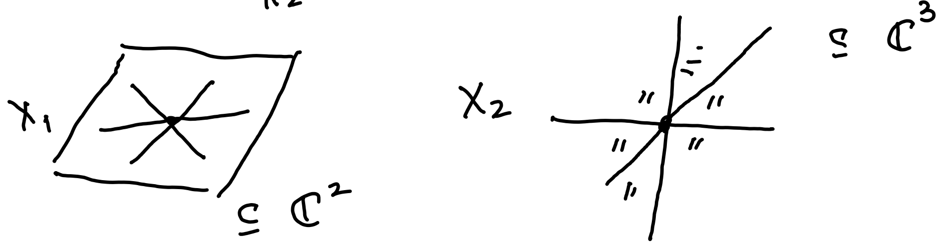
$\operatorname{tr.deg}(k(X)/k) = \dim_k X$

( Pf: Reduce to hypersurface case).

## NICE EXAMPLE

Consider $X_1 = \mathbb{V}(x \cdot y \cdot (x-y))$. and

$X_2 = \mathbb{V}(xy, yz, zx)$




$\boxed{X_1 \not\cong X_2}$ because if they were isomorphic, the tangent space dimension would be preserved.

## AFFINE NULLSTELLENSATZ

THEOREM: (1) Every maximal ideal of $k[z_1, .., z_n]$ is of the form $(z_1 - a_1, .., z_n - a_n)$ for $a_i \in k$.

(II) If $I \subsetneq k[\underline{z}]$ then $\mathbb{V}(I) \neq \emptyset$.

Proof $\boxed{k \text{ uncountable}}$ (See Reid or ... for the general)

(1) Every ideal of this form is maximal.

Let $\mathfrak{m} \subseteq k[\underline{z}]$ be a maximal ideal, and

$$K = k[\underline{z}]/\mathfrak{m} \quad \text{and} \quad a_i = z_i + \mathfrak{m} \in K.$$

Then $K = k[a_1, \dots, a_n]$. If $K = k$ then $a_i \in k$,

$z_i - a_i \in \mathfrak{m}$ and we're done.

Otherwise, let $t \in K \setminus k$. As $k = \bar{k}$, we have

$$k \subsetneq k(t) \subseteq K \quad \text{and } t \text{ is transcendental.}$$

Now, let $U_m$ be the $k$-vs spanned by

$$\{a_1^{m_1} \dots a_n^{m_n}\} \quad \boxed{\sum m_i = m}$$

$\dim U_m < \infty$ and $K = \bigcup_m U_m$. Now,

$\frac{1}{t-c}$ & for $c \in k$ are linearly independent

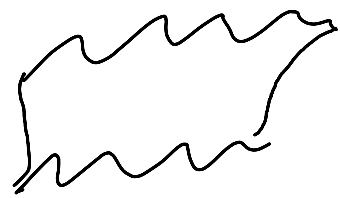
over $k$. But that's a contradiction.

(ii) By ACC, every ideal is contained in a maximal ideal, so $\mathbb{V}(I)$ contains $(a_1, \dots, a_n)$.

$\square$

**FOR THE REST OF THE COURSE WE'LL STUDY CURVES.** (minus digressions)

**DIGRESSION** Let $X \subseteq \mathbb{P}^3$ be a surface of degree $d \geq 4$. Assume $X$ is smooth and the coefficients of $f$ ($X = \mathbb{V}(f)$, $f \in \mathcal{O}_{\mathbb{P}^3}(d)$) are chosen generically.



**THEOREM:** $X$ contains no lines.

Pf: Needs the Grassmannian. (1) The space of surfaces is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(d)) = \mathbb{P}^N$. (2) Lines in $\mathbb{P}^3$ are planes in $k^4 \cong G(2,4) = \mathbb{G}$

$$\mathcal{Y} = \{(S, [\ell]) \mid S \text{ contains } \ell\} \subseteq \mathbb{G} \times \mathbb{P}^N$$

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{G} & & \mathbb{P}^N \end{tikzcd}$$

Dimension of $\mathbb{G}$ is 4; $\dim \pi_1^{-1}(pt) \cong \mathbb{P}^{N-d-1}$

$\dim \mathcal{Y} = N + 3 - d$. Once $d \geq 4$, $\pi_2$ has small image.

$\square$

# RATIONALITY OF A CUBIC SURFACE

**PROPOSITION**: $S = \mathbb{V}(Z_0^2 Z_1 + Z_1^2 Z_2 + Z_2^2 Z_3 + Z_3^2 Z_0) \subseteq \mathbb{P}^3$ is rational

**Proof**: $\varphi : S \dashrightarrow \mathbb{P}^2$; $[Z_i] \mapsto [Z_0 Z_3 : Z_1 Z_2 : Z_2 Z_3]$

$\psi : \mathbb{P}^2 \dashrightarrow S$

$$[X_0 : X_1 : X_2] \longmapsto [X_0 X_2 (X_0 X_2 + X_1^2) : -X_1 (X_0^2 X_1 + X_2^3) : X_2^2 (X_0 X_2 + X_1^2) : -X_2 (X_0^2 X_1 + X_2^3)]$$

Check everything by hand, no tricks

$\square$

---

Let $X$ be variety of dimension 1

**PROPOSITION**: If $X$ is irreducible and $Z \subseteq X$ is a proper subvariety then $Z$ is finite.

Proof: Enough to show for $X \subseteq \mathbb{A}^n_k$ an irreducible affine curve, enough also to assume

$W \subseteq X$ is irreducible.

We have $I(X) \subsetneq I(W)$ by Nullstellensatz.

$W \xhookrightarrow{\varphi} X$ induces $\varphi^* : k[X] \longrightarrow k[W] = \dfrac{k[\underline{z}]}{I(W)}$

If $W$ is not a point then $k[W] \neq k$.

Take $t \in k[W] \setminus k$, transcendental over $k$.

Take 1. $x$ in $k[X]$ w/ $\varphi^*(x) = 0$

2. $y$ in $k[X]$ w/ $\varphi^*(y) = t$.

CLAIM: $x$ and $y$ are algebraically independent.

because of their vanishing properties.

But $\operatorname{tr.deg}(k(X)/k) = 1$, which is a contradiction

$\square$

Let $X$ be an irreducible curve

- Function field: $k(X)$; $k(X)/k(t)$ is finite.

- Local ring: $\mathcal{O}_{X,p} = \{ f/g \mid g(p) \neq 0 \} \subseteq k(V)$ for $p \in X$. Also "a local ring"

Note: $m_p = \{ f \in \mathcal{O}_{X,p} \mid f(p) = 0 \}$ this is the unique maximal ideal. because the non-units form the unique maximal ideal.

$\boxed{\text{THEOREM}}$ If $p \in X$ is a smooth point then $m_p$ is a principal ideal. * Converse holds

PROOF: Assume $p \in X_0 \subseteq \mathbb{A}^n_k$ and $p = (0, \ldots, 0)$

$$k[X_0] = k[z_1, \ldots, z_n] / I(X_0) = k[x_1, \ldots, x_n] \text{ (for } x_i \text{ the image of } z_i)$$

$$\mathcal{O}_{X,p} = \{ f/g \mid f, g \in k[X_0],\ g \notin (x_1, \ldots, x_n) \}$$

$$\mathfrak{m}_p = \{ f/g \mid f \in (x_1, \dots, x_n),\ g \notin (x_1, \dots, x_n) \}$$

$$= x_1 \mathcal{O}_p + \dots + x_n \mathcal{O}_p$$

- Change of coordinates: $T_p^{aff} = \{ z_2 = \dots = z_n = 0 \}$

$\boxed{\text{We will show } \mathfrak{m}_p = (x_1)}$.

There exist $f_2, \dots, f_n \in I(X_0)$ such that

$$f_j = z_j - h_j \quad (2 \leq j \leq n)$$

where $h_j$ has no terms of degree $\leq 1$.
i.e no linear part. So in $\mathcal{O}_p$ we have

$$x_j = h_j(x_1, \dots, x_n) \in (x_1^2, x_1 x_2, \dots, x_n^2)$$
$$= \mathfrak{m}_p^2$$

Thus, $x_1$ generates $\mathfrak{m}_p / \mathfrak{m}_p^2$.

Does it generate $\mathfrak{m}_p$?

YES: Nakayama's Lemma.

LEMMA (Nakayama) Let $R$ be a local ring w/ maximal ideal $m$ and let $M$ be a fg module. If $mM = M$ then $M$ is $0$ + Wiki proof

Corollary: If $t_1, \ldots, t_n \in m$ generate $m$ iff their images generate $m/m^2$. as a $R/m$ vect. space.

Proof: Let $n \subseteq m$ be $\langle t_1, \ldots, t_n \rangle$. If the images of $t_i$ generate in $m/m^2$ then

$$n + m^2 = m + m^2 \Rightarrow n + m^2/n = m + m^2/n$$

$\Rightarrow m(m/n) = m/n$, so $m/n = 0$ so $\boxed{m = n}$ $\square$

$\boxed{\# m/m^2}$

Corollary: Let $p \in X$ be smooth. Then $\mathcal{O}_{X,p}$ is a DVR, so there exists

$$\partial_p : K(X)^* \longrightarrow \mathbb{Z} \quad \text{st}$$

$$\mathcal{O}_p = \{ f \mid \partial_p(f) \geq 0 \}$$

$$m_p = \{ f \in \partial_p(f) > 0 \}. \quad \text{If } \in k(X)^*$$

and $\pi_p$ = local par. $f = \pi_p^{\partial_p(f)} \cdot u$, $u \in \mathcal{O}_p^*$

PROOF: we know $m_p = (\pi_p)$ so $m_p^n = (\pi_p^n)$.

Consider $J = \bigcap_n (\pi_p^n)$. Since $m_p J = J$,

so $J = 0$.

- This defines $\nu_p(f)$ for $f \in \mathcal{O}_p \setminus \{0\}$.
- If $f \in k(X) \setminus \mathcal{O}_p$, the $f^{-1} \in \mathcal{O}_p$; $\nu_p(f^{-1}) = -\nu_p(f)$

$\square$

COROLLARIES

1. If $X$ is a nonsingular projective curve, any $\phi : X \dashrightarrow \mathbb{P}^m$ extends to a morphism.

PROOF: Let $\phi = (\varphi_0, \ldots, \varphi_m)$ and $p \in X$. Pick a local parameter $t$ at $p$. Let

$$n = \min \{ \operatorname{ord}_p (\varphi_i) \}. \text{ Then}$$

$$(t^{-n} \varphi_1(p), \ldots, t^{-n} \varphi_m(p)) = (\varphi_1, \ldots, \varphi_m)$$

is regular, hence $\varphi$ is a morphism.

$\square$

(11) Any birational map $C_1 \to C_2$ of smooth curves is an isomorphism.

THEOREM : The image of a projective variety under a morphism is Zariski closed.

Proof : Not given, not examinable.

aside

FINDING LOCAL PARAMETERS

Key example: if an affine plane curve $\{f \in k[x,y] ; C = \mathbb{V}(f)\}$, $p$ smooth then $x - x(p)$ is a ~~smooth~~ local parameter iff

$$\frac{\partial f}{\partial y}(p) \neq 0.$$

Consequence of Theorem : Any morphism of ~~proj~~ projective curves is either constant or surjective.

PROPOSITION (Morphisms) Let $\varphi : X \longrightarrow Y$ be a non-constant morphism of projective curves.

- $\varphi$ is finite
- $\varphi^* : k(Y) \longrightarrow k(X)$ is a finite extension

PROOF: • First statement is obvious.

- $\varphi(X)$ is infinite, hence dence, so $\varphi^*$ is well-defined. $k(Y)$ sits inside $k(X)$, both with tr. deg equal to 1 'finite extensions of $k(t)$.

$\square$

- The degree of $\varphi$ $\deg(\varphi)$ is given by

$$\deg(\varphi) = [k(Y) : k(X)].$$

We think of this as the "generic" number of preimages.

Given $\varphi$ and $p \in X$, the ramification $e_p = \nu_p(\varphi^* \pi_q)$ for $q = \varphi(p)$.

THEOREM

- Let $\varphi : X \longrightarrow Y$ be a morphism of projective curves

(I) $\varphi$ is surjective

(II) If $X$ & $Y$ are smooth then

$$\sum_{\substack{p \in X \\ p \mapsto q}} e_p = \deg(\varphi).$$

(III) $e_p$ is generically equal to 1

No proofs — some explanations.

CONSEQUENCES

Corollary: • If $X$ is smooth proj. irred and $f \in k(X)^*$ then $f$ is regular ~~at~~ at all $p \in X$ implies that $f$ is constant.

Consider $\phi : X \longrightarrow \mathbb{P}^1$

$(1:f)$. Then $\varphi(p) = \infty = (0:1)$

iff $f$ is not regular at $p$. But if $\varphi(X)$ misses

$\infty$ then $\phi$ is constant so $f$ is also.

Corollary Given $f \in k(X)^*$ the set of points where

$\nu_p(f) \neq 0$ is finite and $\sum_p \nu_p(f) = 0$

Proof: Assume non-constant. We have a

morphism $\varphi = (1:f) : X \longrightarrow \mathbb{P}^1$.

Near $0$, say $t$ is a local parameter. So

- $f(p) = 0 \Rightarrow e_p = \nu_p(\varphi^* t) = \nu_p(f)$

- If $f(p) = \infty$ $\frac{1}{t}$ is a parameter so

$$e_p = \nu_p\left(\varphi^* \frac{1}{t}\right) = -\nu_p(f).$$

$\deg(\varphi) = \sum$ zeroes $= \sum$ poles

# DIVISORS ON CURVES: the basics...

- Maps to projective space
- Rational functions with bounded poles
- Divisors associated to rational functions
- Equivalence of divisors. & hyperplane sections.

---

Digression: Image is closed for projective varieties.

- $\varphi: X \to Y$, the graph $\Gamma_\varphi$ is closed.
- $X$ proj., $X \times Y \xrightarrow{\pi} Y$ is closed

  ↳ $X = \mathbb{P}^n$ is enough

  $Y = \mathbb{A}^m$

↳ $Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$, then $y_0 \in \pi(Z)$ iff

$V(f_1(x, y_0), \ldots, f_k(x, y_0)) \neq \emptyset$..

$\pi(Z) = \bigcap_{s \gg n} T_s$ where $T_s = \{y_0 \mid m_{irr}^s \not\subseteq I_{y_0}\}$

Ideal at $y_0$

Fix on $s$. Let $k_i = \deg(f_i(x, y_0))$.

$\Rightarrow$ Every degree $s$ monomial is a lin. comb. of $f_i$'s w/ hom. deg $s-k$ coeffs.

so $S = \{f_i h_i\}$ spans $V \leftarrow \mathcal{O}(s)$.

. If ~~dim~~ $|S| < \dim V$ false.

. Otherwise: write poly's of $S$ ~~to~~ as linear comb. of monomials, we get a matrix of $|S| \times |V|$ dim.

Want: small rank $|V| \times |V|$ minors should vanish.

GOAL: Understand maps from curves to $\mathbb{P}^r_k$

Given a curve $X$ ($\subseteq \mathbb{P}^N_k$) and a morphism

$$X \longrightarrow \mathbb{P}^r_k$$ we obtain interesting

functions on open sets of $X$:

Pick a linear function $F \in \mathcal{O}_{\mathbb{P}^r}(1)$ and

consider
$$\begin{array}{ccccc} \mathbb{P}^r \setminus \mathbb{V}(F) & \xrightarrow{\sim} & \mathbb{A}^r & \rightrightarrows & \mathbb{A}^1 \\ \uparrow \varphi & & & \vdots & \\ X & & & & \end{array}$$

$\boxed{r \text{ rational functions}}$

Where are these defined?

- Away from $\varphi^{-1}(\mathbb{V}(F) = H)$ away from a hyperplane.
- Let the points be $p_1, \ldots, p_m := X \cap H$.

How does this work again? $X \xrightarrow{\varphi} \mathbb{P}^r$

$\varphi = [\varphi_0, \ldots, \varphi_r]$. Away from, say, $\{X_r = 0\}$

we have $\left(\frac{\varphi_0}{\varphi_r}, \frac{\varphi_1}{\varphi_r}, \dots, \frac{\varphi_{r-1}}{\varphi_r}, 1\right)$

$\varphi : X \setminus \{p_1, \dots, p_r\} \to \mathbb{A}^r$. But there is something more.

---

If we have $f \in k(X)^*$ w/ $\mathrm{ord}_p(f) < 0$ we'll say $f$ has a <u>pole</u> at $p$ or order $-\mathrm{ord}_p(f)$

---

Each of $\left\{\frac{\varphi_i}{\varphi_r}\right\}$ are rational functions with poles on $\{p_1, \dots, p_m\}$ and regular away.

$\boxed{\text{Keep track of orders}}$ $\Sigma a_i p_i = D$ is a <u>divisor</u>. This organizes $k(X)$ into managable pieces $L(D) = \mathcal{O}_X(D)$ of rational functions with poles bounded by $D$

DEFINITION: A divisor on $X$ is an element of $\bigoplus_{p \in X} \mathbb{Z}p$, i.e. a finite $\mathbb{Z}$ linear combination of points of $X$. $\boxed{\mathrm{Div}(X)}$

We have a degree homomorphism

$$\mathrm{Div}(X) \longrightarrow \mathbb{Z}$$ whose kernel is degree 0.

$\boxed{\text{How do degree 0 divisors arise?}}$

Let $f \in k(X)^*$, then $\mathrm{div}(f) = \sum \nu_p(f)\, p$

These form the $\boxed{\text{principal divisors}}$

$$\mathrm{Pic}(X) = \mathrm{Cl}(X) = \mathrm{Div}(X) / \mathrm{Prin}(X).$$

$D$ & $D'$ st $D - D'$ is principal are linearly equivalent

PROPOSITION: Principal divisors are degree zero.

Proof: Given $f \in k(X)^*$ take

$$\varphi : X \xrightarrow{(1:f)} \mathbb{P}^1 .$$

Take a local par. $t$ near $0$, then the zeroes are finite

and $\sum_{p \mapsto 0} e_p = \deg(\varphi)$

$$= \sum_{p \mapsto \infty} -e_p \quad \text{using } 1/t.$$

Everywhere else, $f$ doesn't vanish.

PROPOSITION (Exercise) Every divisor on $\mathbb{P}^1$ is principal.

---

Let $H = \mathbb{V}(L) \subseteq \mathbb{P}^n$ and $X \hookrightarrow \mathbb{P}^n$.

$\operatorname{div}(L) := \sum n_p p$, take $z_i(p) \neq 0$ and

$n_p := \nu_p(L/z_i)$

Similarly, hypersurface sections in higher degree

$\boxed{\text{What is linear equivalence?}}$ Two hyperplane sections in the same embedding.

**DEF**: Degree of $X \subseteq \mathbb{P}^r$ is the degree of any hyperplane section.

- Given $F \in \mathcal{O}_{\mathbb{P}^2}(d)$, $\mathbb{V}(F) \cap X$ is linearly equivalent to $\mathbb{V}(L_d)$ for $L_d$ a product of linear forms.

$\boxed{\text{Consequence: Bezout's Theorem}}$ Two distinct irreducible plane curves intersect in $(\leq)$ $c \cdot d$ point where $c$ & $d$ are the degrees.

$\boxed{\text{EFFECTIVITY}}$

$$L(D) = \{ f \in k(X)^* \mid f = 0 \text{ or } \operatorname{div}(f) + D \geq 0 \}$$

$\boxed{\text{PROPOSITION:}}$ Let $\ell(D) = \dim_k L(D)$

(i) If degree $D < 0$ $\ell(D) = 0$ (ii) For any $p \in X$, $\ell(D) \leq \ell(D-p) + 1$ (iii) $\deg(D) \geq 0$, $\boxed{\ell(D) \leq d+1.}$

<u>PROOF:</u> Let $D \in \mathrm{Div}(X)$ be a divisor and

$$l(D) = \dim L(D) = \{0\} \cup \{ f \in k(X) \mid \mathrm{div}(f) + D \geq 0 \}$$

(I) Suppose $f \in L(D)$ nonzero. Then

$\mathrm{div}(f) + D \geq 0$. But degree of $D$ is negative. and $\deg(\mathrm{div}(f)) = 0$

(II) Let $p \in X$ and $n = [D]_p$ (coefficent)

Define $\alpha : L(D) \longrightarrow k$

$$f \longmapsto \pi_p^n \cdot f.$$

The kernel has a pole of smaller order, so dim drops by 1

(III) Apply (II) repeatedly

$\square$

<u>Remark</u>: For $D \sim E$, $L(D) \xrightarrow[\cdot g]{\sim} L(E)$

where $D - E = \mathrm{div}(g)$.

$\boxed{\text{What do these look like?}}$ If we take $D = n \cdot [\infty]$ on $\mathbb{P}^1$ for $x = z_1 / z_0$, $L(D)$ is spanned by

$1, x, x^2, \dots, x^n$ so $l(D) = n+1$

$$\left\{ \begin{array}{c} \text{INTERLUDE} \\ 75\tfrac{1}{2} \end{array} \right\}$$

## $\boxed{\text{How does } \ell(D) \text{ behave?}}$

- For $D = m\cdot(\infty)$ on $\mathbb{P}^1$ $\ell(D) = m+1$ as we saw last time.

- In fact, for $m$ points $p_1, \dots, p_m$ $\boxed{\sum p_i = D}$ $\ell(D)$ is still $m+1$.

- Let $E = \overline{\mathbb{V}(y^2 - f(x))}$ where $f(x)$ is a cubic, projective closure in $\mathbb{P}^2_k$. Take $E \xrightarrow{\varphi} \mathbb{P}^1$ given by $[1:x]$. Take $\varphi^{-1}(\infty) = D$, a degree 2 divisor (why?). At least one non-constant rational function, so

$$\ell(D) \geq 2.$$

If $p \in E$ then $L(p) = \{0\}$. why? If we had $f \in L(p)$ non-constant, we would have $f^{-1}(\infty) = p$. So $E \xrightarrow{\varphi} \mathbb{P}^1$ st in fact, every $q \in \mathbb{P}^1$ has a unique preimage. this would force $E \cong \mathbb{P}^1$ but it is not (Ex Sh III).

Fact: If $\deg(D) \geq 2$ $\boxed{\ell(D) = d}$ $\left\{ \text{Theory of elliptic functions} \right\}$

.. (A word about the sum of residues being 0)

**DIFFERENTIALS** Let $K/k$ be a field extension

Informally, a <u>differential</u> is a finite sum of expression $f\,dx$. $x, f \in K$

**DEFINITION** The space of Kähler differentials is the quotient $M/N$ where

$M =$ (K-vector space generated by symbols $\delta x$, $x \in K$)

$N =$ (<u>subspace</u>: $\delta(x+y) = \delta(x) + \delta(y)$
$\delta(x.y) = \delta(x)y - \delta(y)x$
$\delta(\lambda) = 0$ for $\lambda \in k$)

$\Omega_{K/k} = \Omega_K = M/N$

The map $\left.\begin{array}{l} K \longrightarrow \Omega_K \\ f \longmapsto df \end{array}\right\}$ EXTERIOR DIFFERENTIATION

**DEFINITION** A deri<u>va</u>tion over $k$ is any map of vector spaces $D: K \longrightarrow U$, $U$ is a K-vector space satisfy the product rule.

$\boxed{\text{TAUTOLOGY}}$ For any $k$-derivation $D : K \to U$ then there is a factorization

$$\begin{tikzcd} \Omega_K \arrow[rr, "\lambda :"] & & U \\ & K \arrow[ul, "d"] \arrow[ur, "D"'] \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

and these are all of them.

Proof: Exercise. □

$\boxed{\text{How to span } \Omega\text{?}}$

LEMMA: • If $f = g/h \in k(X_1, \dots, X_n)$ then for $y = f(x_1, \dots, x_n) \in K$ then $dy = \sum \frac{\partial f}{\partial X_i}(x_1, \dots, x_n) dx_i$

• If $K = k(x_1, \dots, x_n)$ then $\{dx_i\}$ spans $\Omega_K$.

Proof: (I) Calculus (II) Immediate □

LEMMA: Let $K/k(t)$ be finite and separable and $t$ is transcendental, then $\Omega_K$ spanned by $dt$.

PROOF : If $K = k(t)$ $\dim \Omega_K \leq 1$, so it suffices to exhibit a derivation that is nonzero.

Take $D = \frac{d}{dt}$.

If not, then $K_0 = k(t)$, $K = K_0(\alpha)$ by primitive element. Let $h \in K_0[X]$ be the min. polynomial. spanning by $dt$, $d\alpha$ is easy. Also:

$$0 = d(h(\alpha)) = D_t h(\alpha)\, dt + h'(\alpha)\, d\alpha$$

so $dt$ is enough.

To show dimension is nonzero, build an extension from $K_0$ to $K$ of $d/dt$

Define : $D : K_0[X] \longrightarrow K \cong K_0[X]/h$

- $D(f) = D_t(f)$ for $f \in K_0$
- $D(X) = -\dfrac{D_t h(\alpha)}{h'(\alpha)}$ • $D(X^n) = n\alpha^{n-1} D(X)$

Check this vanishes on $h$, so this gives nonzero $\square$

Given $p \in X$, we can take

$$\Omega_{X,p} \subseteq \Omega_K \quad \text{as those} \quad \omega = \sum f_i \, dg_i$$

$f_i, g_i \in \mathcal{O}_{X,p}$.

THEOREM: The module $\Omega_{X,p}$ is free over $\mathcal{O}_{X,p}$, generated by $d\pi$ for $\pi$ a local parameter.

why? We get divisors out of this!

Given $\omega \in \Omega_{K(X)}$ and $p \in X$, let

$$\nu_p(\omega) = \nu_p(f) \quad \text{for} \quad \omega = f\, d\pi$$

$\rightsquigarrow \boxed{\mathrm{div}(\omega)}$

Proof of theorem · $\Omega$ is finitely generated by formal calculus. Now apply Nakayama's Lemma. □

EXAMPLE $X = \mathbb{P}^1$ with $\omega = dt$. Then

$\nu_p(dt) = 0$.

$$\nu_\infty(dt) \rightsquigarrow dt = -t^2 \, d(1/t)$$

$$\nu_\infty(dt) = -2[\infty]$$

$\boxed{\text{MAJOR DEFINITIONS}}$ • $\operatorname{div}(\omega) = K_X$ is the canonical divisor of $X$.

• the genus of $X$ is $g(X) := \ell(K_X)$.

$\boxed{\text{Topology and the genus}}$ Remarks on number of holes over $\mathbb{C}$, curvature

Next time: If $X \subseteq \mathbb{P}^2$ is a plane curve we will calculate $K_X$ explicitly as $(\deg(X) - 3) \cdot H$

---

THE MAIN THEOREMS

• (Degree-genus formula): If $X \subseteq \mathbb{P}^2$ is a smooth degree $d$ curve, then $g(X) = \frac{(d-1)(d-2)}{2}$ (* question of Ex II)

• (Riemann-Roch): For $D \in \operatorname{Div}(X)$ with $g = g(X)$ $d = \deg(D)$, we have $\boxed{\ell(D) - \ell(K-D) = d - g + 1}$

$\rightsquigarrow$ $\deg(K) = 2g-2$, for $D \gg 0$ $\ell(D) = d - g + 1$, ...

OTHER FACTS

- For any genus $g$, there exists a curve $X$ with $g(X)=g$ such that there exists a degree 2 divisor $D$ on $X$ with $\ell(D)=2$. HYPERELLIPTIC CURVES

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## AN EXPLICIT CALCULATION (good for health!)

Let $C = \mathbb{V}(y^2 - (x-e_1)(x-e_2)(x-e_3))$. Take the rational function $y$ on $X$ and we claim $y$ has a pole of order 3 at $[0:1:0]$ and zeroes at $[e_i:0:1] = P_i$

---

To see the pole and its order homog. to

$$z_1^2 z_2 = (z_0 - e_1 z_2)(z_0 - e_2 z_2)(z_0 - e_3 z_2)$$

$$\quad \| $$

$$z_1^2 z_2 = z_0^3 + a z_0 z_2^2 + b z_2^3$$

and set $z_1 = 1$ to get

$$\boxed{u = v^3 + a u^2 v + v^3}$$

the point is now $(0,0)$. What does the curve look like? Well $v$ is a local parameter and $u$ has a triple zero.

Relate the two: $\boxed{u = v^3 + av^7 + \cdots}$

Now, $y = \frac{1}{u}$ and $x = \frac{v}{u}$.

Calculate $\operatorname{div}(x - e_i)$ on $y^2 = (x-e_1)(x-e_2)(x-e_3)$

---

PROPOSITION: Let $X = \mathbb{V}(F) \subseteq \mathbb{P}^2$ be a curve of degree $d \geq 1$. Then $K_X = (d-3)\cdot H$. A basis for $L(K_X)$ is given by $\left\{ \frac{x^r y^s}{\partial f/\partial y} dx \,\middle|\, 0 \leq r+s \leq d-3 \right\}$

for $\{f = 0\}$ an affine equation.

PROOF:

$\boxed{\text{STEP 1} \quad \text{Affine patch calculation}}$ Choose coordinates such that $[0:1:0] \notin X$. Dehomogenize away from $\{Z_0 = 0\}$ and set

$$\boxed{x = \frac{Z_1}{Z_0} \quad \text{and} \quad y = \frac{Z_2}{Z_0}}$$

These are rational functions on $X$. They are related: $f(x,y)=0$ where $f(T_1,T_2) = F(1,T_1,T_2)$

Equation in $\Omega_X$: $\dfrac{\partial f}{\partial T_1}(x,y)\,dx + \dfrac{\partial f}{\partial T_2}(x,y)\,dy = 0$

Write a differential:

$$\omega = \frac{dx}{(\partial f/\partial T_2)(x,y)} = \frac{-dy}{(\partial f/\partial T_2)(x,y)}$$

Claim: $\boxed{\operatorname{div}(\omega) = (d-3)H_\infty \qquad H_\infty = \{Z_2 = 0\}}$

$\boxed{\text{STEP 2}}$ Finding local parameters away from $\infty$.

- If $\dfrac{\partial f}{\partial T_2}(P) \neq 0$ then $x - x(P)$ is a local parameter. Otherwise $y - y(P)$ is a local parameter.

$\boxed{\text{CONCLUSION}}$ $\nu_p(\omega) = 0$ for all $p \in X \cap \{Z_2 \neq 0\}$.

$\boxed{\text{STEP 3}}$ The calculation at $\infty$. Any point at $\infty$, is contained in the piece $\{Z_2 = 0\}$

The equation is : $g(z,w)=0$ where

$$g(S_1, S_2) = F(S_1, S_2, 1) \quad \text{and}$$

$$z = \frac{Z_0}{Z_2} \quad \text{and} \quad w = \frac{Z_1}{Z_2} = x/y$$

$$\frac{Z_0}{Z_2} = 1/y$$

Take a (different) form

$$\eta = \frac{dz}{(\partial g/\partial S_2)(z,w)} = \frac{-dw}{(\partial g/\partial S_1)(z,w)}$$

The preceding argument shows $\nu_p(\eta)=0$

$\boxed{\text{RELATE } \omega \text{ and } \eta}$ Use the rules of calculus

and homogenization to show

$$\boxed{\omega = z^{d-3} \cdot \eta} \quad T_2^d\, g\left(\frac{1}{T_2}, \frac{T_1}{T_2}\right) = f(T_1, T_2)$$

$\boxed{\nu_p(a \cdot b) = \nu_p(a) + \nu_p(b)}$ so $\mathrm{div}(\omega) = 3 \cdot \mathrm{div}(Z_0)$

# BASICS WITH GENUS

- Calculation from before, on $\mathbb{P}^1$ we have

$$\operatorname{div}(dt) = -2\cdot[\infty].$$ In particular

$$L(K_{\mathbb{P}^1}) = 0$$

- Let $X = \mathbb{V}(F)$ nonsingular $F \in \mathcal{D}_{\mathbb{P}^2}(3)$. Say say affine equation is $\boxed{y^2 - g(x)}$ with

$g(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$. Observe

$$\boxed{2y \cdot dy = g'(x)\,dx}$$ Take $\boxed{\omega = \frac{dx}{y}}$

Then $\mathcal{O}_p(\omega) = 0$ so $g(X) = 1$.

$\boxed{\text{How?}}$ 3 types of points to check. • If $y(p) \neq 0$ then $x - x(p)$ is a local parameter. • If $y_p = 0$ then $x(p) = \lambda_i$, then $\partial f/\partial x(p) \neq 0$ so $y$ is a loc. par.

- If $p = [0:0:1]$ change coordinates and use calcs.

l. $d_1, d_2 \in \mathbb{Z}_{>0}$ and $F$ be a bihomogeneous poly. of degree $(d_1, d_2)$ and $X = \mathbb{V}(F) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ a smooth curve.

PROPOSITION: $K_X = (d_1 - 2)H_1 + (d_2 - 2)H_2$ where $H_1$ and $H_2$ are divisors of the two fibers of projections

**GENUS FORMULAE**

- Degree $d$ plane curve has genus $\binom{d-1}{2}$.
- Bidegree $(d_1, d_2)$ has genus $(d_1 - 1)(d_2 - 1) \longrightarrow$ Note to self: put on Ex. sh. $\overline{\text{IV}}$.

**GROUP LAW ON THE ELLIPTIC CURVE:**

$P, Q, P_0 \in E$. $\ell(P + Q - P_0) = 1$ so $P + Q - P_0 \sim R$. Set $\boxed{P + Q = R}$

THEOREM: The addition operation turns $E$ into an abelian group (variety). The map $E \to \mathrm{Pic}^0(E)$ given by $p \longmapsto [p - P_0]$ is an isomorphism

<u>PROOF</u>: Let $AJ : E \to \mathrm{Pic}^0(E)$ be the map sending $p$ to $[p - p_0]$.

Injectivity: $AJ(p) = AJ(q)$ means $p - p_0 \sim q - p_0$ so $p \sim q$. But then $p - q = \mathrm{div}(f)$

But there is a unique (up to scalar) function w/ a pole at $p$ so $\boxed{p = q.}$ since $l(p) = k$.

Surjectivity: Let $D \in \mathrm{Div}^0(E)$. Then $D + p_0$ has degree 1 so $l(D + p_0) = 1$. Therefore, $D + p_0$ is equivalent to an effective divisor.

$P \sim D + p_0$. But then $[D + p_0] = [p]$

so $[D] = AJ(p)$.

Homomorphism property is apparent by inspection. $\square$

ASIDE ON ABEL-JACOBI (DIDN'T GO EXACTLY LIKE THIS IN LECTURE)

<u>Fact</u>: $\mathrm{Jac}(X)$ is a projective variety with an abelian group structure. AJ always embeds the curve. More generally $X^n \to \mathrm{Jac}(X)$.

We have

$$\begin{array}{ccccc} X^n & \longrightarrow & \mathrm{Sym}^n(X) & \xrightarrow[AJ]{} & \mathrm{Jac}(X) \\ & & \| & & \\ & & X^n / S_n & & \\ (p_1, \dots, p_n) & & \longmapsto & & \sum p_i - np_0. \end{array}$$

$\boxed{\text{What are the fibers?}}$

$$AJ^{-1}([D]) = \{ E \in \mathrm{Sym}^n(X) \mid D \sim E \}$$

Once $n \gg 0$, i.e. $n > 2g-2$, the fibers of $AJ$ are $\mathbb{P}(L(D)) = \mathbb{P}^{n-g}_K$.

$\boxed{\text{FANCY SPEAK}}$ $\mathrm{Sym}^n(X)$ is a projective bundle over $\mathrm{Jac}(X)$. For small $n$, different fibers have different dimensions.

$\boxed{\text{OTHER GEOMETRY}}$ $\mathrm{Jac}(X)$ is smooth of dimension $g$.

Natural divisor (hypersurface) on $\mathrm{Jac}(X)$ is $\Theta = \mathrm{im}\left(\mathrm{Sym}^{g-1}(X)\right)$.

<u>THEOREM</u>: Let $E \subseteq \mathbb{P}$ be the plane cubic

$$V(F), \quad F = Z_0 Z_2^2 - \prod_{i=1}^{3}(Z_1 - \lambda_i Z_0), \quad \lambda_i \neq \lambda_j$$

Take $O_E = [0:0:1]$. The group on $E$ is given

by $P +_E Q +_E R = O_E \iff P, Q, R$ are collinear.

<u>PROOF</u>:

$P +_E Q +_E R = O_E$ iff $P+Q+R \sim 3P_0$ which

holds iff $\operatorname{div}(f) = P+Q+R-3P_0$, with $f \in k(E)$.

Now, $\ell(3P_0) = 3$ and

$$\boxed{L(3P_0) = \langle 1, x, y \rangle = \langle 1, \frac{Z_1}{Z_0}, \frac{Z_2}{Z_0} \rangle}.$$ why?

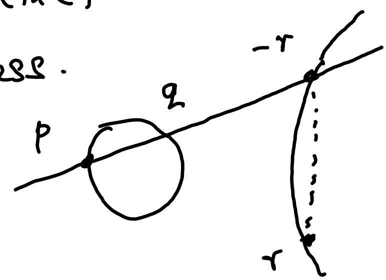
then $f$ must be of the form $\frac{G}{Z_0}$ for $G \in \mathcal{O}_{\mathbb{P}^2}(1)$

and $\operatorname{div}(G) = P+Q+R$

$\square$.

$\boxed{\text{PICTURE}}$ (Didn't draw in lecture)

The "chord-and-tangent" process.




$p+q=r$

# $\boxed{\text{RIEMANN-HURWITZ}}$ (char $k = 0$)

Let $\phi : X \to Y$ be a finite morphism of curves

Let $\omega = f\,dt \in \Omega_Y$ with $k(Y)/k(t)$ finite.

Then $k(X) \,|\, \phi^* k(t)$ is also finite so generated by $\phi^* dt$ (Proof from last week).

DEFINE: $\boxed{\phi^*(f\,dt) := \phi^* d(\phi^*(t))}$ Makes sense?

(GENERA)

Goal: Compare $g(X)$ & $g(Y)$. To get there, compare

$$\nu_p(\phi^*(\omega)) \quad \text{and} \quad \nu_{\phi(p)}(\omega).$$

LEMMA: Let $e_p$ the the ramification index near $p$.

Then
$$\nu_p(\phi^* \omega) = e_p \cdot \nu_{\phi(p)}(\omega) + e - 1$$

Proof: Omitted; basic point is that if $q = \varphi(p)$,

then $d(\varphi^* \pi_q) = d(\pi_p^e) = e \cdot \pi_p^{e-1} \cdot d\pi_p$.

Apply this to local expressions for $\omega$.

$\square$

$\boxed{\text{THEOREM (Riemann-Hurwitz)}}$ Let $\varphi : X \to Y$ have degree $d$. Then

$$2g(X) - 2 = d(g(Y) - 2) + \sum_{p \in X} (e_p - 1)$$

Proof: $2g(X) - 2 = \deg(\operatorname{div}(\varphi^* \omega))$

$$= \sum_{p \in X} \nu_p(\varphi^* \omega)$$

$$= \sum_{q \in Y} \sum_{p \mapsto q} \nu_p(\varphi^* \omega)$$

$$= \sum_{q \in Y} \sum_{p \mapsto q} (e_p \nu_q(\omega) + e_p - 1)$$

$$= \sum_{q \in Y} \Big( d \cdot \nu_q(\omega) + \sum_{p \mapsto q} (e_p - 1) \Big)$$

$$= d(2g(Y) - 2) + \sum e_p - 1$$

$\square$

$\boxed{\text{ALSO}}$ Topological proof via triangulations — choose a triangulation with vertices on the ramification points.

ALSO: In $\mathrm{char}(k) = p$, need $\varphi$ to be separable, otherwise $\varphi^* \equiv 0$. Proof holds in the "tame case", i.e. $p \nmid e_p$ for all $p$. Modified statement in general

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Enumerative Geometry (aside)

Let $\alpha + \beta = 3d + g - 1$. How many irreducible degree $d$ genus $g$ curves in $\mathbb{P}^2$ pass through $\alpha$ points and $\beta$ lines in general position?

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- If $\beta = 0$, these numbers all known via physics formulae (Gromov-Witten theory)

- Ex: $d=1$, $g=0$, line through 2 points
  $d=4$, $g=3$ (smooth), $\alpha=0$ $\beta=14$, $\boxed{23011191144}$
  $d=5$ open. (Dhruv's failed thesis problem)

---

Note to self:

→ Don't forget basis for $\Omega_X$ $X \subseteq \mathbb{P}^2$ $\langle x^r y^s \frac{\partial f}{\partial y} dx \rangle$
didn't have time to state

## APPLICATIONS



THEOREM (Lüroth) Let $k \subseteq L \subseteq k(t)$ be a tower with $k = \bar{k}$. Then $k$ is purely transcendental.

LEMMAS: Let $X$ be a curve

(i) If $p, q \in X$ st $p \sim q$ then $X \cong \mathbb{P}^1$.

(ii) If $g(X) = 0$ then $X \cong \mathbb{P}^1$.

(iii) If $X \to Y$ is finite $g(X) \geq g(Y)$.

Proofs: (i) Take $f \in L(p-q)$ with $X \to \mathbb{P}^1$ given by $[1:f]$. This is birational (2) Apply (1) and Riemann-Roch

(iii) Apply Riemann-Hurwitz.

---

**PROOF OF LÜROTH** Assume $L \neq k$ so $\mathrm{tr.deg}(k) = 1$. There exists a curve $X$ st $k(X) = L$ (why?) then $L \hookrightarrow k(t)$, after clearing denominators gives $\mathbb{P}^1 \to X$ so $X \cong \mathbb{P}^1$, so $k(X)$ is pure transcendental.

$\square$

# FERMAT'S LAST THEOREM FOR POLYNOMIALS

THEOREM: Let $k = \bar{k}$ with $\mathrm{char}(k) = 0$. If $f, g, h \in k[x, y]$

st $\boxed{f^n + g^n = h^n}$ nonconstant and homogeneous

then $n \leq 2$.

Proof: The genus of $\underset{\overset{\shortparallel}{Y}}{\mathbb{V}(z_0^n + z_1^n - z_2^n)}$ is

$\frac{1}{2}(n-1)(n-2)$ so if we had such

$f, g, h$ we'd get a map $\mathbb{P}^1 \to Y$ so

$g(Y) = 0$ i.e. $n = 1$ or $2$.

$\square$

Exercise: Calculate the genus via Riemann-Hurwitz

$\rightsquigarrow$ Take $Y \longrightarrow \mathbb{P}^1$

$[z_0, z_1, z_2] \longmapsto [z_0, z_1]$. Fixing two of

those, find $n$ preimages (when $z_0^n + z_1^n \neq 0$)

we get $n$ points with ramification $n$.

Final topics: • Canonical morphism & embedding criteria • Equations for a genus 1 curve • Canonical is an embedding for non-hyperelliptic curves

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• Let $\iota: X \hookrightarrow \mathbb{P}^n$ be an embedded projective curve of degree $d$ not contained in any hyperplane.

Given $H = \mathbb{V}(Z_0)$ (or any hyperplane) consider the divisor $\boxed{H \cap X := \operatorname{div}(Z_0)}$ generically this is just $\sum_{p \in H \cap X} [p]$.

Let $D =$ divisor $H \cap X$

we have a morphism

$$\iota^*: \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow L(D)$$

$$F = \sum \lambda_i Z_i \longmapsto F/X_0 \qquad \boxed{\text{why injective?}}$$

$\boxed{\text{CONVERSELY}}$

Given a basis $\varphi_0, \dots, \varphi_r \in L(D)$ we'd obtain

$$\varphi_D: X \longrightarrow \mathbb{P}^r \qquad \text{(How?)}$$

Note: How does $D \sim D'$ affect $L(D')$?

How to tell if $\varphi$ is an embedding? 

That is $\operatorname{im}(\varphi) \cong X$?

---

Two observations

- Given $p, q \in X$ distinct, there exist linear $F$ & $G$ st $F(p) \neq 0$ & $G(p) = 0$, $G(q) \neq 0$.

Thus,
$$\begin{cases} \iota^*(F) \in L(D) \setminus L(D-p) \\ \iota^*(G) \in L(D-p) \setminus L(D-p-q) \end{cases}$$

Therefore $\quad \ell(D-p-q) \leq \ell(D) - 2$

But every point drops the dim $L(D)$ by at most 1. so $\boxed{\ell(D-p-q) = \ell(D) - 2}$

- As $p$ is smooth it has a tangent line $T_p$ and there exists $F$ st $F(P) = 0$ but $T_p \not\subseteq V(F)$.

Therefore the multiplicity of $p$ in $\operatorname{div}(F) = 1$ so
$$\iota^*(F) \in L(D-p) \setminus L(D-2p)$$

$\boxed{\text{Embedding criterion}}$ For any $p, q \in X$ we have

$$l(D-p-q) = l(D) - 2 \qquad (*)$$

"separates points and tangent vectors".

THEOREM: $\varphi_D$ is an embedding if and only if (*) is satisfied. i.e. if D is $\boxed{\text{very ample}}$.

Proof: One side is obvious, other side is omitted;

remark: over $\mathbb{C}$, this gives an embedding i.e. $\varphi_D(X)$ is a curve. By (*) it is smooth so $k(X) = k(\varphi_D(X))$ and we're done □

COROLLARY: If $\deg D > 2g$ then $\varphi_D$ embeds.

COROLLARY: If $X$ is a curve of $g(X) \geq 3$ then $D = 2 \cdot K_X$ is an embedding.

Terminology: If $mD$ is very ample for some $m \geq 1$ then D is $\boxed{\text{ample}}$.

Let $K = K_X$ be the canonical. $L(K)$ is the vector space of holomorphic (= regular) differential forms.

Definition of a hyperelliptic curve: $g(X) \geq 2$ and either

① $\exists\ X \to \mathbb{P}^1$ degree 2

② $\exists\ D \in \mathrm{Div}(X)$ st $\ell(D) = \deg(D) = 2$

THEOREM: If $X$ is non-hyperelliptic then

$$\varphi_K : X \to \mathbb{P}^{g-1} \text{ is an embedding.}$$

Proof: If $\varphi_K$ is not an embedding there exist $p, q \in X$ st $\ell(K - p - q) \geq g-1$. Apply Riemann-Roch to $D = p+q$.

$$\ell(D) = \ell(K - D) + 1 - g + \deg(D)$$

$g \geq 2$ so $\ell(D) \geq 2$. Take $f_0, f_1 \in L(D)$ to obtain $X \to \mathbb{P}^1$ of degree 2.

□

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Low genus, hyperelliptic

- If $g = 0$, $D = n \cdot p$, then $\mathbb{P}^1 \to \mathbb{P}^n$ is the $n$'th Veronese.

If $g=1$ take $(E, p_0)$. Consider $D = 3p_0$

$\ell(np) = n$ by Riemann-Roch $n \geq 1$. So

$L(p_0) = k \subsetneq L(2p_0) = \langle 1, x \rangle$, $v_{p_0}(x) = -2$

$\subsetneq L(3p_0) = \langle 1, x, y \rangle$, $v_{p_0}(y) = -3$.

Then $L(4p_0) = L(3p_0) \oplus k \cdot x^2$

$L(5p_0) = L(4p_0) \oplus k\, xy$

But $x^3$ & $y^2$ have val. $-6$ so lie in

$L(6p_0) \setminus L(5p_0)$. Thus,

there is a linear dependence in

$\{1, x, x^2, x^3, y, y^2, xy\}$

$\rightsquigarrow$ Take $f(x,y) \in k[x,y]$. Let $E = \overline{V(f)}$ in $\mathbb{P}^2$.

Now, $\varphi_{3p_0} : X \longrightarrow \mathbb{P}^2$ and its image lies in

$\mathbb{V}(F)$ where $F(1, x, y) = y^2 + a_1 xy + a_3 y$

$$- (x^2 + a_2 x^2 + a_4 x + a_6).$$

w/ $p_0 = [0:0:1]$. $\boxed{\text{Weierstrass}}$

As $g(X) = 1$, the image must be $\mathbb{V}(F)$, nonsingular.

$\square$.

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THEOREM: Every hyperelliptic curve can be embedded in a quadric surface. ~~of~~ in bidegree $(2, g+1)$

Hyperelliptic curve: $X \longrightarrow \mathbb{P}^1$ : Riemann-Hurwitz says there are exactly $2g+2$ branch / ramification points.

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FINAL LECTURE: • One structure (a scheme) • one space worth studying ($M_g$) • One formula / $\chi(M_g) = \dfrac{\zeta(1-2g)}{2-2g}$.