ALGEBRAIC GEOMETRY, SHEET IV: LENT 2022

1. Let X be the projective closure of the affine curve $y^3 = x^4 + 1$. Prove that this curve is smooth and prove that it has a unique point at infinity. Calculate the zeroes and poles of the differential
$$\omega = \frac{dx}{y^2}.$$
2. Let V be a curve and $p \in V$. Prove that there exists a non-constant rational function on V that is regular away from p.
3. Let V be a curve and $p \in V$. Prove that the variety $V \setminus \{p\}$ is affine.
4. Let V be a curve of genus $g \geq 2$. Prove that V admits a degree 2 morphism to $\mathbb{P}^1$ if and only if there exists an effective divisor D on V of degree 2 such that $\ell(D) \geq 2$.
5. Prove that a smooth plane quartic curve is not hyperelliptic by examining the map determined by the canonical divisor.
6. ($\star$) Let F be a bihomogeneous polynomial of bidegree (d_1, d_2) in in 4 variables X_0, X_1 and Y_0, Y_1. Assume that $\mathbb{V}(F) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth curve V.[1] By adapting the calculation for $\mathbb{P}^2$ from lectures, calculate the degree of the canonical divisor of V and deduce that the genus of C is $(d_1 - 1)(d_2 - 1)$. Deduce that there exists a curve of genus g for all $g \in \mathbb{Z}_{\geq 0}$.
7. Let Q_1 and Q_2 be two smooth quadric surfaces in $\mathbb{P}^3$. Assume that their intersection $Q_1 \cap Q_2$ is a smooth curve. Calculate the genus of this curve. [*One way to go about this is via the geometry of the Segre embedding and using the previous question*].
8. Prove that if V and W are smooth projective curves then $\mathbb{C}(V) \cong \mathbb{C}(W)$ if and only if V is isomorphic to W.
9. Let $\varphi : V \to W$ is a morphism of smooth curves. Given a point $q \in W$, define the pullback $\varphi^\star([q])$ of the divisor $[q]$ as $\sum_{p \mapsto q} e_p[p]$ where e_p is the ramification index. The pullback on divisors is defined by linear extension. Prove that this determines a well-defined map on class groups:
$$\varphi^\star : \mathrm{Cl}(W) \to \mathrm{Cl}(V).$$

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[1] From the end of Sheet III, we have two ways of thinking about subvarieties $\mathbb{P}^1 \times \mathbb{P}^1$: via the Segre embedding or by vanishing loci of bihomogeneous polynomials. These coincide by the work on the previous Sheet III, and as a consequence $\mathbb{V}(F)$ has the structure of a projective variety in the usual sense that we've defined.

10. ($\star$) Construct a smooth projective variety S of dimension 2 and a morphism $\pi : S \to \mathbb{P}^1$ such that (i) away from a finite set of points on $\mathbb{P}^1$, the π-preimage of $p \in \mathbb{P}^1$ is a smooth curve of genus 1, and (ii) there exists a point $q \in \mathbb{P}^1$ such that $\pi^{-1}(q)$ is a singular curve[2].

[2] This is an example of a type of surface called an elliptic fibration