ALGEBRAIC GEOMETRY, SHEET II: LENT 2023

Projective Space Basics

1. Prove that any two distinct lines in $\mathbb{P}^2$ intersect at a single point.
2. Let V be a hypersurface in $\mathbb{P}^n$ and let L be a projective line in $\mathbb{P}^n$. Show that V and L intersect in a non-empty set.
3. Given distinct points $P_0, \ldots, P_{n+1}$ in $\mathbb{P}^n$, no $(n+1)$ of which are contained in a hyperplane, show that there is a change of coordinates on $\mathbb{P}^n$ so that the points are given by $(1 : 0 : \cdots : 0), \ldots, (0 : 0 : \cdots : 0 : 1)$ and $(1 : \cdots : 1)$.
4. A *coordinate line* in $\mathbb{P}^2$ is the vanishing locus of a homogeneous coordinate function. Write down the projective closures of the following affine plane curves and calculate their intersections with the three coordinate lines in $\mathbb{P}^2$.
   (i) $xy = x^6 + y^6$.
   (ii) $x^3 = y^2 + x^4 + y^4$
5. Give an example of a smooth affine variety $V \subset \mathbb{A}^n$ whose projective closure $\overline{V} \subset \mathbb{P}^n$ is not smooth. Give an example of a reducible projective variety $W \subset \mathbb{P}^n$ whose intersection with a standard affine patch $\mathbb{A}^n \subset \mathbb{P}^n$ is nonempty and irreducible.
6. Prove that the following two topologies on $\mathbb{P}^n$ are equivalent:
   (i) A set is closed if and only if it is the vanishing set of a homogeneous ideal.
   (ii) A set $Z \subset \mathbb{P}^n$ is closed if and only if it its preimage under $\mathbb{C}^{n+1} \setminus \{\underline{0}\} \to \mathbb{P}^n$ is closed in the Zariski subspace topology on $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$.

Some Projective Varieties

7. The *Segre surface* $\Sigma_{1,1} \subset \mathbb{P}^3$ is given by $\mathbb{V}(X_0X_3 - X_1X_2)$. Find a pair of disjoint lines that are contained in $\Sigma_{1,1}$. Find a pair of intersecting lines that are contained in $\Sigma_{1,1}$.
8. Consider $V = \{(t, t^2, t^3) : t \in \mathbb{C}\} \subset \mathbb{A}^3$. Observe that V is the vanishing locus of $y_2 - y_1^2$ and $y_3 - y_1^3$. Show that the vanishing locus in $\mathbb{P}^3$ of $X_2X_0 - X_1^2$ and $X_0^2X_3 - X_1^3$ is not irreducible. Calculate generators for the ideal of the projective closure of V.
9. Consider the *cubic surface* $S \subset \mathbb{P}^3$ given by $\mathbb{V}(Z_0^3 - Z_1^3 + Z_2^3 - Z_3^3)$. Find a line ℓ contained on this surface[1]. ($\star$) Find a projective plane $P \subset \mathbb{P}^3$ that contains your line ℓ, such that $S \cap P$ is the union of three lines.

Rational maps and Morphisms

10. The *Cremona transformation* is the map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$(X_0 : X_1 : X_2) \mapsto (X_1X_2 : X_0X_2 : X_1X_0)$$

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[1]This is part of a famous geometry. A (smooth) cubic surface contains exactly 27 lines no matter what the equation is. How many lines can you find?

Let ℓ be the line $\mathbb{V}(X_0 + X_1 + X_2)$ and let $U \subset \mathbb{P}^2$ be a nonempty open in $\mathrm{dom}(\varphi)$. Calculate ideal of the Zariski closure of $\varphi(U \cap \ell)$.

11. Let $Q \subset \mathbb{P}^{n+1}$ be an irreducible quadric hypersurface. Prove that Q is birational to $\mathbb{P}^n$ and use this to calculate the function field of Q.
12. *(Weighted projective space,⋆)* Let $\underline{w} = (w_0, \ldots, w_n)$ be a tuple of positive integers. The <u>weighted projective space</u> $\mathbb{P}(\underline{w})$ is defined by

$$\mathbb{P}(\underline{w}) := \frac{\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}}{\sim}$$

where $\sim$ is the relation that declares $(a_0, \ldots, a_n) \sim (\lambda^{w_0} a_0, \ldots, \lambda^{w_n} a_n)$ for any scalar $\lambda \in \mathbb{C}^\star$. Define homogeneous coordinates on $\mathbb{P}(\underline{w})$ in analogy with $\mathbb{P}^n$. Let X_0, X_1, X_2 be such coordinates on $\mathbb{P}(1, 1, 2)$. Prove that

$$\mathbb{P}(1,1,2) \to \mathbb{P}^3 \quad ; \quad (X_0 : X_1 : X_2) \mapsto (X_0^2 : X_1^2 : X_0 X_1 : X_2)$$

is well-defined. Prove the image is Zariski closed and calculate its homogeneous ideal.