

## ALGEBRAIC GEOMETRY, SHEET I: LENT 2023

### Topological Spaces of Varieties

1. Describe all the open sets in the topological subspace  $\mathbb{V}(XY) \subset \mathbb{A}^2$  in the Zariski topology.
2. Equip  $\mathbb{A}^1 \times \mathbb{A}^1$  with the product topology, where each factor is given the Zariski topology. Characterize all the closed sets in this topology.
3. Prove that  $\mathbb{A}^2$  with the Zariski topology has the property that every open cover has a finite subcover. (If you have trouble with this, start with  $\mathbb{A}^1$ ).
4. Let  $V$  and  $W$  be affine varieties and let  $V \rightarrow W$  be a morphism. Verify that it is continuous in the Zariski topology. Suppose  $\varphi : \mathbb{A}^1 \rightarrow \mathbb{A}^1$  is a set theoretic map that is continuous in the Zariski topology. Is it necessarily true that  $\varphi$  is a morphism?
5. Identify  $\mathbb{A}^{n^2}$  with the set of complex  $n \times n$  matrices. Prove that the subset  $GL(n, \mathbb{C})$  of invertible matrices is Zariski dense. Prove that the set of matrices with  $n$  distinct eigenvalues is also Zariski dense.

### Irreducible Components

6. Prove that in a unique factorization domain, every irreducible element is prime. Using this, prove that if  $f$  is an irreducible polynomial in  $\mathbb{C}[\underline{X}]$  then  $\mathbb{V}(f)$  is irreducible. Consider the variety

$$V = \mathbb{V}(X^2 - YZ) \subset \mathbb{A}^3.$$

Find an element in  $\mathbb{C}[V]$  that is irreducible but not prime.

7. Let  $V \subset \mathbb{A}_k^n$  be an affine variety. Suppose  $V = V_1 \cup \dots \cup V_n$  and  $V = V'_1 \cup \dots \cup V'_m$  are two decompositions into irreducible varieties. Assume that no  $V_i$  is contained in  $V_j$  for  $i \neq j$  and similarly for the  $V'_i$  – i.e. the decompositions are non-redundant. Prove that  $n = m$  and the two decompositions coincide up to reordering.
8. Consider the ideal

$$I = \langle X^2 + Y^2 + Z^2, X^2 - Y^2 - Z^2 + 1 \rangle$$

in  $\mathbb{C}[X, Y, Z]$ . Let  $V$  be the variety  $\mathbb{V}(I)$ . Calculate the irreducible components of  $V$ .

### Morphisms of varieties

9. Prove that the affine curve  $V$  given by  $\mathbb{V}(XY - 1)$  in  $\mathbb{A}^2$  is not isomorphic to  $\mathbb{A}^1$ . Calculate all morphisms

$$\mathbb{A}^1 \rightarrow V$$

10. Let  $V$  and  $W$  be affine varieties in  $\mathbb{A}^n$  and  $\mathbb{A}^m$  respectively. Prove that the product  $V \times W \subset \mathbb{A}^{n+m}$  is also an affine variety. Prove that the projection

$$V \times W \rightarrow V$$

is a morphism. (More difficult: the product of irreducible varieties is also irreducible).

11. Let  $\pi : \mathbb{A}^3 \rightarrow \mathbb{A}^1$  be the projection onto the first coordinate. For each point  $z$  in  $\mathbb{A}^1$ , the set theoretic preimage  $\pi^{-1}(z)$  is isomorphic to  $\mathbb{A}^2$ . Denote this preimage by  $\mathbb{A}_z^2$ . Construct a variety  $V \subset \mathbb{A}^3$  and a morphism

$$\pi : \mathbb{A}^3 \rightarrow \mathbb{A}^1$$

with the property that if  $z \neq 0$ , then  $\pi^{-1}(z) \cap V$  is a union of two intersecting lines in  $\mathbb{A}_z^2$ , but  $\pi^{-1}(0) \cap V$  is a union of two parallel lines in  $\mathbb{A}_0^2$ .

12. ( $\star$ ) Let  $V \subset \mathbb{A}^2$  be the union of the  $X$ -axis,  $Y$ -axis, and diagonal line  $X = Y$ . Calculate generators for  $I(V)$ . Let  $W \subset \mathbb{A}^3$  be the union of the  $X, Y$  and  $Z$  axes. Calculate generators for  $I(W)$ . Show that  $V$  is not isomorphic to  $W$ .<sup>1</sup>

**Local Geometry.** The final question is meant to give you some intuition, but is to be considered non-examinable.

13. Consider the polynomial  $F = Y^2 - X^2(X + 1)$  and sketch the set of real solutions to the equation  $F = 0$  in  $\mathbb{R}^2$ . Call this set  $C$ . You may use a computer program to do this for you.

Let  $\mathbb{D}_\epsilon$  be an open ball around  $(0, 0)$  in  $\mathbb{R}^2$  in the standard Euclidean topology of some small radius  $\epsilon$ . Observe that  $\mathbb{D}_\epsilon \cap C$  is homeomorphic to a union of two axes in  $\mathbb{R}^2$ .

Consider the ring  $\mathbb{C}[[X, Y]]$  of formal power series in two variables. Prove that there exists an element  $G(X, Y)$  in this ring such that  $G(X, Y)^2 = (1 + X)$ . Deduce that the element  $Y^2 - X^2(X + 1)$  can be factorized as:

$$Y^2 - X^2(X + 1) = (Y - XG)(Y + XG).$$

Note that  $G$  is an invertible element in this ring and meditate on the relationship between the two parts of this problem.

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<sup>1</sup>One path is as follows. If  $p$  is a point on  $V$ , let  $\mathfrak{m}_p$  be the set of elements in the coordinate ring that vanish at  $p$ . The quotient  $\mathfrak{m}_p/\mathfrak{m}_p^2$  is a  $\mathbb{C}$ -vector space. What are the possible dimensions of this vector space for different choices of point  $p$  on  $V$ ? How about on  $W$ ?