# Part III Algebraic Geometry

Example Sheet IV, 2023.

1. Let $k$ be a field and $X = \mathbb{P}^2_k \setminus \{p\}$ where $p$ is some fixed point. Consider the natural morphism
$$\pi : X \to \operatorname{Spec} k.$$
The structure sheaf $\mathcal{O}_Z$ of any closed subscheme $Z$ in $X$ can be viewed as a coherent $\mathcal{O}_X$-module. Give an example of a $Z$ such that $\pi_\star \mathcal{O}_Z$ is not coherent. ($\star$) Can you classify all $Z$ such that $\pi_\star \mathcal{O}_Z$ is coherent?

2. A morphism $f : X \to S$ is *finite* if $S$ has a cover by affines $V_i = \operatorname{Spec} B_i$ such that for each $i$, the scheme $f^{-1}(V_i)$ is also affine, isomorphic to, $\operatorname{Spec} A_i$ in such a way that $A_i$ is a finitely generated module over $B_i$.

Assume that $X$ and $S$ are noetherian. Using the valuative criterion for properness, show that finite morphisms are proper. (You may freely use the fact that discrete valuation rings are integrally closed in their fraction field).

3. Let $f : X \to S$ be a finite morphism. Prove that $f_\star \mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. ($\star$) Prove that if $\mathcal{F}$ is any coherent sheaf on $X$, then $f_\star \mathcal{F}$ is also coherent.

4. Let $D$ be a degree 0 Weil divisor on $\mathbb{P}^1_k$. Construct a rational function $f$, i.e. a section of the structure sheaf over the generic point, such that the divisor associated to $f$ is $D$. Deduce that every degree 0 divisor on $\mathbb{P}^1_k$ is principal.

5. Let $k$ be a field and consider the *quadric cone* $X = \operatorname{Spec} k[x, y, z]/(xy - z^2)$. Let $Z = \mathbb{V}(x, z)$. In this problem you may assume that $X$ is normal (or take it as a commutative algebra exercise). Prove that $Z$ is an integral closed codimension 1 subscheme. Prove that $X \setminus Z$ is the spectrum of a unique factorization domain. Using the excision exact sequence, prove that $[Z]$ generates the class group $Cl(X)$.

6. Continuing in the context of the previous problem, prove that the class $[Z]$ is not 0 in the class group, but that $2[Z]$ is 0. Deduce that the class group of the cone $Cl(X)$ is $\mathbb{Z}/2$.

7. ($\star$) Let $X$ be an integral separated scheme of finite type over a field $k$, regular in codimension 1. Prove that the class group of $X$ and $X \times \mathbb{A}^1$ are isomorphic.

8. Assuming the conclusion of the previous question, prove that $Cl(X \times \mathbb{P}^n) = Cl(X) \oplus \mathbb{Z}$. Calculate the class group of $\mathbb{P}^1 \times \mathbb{P}^1$.

9. In lectures, we sketched the construction of a line bundle associated to a Cartier divisor. You may review this construction by examining the two definitions before Propositions 6.11 and 6.13 in Chapter II of Hartshorne. Let $H$ be any hyperplane in $\mathbb{P}^n_k$. Prove that the sheaf associated to the Cartier divisor $[H]$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)$, defined earlier via graded modules[1].

10. Let $X = \operatorname{Spec} A$ be affine, and let
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
be an exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that
$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$
is exact. [You may freely use the following fact about quasi-coherent sheaves: if $\mathcal{F}$ is a quasi-coherent sheaf on a scheme $X$, and $U \subseteq X$ is open affine, then $\mathcal{F}|_U$ can be written as a cokernel of a morphism between free sheaves on $U$. This is an analogue of the type of result concerning locally Noetherian schemes and locally finite type morphisms. These definitions state the existence of an affine open cover with certain properties and then one shows that the required properties hold for any affine open. You may find a proof of this fact in Hartshorne II, §5.]

11. Let $f$ be a homogeneous polynomial of degree $d$ in $n + 1$ variables and let $D$ be the scheme theoretic vanishing locus of $f$ in $\mathbb{P}^n$. Let $\mathcal{I}_D$ be sheaf of ideals (i.e. the subsheaf of the structure) whose sections over $U$ are those functions that vanish on $D \cap U$. Construct an isomorphism between $\mathcal{I}_D$ and $\mathcal{O}_{\mathbb{P}^n}(-d)$. Describe the quotient sheaf $\mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_D$.

12. Let $X$ be the vanishing of a homogeneous polynomial in $\mathbb{P}^n_k$, with $n \geq 3$. Using the cohomology of projective space, the exact sequence in Problem 7, and the long exact sequence in cohomology, show that $H^1(X, \mathcal{O}_X)$ vanishes.

[1] Beware that sign conventions vary when it comes to $\mathcal{O}(1)$ and $\mathcal{O}(-1)$, but only one will make any sense given the context, so it's not a serious source of confusion.

13. Let $X$ be the closed subscheme of $\mathbb{P}^2_k$ defined by the scheme theoretic vanishing locus of a homogeneous polynomial of degree $d$. Assume that $(1,0,0) \notin X$. Then show $X$ can be covered by the two affine open subsets $U = X \cap D_+(x_1)$, $V = X \cap D_+(x_2)$. Now compute the Čech complex explicitly and show that

$$\begin{aligned} \dim H^0(X, \mathcal{O}_X) &= 1 \\ \dim H^1(X, \mathcal{O}_X) &= (d-1)(d-2)/2 \end{aligned}$$

where $d$ is the degree of $f$. Compare with the degree-genus formula for a plane curve.

14. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, $U = X \setminus \{(x,y)\}$ (removing the maximal ideal corresponding to the origin). By choosing a suitable affine cover of $U$, show that $H^1(U, \mathcal{O}_U)$ is naturally isomorphic to the infinite dimensional $k$-vector space with basis $\{x^i y^j \mid i,j < 0\}$. Thus in particular $U$ is not affine.