Part III Algebraic Geometry

Example Sheet III, 2023.

1. We say a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is *Cartesian* or *a fiber square*, if the induced map $A \to B \times_D C$ is an isomorphism. Consider the commutative diagram:

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \longrightarrow & S' & \longrightarrow & S. \end{array}$$

There are three squares to consider: the left, the right, and the outer. Show that (i) if the left and the right squares are Cartesian then then outer is too, and (ii) if the outer square and the right square are Cartesian, the left is too. (Note: If you wish you may assume that all the objects above are schemes and all the arrows are morphisms of schemes. But you will not need to use this! This is true in any category).

2. (Without using the valuative criteria) Let $X \to S$ be a separated morphism of schemes and let $S' \to S$ be any morphism. Prove that if $X \to S$ is separated then $X \times_S S' \to S'$ is also separated. Deduce the analogous statement for proper morphisms.

   (Hint: You need to show that $X' \to X' \times_{S'} X'$ is a closed map. Use the previous problem to relate this to $X \to X \times_S X$.)

3. (Without using the valuative criteria) Prove that compositions of separated morphisms are separated. Deduce the analogous statement for proper morphisms.

4. Let $X \to S$ be a closed immersion and let $Y \to S$ be any morphism. Prove that the base change $X \times_S Y \to Y$ is also a closed immersion. Deduce that closed immersions are proper. In particular, deduce that any closed subscheme $Z \subset \mathbb{P}^n_k$ is proper over $\operatorname{Spec} k$.[1]

5. Let Z be the closed subscheme of $\mathbb{P}^2_{\mathbb{C}[\![t]\!]}$ given by the vanishing locus of the homogeneous ideal $(xy - tz^2)$, where x, y, z are the homogeneous coordinate functions on $\mathbb{P}^2_{\mathbb{C}[\![t]\!]}$. (i) Calculate[2] the scheme theoretic fibers of Z over each of the two points of $\operatorname{Spec} \mathbb{C}[\![t]\!]$. (ii) For each integer $n \geq 1$, inductively define Z^n to be the fiber product of Z with Z^{n-1} over $\operatorname{Spec} \mathbb{C}[\![t]\!]$. Calculate the fiber of Z^n over the closed point of $\operatorname{Spec} \mathbb{C}[\![t]\!]$. Calculate the number of irreducible components of this closed fiber as a function of n.

6. ($\star$) Let k be a field. Let V be the 4-dimensional vector space of 2×2-matrices with entries in k. Analogously, view $k[x, y, z, w]$ as the graded ring of polynomial functions on V. Prove that the set of *rank* 1 matrices in V is Zariski closed defined by an ideal I, and therefore has the structure of an affine scheme. Prove that $I \subset k[x, y, z, w]$ is a *homogeneous* ideal. Let $Z \subset \mathbb{P}^3_k$ be the resulting closed subscheme. Prove that Z is isomorphic to $\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k$.

7. Let X be the line with 2-origins over $\operatorname{Spec} k$, i.e. glue two copies of $\mathbb{A}^1_k$ to itself by identifying the complement of the origin in the obvious way. Identify the fiber product $X \times_{\operatorname{Spec} k} X$ by describing it as a gluing of affine schemes.

8. Explicitly verify that the valuative criterion for universal closedness fails for the scheme X constructed in the previous problem, for the obvious morphism to $\operatorname{Spec} k$.

9. In lecture, we have claimed/will soon claim that given a module M over a ring A, there is an associated sheaf M^{sh} on $\operatorname{Spec}(A)$ whose value over a distinguished open U_f is identified with the localization of M at f. We have also claimed that if $A_\bullet$ is an $\mathbb{N}$-graded ring and $M_\bullet$ is a graded $A_\bullet$–module, there is an analogous sheaf on $\operatorname{Proj}(A_\bullet)$ whose sections, over a distinguished open associated to a positive degree element, are the degree 0 elements in the localization of $M_\bullet$ at f. Give precise definitions of these sheaves and verify that they do indeed form sheaves of modules over the structure sheaf.

[1] This should match the fact from topology that closed subsets of a compact set are compact.

[2] In this context, calculate means to describe the scheme up to isomorphism using basic operations: standard rings, Spec and Proj, and gluing constructions, as explicitly as possible.

Given a graded ring $A_\bullet$ and a graded module $M_\bullet$, define the sheaf $M_\bullet(1)$ to be graded module whose degree n piece is the degree $n+1$ piece of $M_\bullet$. Let X be $\mathrm{Proj}(A_\bullet)$. The sheaf associated to $A_\bullet(1)$ is denoted $\mathcal{O}_X(1)$. Take $A_\bullet$ to be $\mathbb{C}[x,y,z]$ with the standard grading. Calculate the sections of this sheaf over the standard distinguished open sets in $\mathbb{P}^2_{\mathbb{C}}$. Repeat the exercise for the sheaves $\mathcal{O}_X(d)$ for all integers d. For which d does $\mathcal{O}_{\mathbb{P}^2}(d)$ have non-zero global sections?

10. Let X be a Noetherian scheme, $f : \mathcal{F} \to \mathcal{G}$ a morphism of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that $\ker f, \mathrm{coker} f$ and $\mathrm{im} f$ are quasi-coherent. If $\mathcal{F}$ and $\mathcal{G}$ are coherent, show $\ker f, \mathrm{coker} f$ and $\mathrm{im} f$ are coherent.

    Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_Y$-modules. Show that $f^*\mathcal{F}$ is a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules.

    Show by example that if $\mathcal{G}$ is a coherent sheaf on X, then $f_*\mathcal{G}$ need not be a coherent sheaf on Y. [Note: $f_*\mathcal{G}$ is always quasi-coherent, but this is harder to prove.]

11. Let $i : Z \to X$ be a closed immersion of schemes. Recall this means i is a homeomorphism of Z onto a closed subset of X, and the map $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. We write $\mathcal{I}_{Z/X} = \ker i^\#$.

    a) Show that $\mathcal{I}_{Z/X}$ is a *sheaf of ideals* (or *ideal sheaf*) of $\mathcal{O}_X$, i.e., $\mathcal{I}_{Z/X}(U)$ is an ideal in $\mathcal{O}(U)$ for each $U \subseteq X$ open.
    b) Show that $\mathcal{I}_{Z/X}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, and is coherent if X is Noetherian.
    c) Show that there is a one-to-one correspondence between quasi-coherent sheaves of ideals of X and closed subschemes of X.

12. Let k be a field and let $H \hookrightarrow \mathbb{P}^n_k$ be the closed immersion corresponding to the inclusion of a hyperplane, i.e. the vanishing of a linear homogeneous polynomial. Let $\mathcal{I}$ be the ideal sheaf of $H \hookrightarrow \mathbb{P}^n_k$. Prove that $\mathcal{I}$ is isomorphic to the sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$.

13. (Global Spec Construction). Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$–algebras on a scheme X, i.e. a sheaf of commutative rings that is simultaneously a sheaf of $\mathcal{O}_X$-modules. Assume that the underlying sheaf of modules is quasi-coherent. We will now construct the "relative" or "global" Zariski spectrum $\pi : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ as a scheme with a map to X.

    (Set) Given a point $p \in X$, the set $\pi^{-1}(p)$ is defined to be $\mathrm{Spec}\,(\mathcal{A} \otimes \kappa(p))$, where $\kappa(p)$ is the residue field of X at p. Ranging over all p in X defines the points of $\underline{\mathrm{Spec}}\,\mathcal{A}$ with a natural map to X.

    (Topology) Given an affine open U in X, describe a natural bijection between $\pi^{-1}(U)$ and the spectrum of the algebra $\mathcal{A}(U)$. Use this to upgrade $\pi : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ to a continuous map of topological spaces.

    (Functions) Construct a structure sheaf on $\underline{\mathrm{Spec}}\,\mathcal{A}$ by using the identification of sets of the form $\pi^{-1}(U)$ with the spectrum of $\mathcal{A}(U)$ to endow such sets with a natural ring of functions.

    Show that all these data produce a scheme map $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$.

    For any scheme X over S, for each positive integer n, describe a sheaf of algebras on X such that the construction above gives rise to $X \times_S \mathbb{A}^n_S \to X$.

    [If the sheaf of algebras is graded, one can similarly construct a global Proj scheme of X. This is a very useful thing to do; in fact, once you've constructed global Spec, you can patch it together to form global Proj like we did in the ordinary Spec/Proj case.]