Part III Algebraic Geometry

Example Sheet II, 2023. (Edited 12 November 2023 for clarification)

1. Let A be a graded ring and let f be a degree 1 element. In lecture we proved that there is a natural bijection between the subset $U_f \subset \mathrm{Proj}(A)$ of homogeneous primes missing A and ordinary primes in the degree 0 part of the localization A_f. Prove that when equipped with the Zariski topology, this gives a homeomorphism between U_f and $\mathrm{Spec}((A_f)_0)$.

2. Let A be a graded ring and let I be a homogeneous ideal. Prove that the locus $\mathbb{V}(I)$ of homogeneous primes containing I is empty if and only if $\sqrt{I}$ contains the irrelevant ideal A_+.

3. Let k be a field and let $A = k[x]$. Give A the trivial grading, so all elements have degree 0. Describe $\mathrm{Proj}(A)$ in terms of known schemes.

4. Let k be an algebraically closed field. Let A be the ring $k[x, y, z]$ with the following grading: the degree all elements of k and of x is 0 and the degree of y and z is 1; the rest is determined by multiplicativity. Give a natural bijection between the closed points of Proj A and the set $(k \cup \{\infty\}) \times k$. (Informal) What variety is this the scheme theoretic version of?

5. Let $(X, \mathcal{O}_X)$ be a scheme and let $Y \subset X$ be a Zariski closed subset. Let $U \subset X$ be open and affine, and let $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ be the ideal consisting of all elements that vanish on $U \cap Y$. Consider the sheaf of rings on Y that sends $Y \cap U$ to $\mathcal{O}_X(U)/\mathcal{I}(U)$. Prove that this assignment forms a sheaf of rings on Y and verify that Y is a scheme. (This is called the "reduced induced scheme structure" on Y).

6. Let k be a field. In this problem, all morphisms are taken to be morphisms over $\mathrm{Spec}\, k$, i.e. all ring maps that appear are taken to be k-algebra homomorphisms. Let $\mathbb{P}^n_k$ be Proj $k[x_0, \dots, x_n]$. Fix a morphism $f : \mathrm{Spec}\, k(\!(t)\!) \to \mathbb{P}^n_k$, and observe there is also a natural morphism
$$i : \mathrm{Spec}\, k(\!(t)\!) \to \mathrm{Spec}\, k[\![t]\!].$$
Prove that f extends uniquely to a morphism $g : \mathrm{Spec}\, k[\![t]\!] \to \mathbb{P}^n_k$ such that $g \circ i = f$. Show that the analogous property does not hold for $\mathbb{A}^n_k$.

7. Maintain the notation $i : \mathrm{Spec}\, k(\!(t)\!) \to \mathrm{Spec}\, k[\![t]\!]$ from the previous question. Give an example of a scheme X and a morphism $f : \mathrm{Spec}\, k(\!(t)\!) \to X$, such that f extends to two distinct morphisms $g_1, g_2 : \mathrm{Spec}\, k[\![t]\!] \to X$ with $g_1 \circ i = g_2 \circ i = f$. (Hint: try the affine line with doubled origin.)

8. Let X be a scheme and Y be an affine scheme. Prove that morphisms $X \to Y$ are in natural bijection with ring homomorphisms from $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Describe all morphisms from $\mathbb{P}^1_{\mathbb{Z}}$ to $\mathbb{A}^2_{\mathbb{Z}}$.

9. Give an example of two locally ringed spaces X and Y and a morphism $X \to Y$ of *ringed spaces* that is not a morphism of *locally ringed spaces*. (Hint: if you want a scheme theoretic example, try to take Y to be the spectrum of $\mathbb{C}[\![t]\!]$, take X to be the spectrum of the fraction field, and make the topological morphism send X to the closed point of Y – notice there is an obvious map from X to Y but it sends X to the non-closed point!)

   We will now define a number of properties of schemes and morphisms of schemes. This material can be found as a mixture of the text and the exercises of Chapter II, §3 of Hartshorne. Consult that text if you get stuck!

10. We say a scheme X is *irreducible* if it is irreducible as a topological space, i.e., whenever $X = X_1 \cup X_2$ with X_1, X_2 closed subsets, then either $X_1 = X$ or $X_2 = X$.

    We say a scheme X is *reduced* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotents.

    We say a scheme X is *integral* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ is an integral domain.

    Show that a scheme integral if and only if it is reduced and irreducible.

11. We say a scheme is *locally Noetherian* if it can be covered by affine open subsets $\mathrm{Spec}\, A_i$ with A_i a Noetherian ring. We say a scheme is *Noetherian* if it can be covered by a *finite* number of open affine subsets $\mathrm{Spec}\, A_i$ with A_i Noetherian.

    Show that a scheme X is locally Noetherian if and only if for every open affine subset $U = \mathrm{Spec}\, A$, A is a Noetherian ring. [Hint: This is II Prop. 3.2 in Hartshorne. Do have a go at this before you look at his proof. At least try to reduce to the following statement before you peek: given a ring A and a finite collection of elements $f_i \in A$ which generate the unit ideal, suppose A_{f_i} is Noetherian for each i. Then A is Noetherian.]

12. A morphism $f : X \to Y$ is *locally of finite type* if there exists a covering Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra.

    The morphism is *of finite type* if the cover of $f^{-1}(V_i)$ above can be taken to be finite.

    Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

    (Finite type is a very reasonable hypothesis to have on in practice, though objects that are only locally of finite type do occur in nature. Morphisms that are not even locally of finite type are typically pathological.)

13. *Examples.* A disconnected scheme is not irreducible. Find an example of a connected but reducible scheme. Give an example of a non-Noetherian ring whose spectrum is a Noetherian topological space. Give an example of a locally finite type morphism that is not of finite type.

14. *Normalization.* A scheme is *normal* if all its local rings are integrally closed domains. Give 3 examples of non-normal schemes.

    Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization* of X. Show that there is a morphism $\tilde{X} \to X$ having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\tilde{X}$. [A morphism $f : Z \to X$ is *dominant* if $f(Z)$ is a dense subset of X.]