Part III Algebraic Geometry

Example Sheet I, 2023

1. Describe the topological spaces $\operatorname{Spec}\mathbb{R}[x]$, $\operatorname{Spec}\mathbb{R}[x,y]$, $\operatorname{Spec}\mathbb{Z}[x]$, and $\operatorname{Spec}\mathbb{C}[\![x]\!]$. In each case, describe the subset of maximal ideals. In each case, do there exist points that are neither open nor closed?

2. Let A be a commutative ring. Recall that a *distinguished open set* in $\operatorname{Spec} A$ associated to $f \in A$ is a subsets of the form $U_f = \{\mathfrak{p} \in \operatorname{Spec} A | f \notin \mathfrak{p}\}$. Prove that the distinguished open sets form a basis for the open sets in the Zariski topology on $\operatorname{Spec} A$.

3. Let A be an integral domain and let η in $\operatorname{Spec} A$ be the point corresponding to the prime (0) – it is called the generic point. Show that η is dense. Show that if $A = \mathbb{Z}$ then η is not open. If $A = \mathbb{C}[\![t]\!]$ then η is open. Is the generic point of $\operatorname{Spec} A$ open for $A = \mathbb{C}[\![x,y]\!]$? How about for $\mathbb{C}[x,y]$?

4. Let A be a ring and suppose $\{f_i\}_{i\in I}$ is a set of elements in A, indexed by some set I. This gives a set of distinguished opens U_{f_i}. Show that these open sets cover $\operatorname{Spec} A$ if and only if the ideal generated by all these elements is (1). Deduce that if these open sets cover $\operatorname{Spec} A$, then in fact, finitely many of them cover $\operatorname{Spec} A$.

5. Describe the map on Zariski spectra induced by the following map on $\mathbb{C}$-algebras:
$$\mathbb{C}[t] \to \mathbb{C}[x,y,z]/(xy - z), \quad t \mapsto z.$$
In particular, if the map on spectra is denoted $\pi : X \to B$ what are the possibilities for the topological space $\pi^{-1}(b)$ for $b \in B$, up to homeomorphism?

6. Given an example of a homomorphism of rings $\varphi : A \to B$ such that the preimage of a maximal ideal is not maximal. Prove that if φ is surjective then the preimage of a maximal ideal is maximal.

7. Let X_1 and X_2 be the Zariski spectra of rings A_1 and A_2. Describe a natural ring whose Zariski spectrum is homeomorphic $X_1 \sqcup X_2$.

8. Let A be the quotient of a polynomial ring in finitely many variables by a prime. Let $m\operatorname{Spec}(A)$ be the set of maximal ideals of A equipped with the Zariski topology. Describe a procedure that reconstructs the full Zariski spectrum $\operatorname{Spec}(A)$ and its topology in terms of the irreducible closed subsets of $m\operatorname{Spec}(A)$.

   Now, apply this procedure with $A = \mathbb{C}[\![x]\!]$ to conclude that some rings “do not have enough maximal ideals”.

9. (Sheafification is functorial) Prove that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there is an induced morphism $f^{\mathrm{sh}} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$ with $(f^{\mathrm{sh}})_p = f_p$.

10. Describe a non-zero presheaf of abelian groups all of whose stalks are 0. Conclude that the sheafification is the constant sheaf 0.

11. (Exactness is stalk local) Show that a sequence $\cdots \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \cdots$ is exact if and only if for every $p \in X$, the corresponding sequence of maps of abelian groups is exact.

12. Show that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism between sheaves, then the sheaf image $\operatorname{im} f$ can be naturally identified with a subsheaf of $\mathcal{G}$.

13. Show a morphism of sheaves is an isomorphism if and only if it is injective and surjective.

14. (f^{-1} and f_* are adjoint functors.) Given a continuous map $f : X \to Y$, sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, construct natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use this to construct a bijection
$$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \to \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}),$$
(i.e., f^{-1} is left adjoint to f_* and f_* is right adjoint to f^{-1}.)

15. (Gluing) Let $\{X_i\}$ be a family of schemes (possibly infinite) and suppose for each $i \neq j$ we are given an open subscheme $U_{ij} \subseteq X_i$. Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$, such that (1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$ and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

    Then show there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i with an open subscheme of X; (2) the $\psi_i(X_i)$ cover X; (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$; and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

    (Note: The last two exercises are basically big tautologies; these kinds of arguments are affectionately called “diagram chases”. You should only do them once in your life, but you have to do them once in your life.)